Quantum Entanglement: Fundamentals, measures and application

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Week-04

Lec 14: Quantum Entanglement Measure-II

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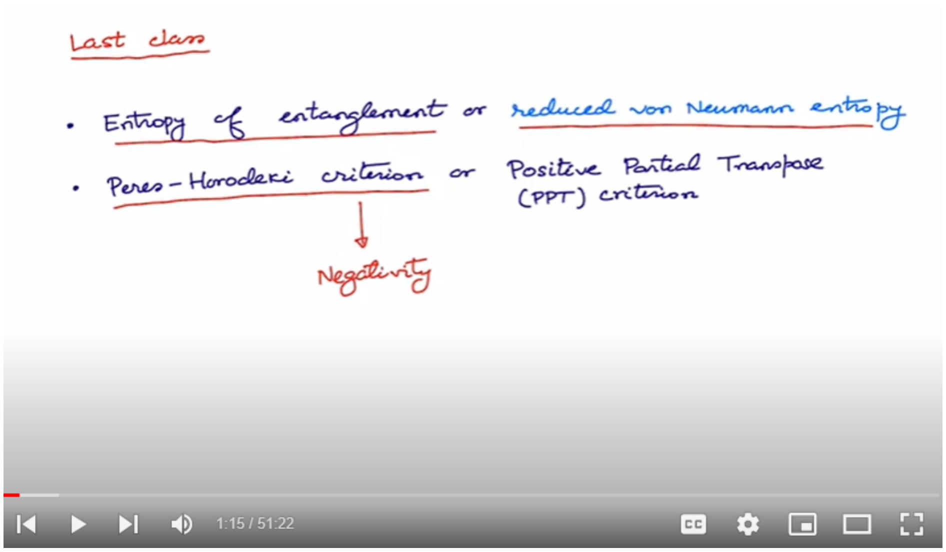

Music Hello, welcome to lecture number 4 of module 3. This is lecture number 10 of this course and in this lecture we will continue discussing entanglement measures. So let us begin. In the last class we discussed about two entanglement measures namely entropy of entanglement which is also known as reduced von Neumann entropy and the so called Peres Horodeki criterion more popularly known as PPT criterion or positive partial transpose and this criterion led to an entanglement measure called negativity and we discussed several example related to entropy of entanglement and negativity.

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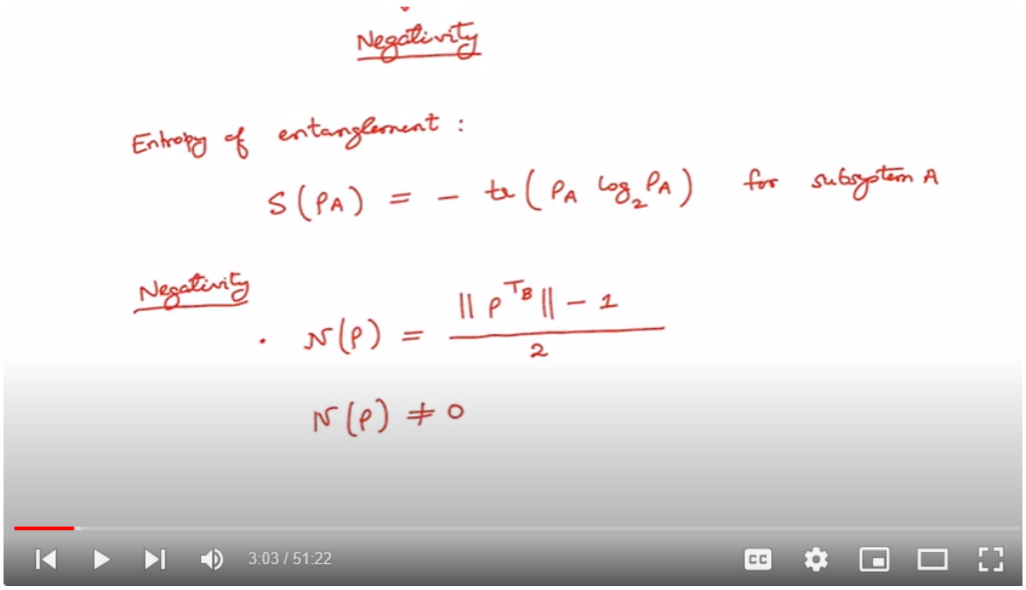

In fact the entropy of entanglement was defined as so this was defined as entropy of entanglement which is one of the most important entanglement measure is given by this S represents entropy it's a function of the reduced density matrix that's why it is called reduced von Neumann entropy trace rho A logarithm of rho A with base 2 log is taken to be is taken over base 2 and it's for subsystem A similar formula we can write for subsystem B as well and on the other hand the formula for the negativity we wrote as it was represented by the symbol italicized n it's a function of the density operator rho and it is given by the trace norm taken over the partially transpose taken partial transpose over the subsystem B and this is a trace norm rho TB is the partially transpose matrix and then you take the transpose trace norm this is the formula and the subsystem B is said to be entangled with the system A we should get negativity to be non-zero the similarly we can find out the negativity with respect to the subsystem A as well okay.

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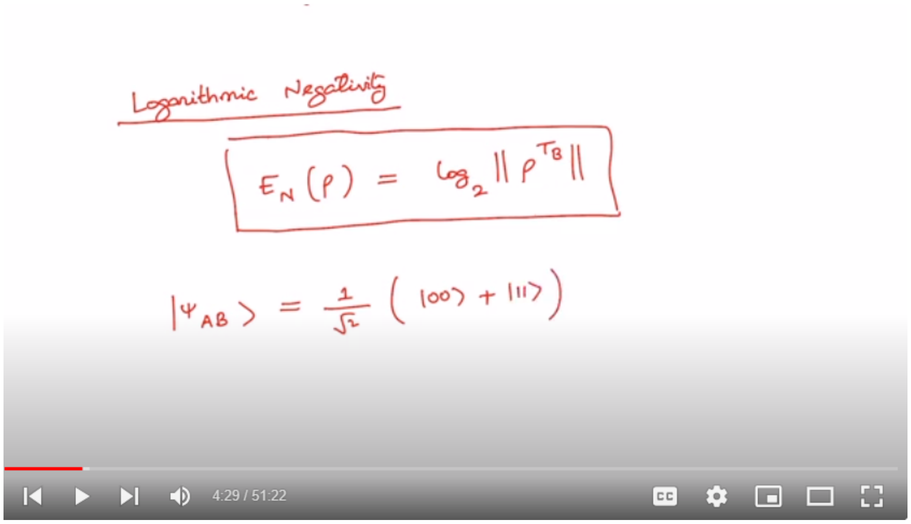

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Now negativity leads immediately to an another convenient measure of entanglement called logarithmic negativity logarithmic negativity and this is represented by the symbol E subscript n it's a function of the density operator and it is defined as logarithm of the trace norm of the partially transpose matrix density operator not density operator after you take the partial transpose over the density operator rho representing the two composite system A and B and you take the partial transpose over the system B or subsystem B right and to this is the formula and this is very useful it's very easy to work out in fact let us do a quick example taken we considered this particular state in the last class as well so let us just consider that once again suppose a composite system is represented by this ket state 1 by root 2 ket 0 0 plus ket 1 1 and we intend to find out the logarithmic negativity with respect to say the subsystem B and we have worked this out in the last class the density operator.

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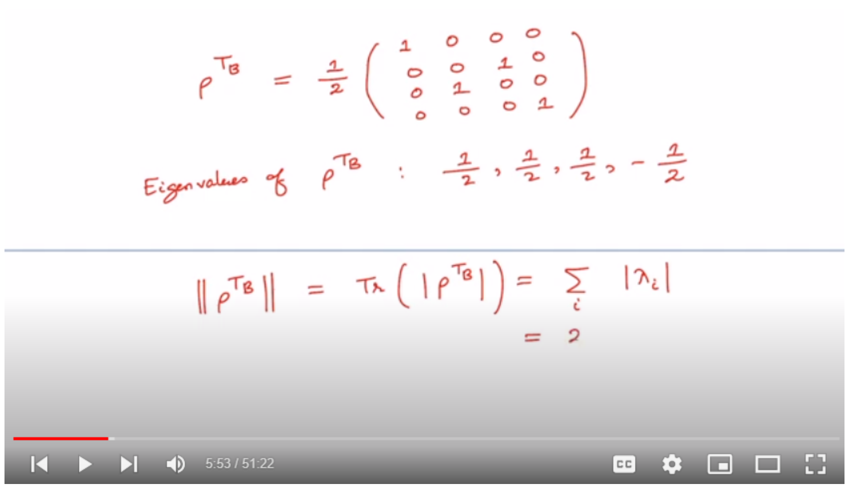

We had we have worked out and then we worked out the partially transposed matrix when we have taken the partial transpose over the system B subsystem B and we found the partially transposed matrix to be of this form it would be here 1 0 0 0 0 0 1 0 0 1 0 0 0 0 0 1 and this will give us the we worked it out the eigenvalues of this matrix eigenvalues of rho T B we got them to be one half one half one half and minus one half this led us to the fact that the negativity was turned out to be non-zero and this matrix rho T B is not a valid density matrix because it is not a semi-positive definite and the trace norm we can calculate rho T B that would be equal to trace of you know modulus of rho T B trace norm if you take and this is basically equal to the sum of all the eigenvalues the magnitude of the eigenvalues so in this case we will get it to be plus 2.

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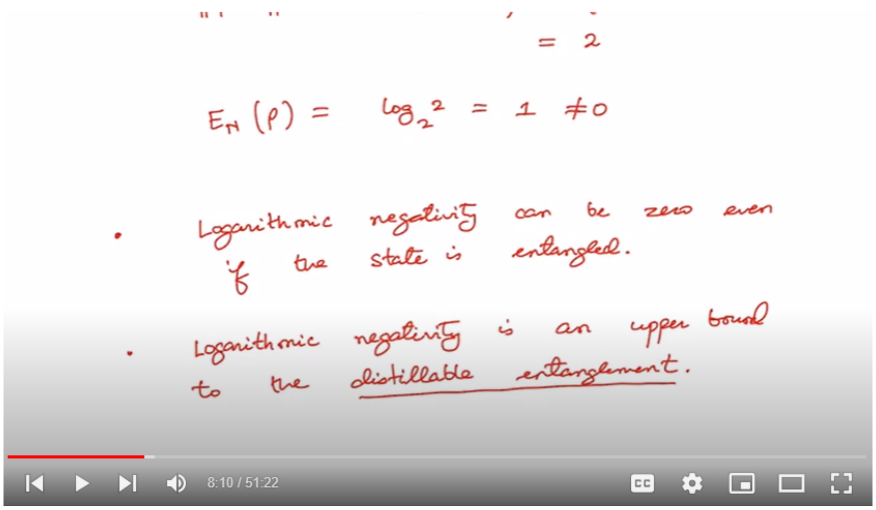

Therefore logarithmic negativity is would turn out to be in this case would be log of 2 base 2 so this would be equal to 1 so you see the logarithmic negativity is a non-zero quantity here which indicates that the subsystem A and B are entangled this is a trivial example but we can work out many many examples and we already know that the state that we have taken is a build state and this is an entangled state and yes the logarithmic negativity also shows that that also gives the correct measure that indeed the system subsystem A and B are entangled but unlike negativity however there's a difference logarithmic negativity can be zero so let me write it unlike negativity logarithmic negativity logarithmic negativity can be zero even if can be zero even if even if the state is entangled state is entangled so one have to be very careful one cannot just conclude that just because you are getting logarithmic negativity to be zero that the state is not entangled but for sure if the logarithmic negativity is not zero the state is entangled but if you find it to be zero sometime what may happen is that the system may be entangled actually and logarithmic negativity another important property of logarithmic negativity is that or effect associated logarithmic negativity that logarithmic negativity is an upper bound is an upper bound to the distillable entanglement to the distillable entanglement now we have not talked about distillable distillable entanglement as yet so we are going to discuss that.

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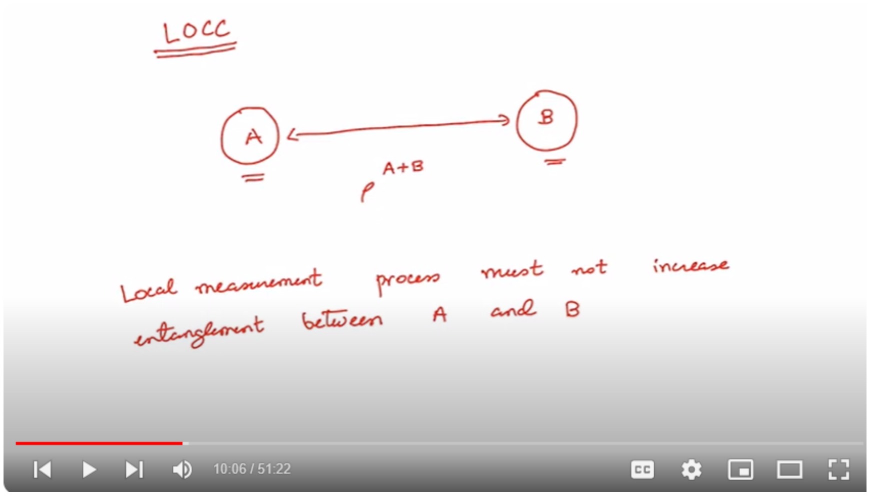

But before i discuss it let me once again clarify the idea of quantification of entanglement in the context of LOCC or local quantum operational classical communication you see we want to know given two subsystem A and B at a far away distance what is the degree of entanglement between them and to know that some experiment which mathematically speaking unitary operation needs to be made on subsystem A or B locally however this local operation should not increase the entanglement between A and B for example the system A and system B suppose their composite system is represented by the density operator rho A plus B and if the system are say to spin half system and you rotate the spin system A as a part of some measurement process this operation should not increase the entanglement between A and B hope you get the idea so that means that local measurements or local unitary operations local measurement measurement process must not this is extremely critical for us must must not increase entanglement between entanglement between A and B because ultimately when we say entanglement measure that means we have to carry out some measurement or unitary operations and because these are measurement and this local measurement should not increase the entanglement between A and B entanglement may decrease actually but it should not increase entanglement between them.

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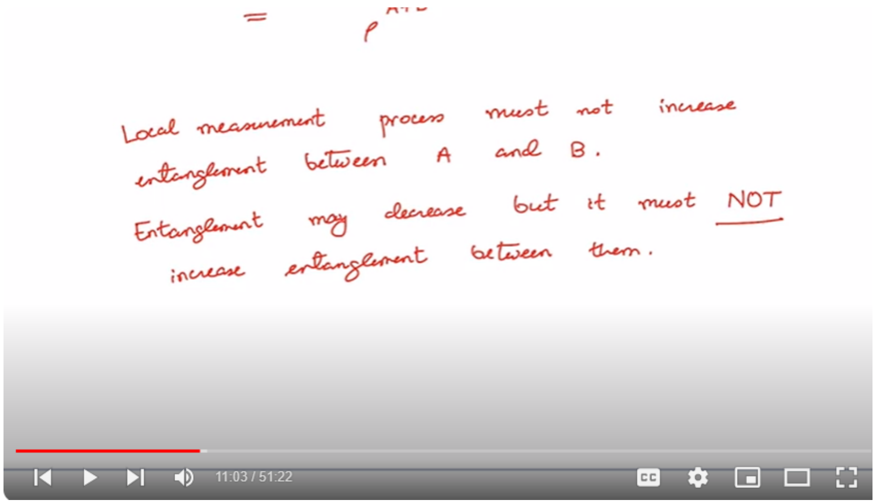

Okay this is very important entanglement may decrease as a result of this process but it should not or it must not if it has to be a proper entanglement measure it must not increase entanglement entanglement between the two subsystem between them okay you can do local operation plus classical communication that means locc you can do but this locc should not increase entanglement this is the general rule for entanglement which we discussed in an earlier class on the properties of entanglement.

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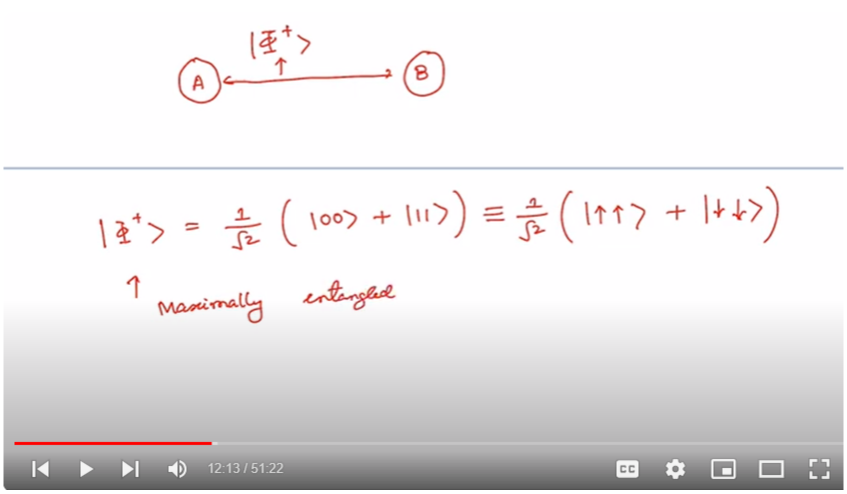

To give you an example let us say we have two qubits two qubits belonging to A and other one qubit belonging to A and other qubit belonging to B here you can consider A and B to be Elise and Bob and they share a Bell state this here a Bell state let us say phi plus so A and B Elise and Bob share a Bell state phi plus and you know that phi plus is is one by root two ket zero zero plus ket two one one or if it is a spin system I can write it as one by root two so spin up up and spin down down right so this is what we mean by phi plus Bell state you already know that this is maximally entangled these bales all Bell states are maximally entangled states this is maximally entangled all right.

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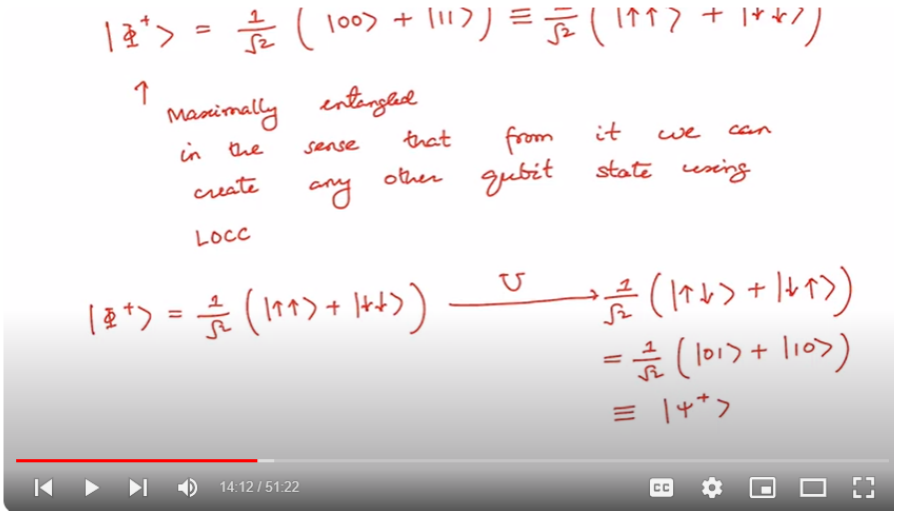

This maximally entangled state in what sense this is maximally entangled state in the sense maximally entangled in the this is important to understand in the sense that from it from it we can create we can create any other any other qubit state qubit state using locc or some clever you know local operations and classical communication protocols or process if we using locc from this we can create for example let us say we we are starting with phi plus which is one by root two in the case of spin half system I can write it as say up up and down down state and by using locc what I can get by some unitary operations I can end up in the state one by root two I just flip the spin of say Bob and then I get accordingly I get some local operations I get this particular state and this is also this is nothing but in our zero one notation it is one by root two ket zero one plus ket one zero which is another Bell state we have we are basically getting another Bell state starting with phi plus sometimes we may get this is from here we are getting a maximally entangled state from a maximally entangled state but sometime we get an entangled state which may not be maximally entangled state this idea of getting other state from a given state through locc means that we can compare the states to find which one is having more entanglement.

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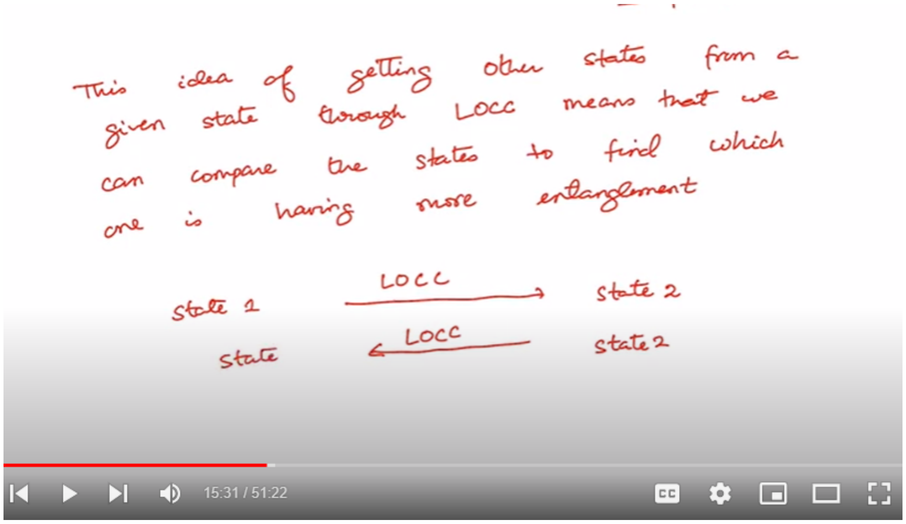

Let me write it this is very critical for us this idea of getting other states other states from a given state from a given state I am talking about entangled state here through locc means that means that we can we can compare we can compare the states to find which one is more entangled which one is having more entanglement. So this is basically the essence of entanglement measure so suppose you go from state one to state two you have started with a state one and you get another state via locc local operation quantum operation and classical communication or you get state one from state two via locc right.

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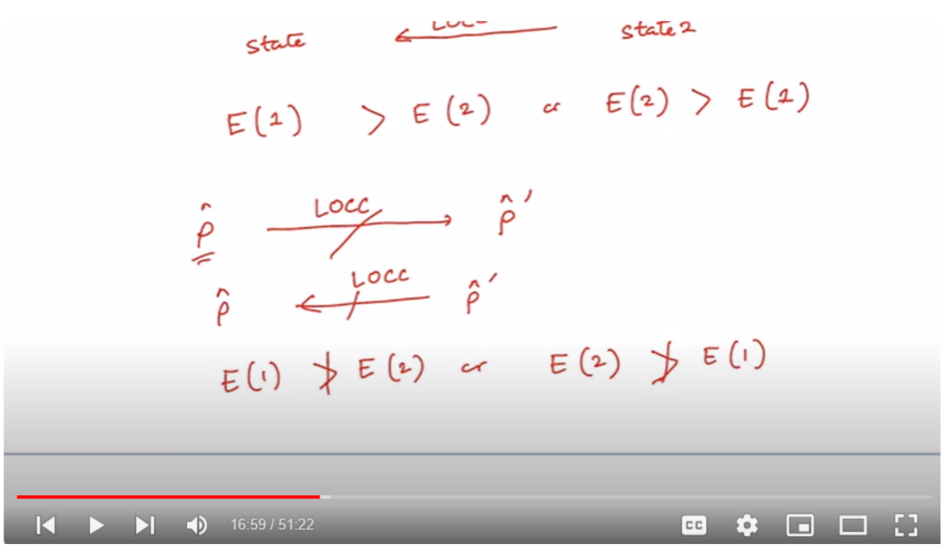

Then the question we can ask is that what about the entanglement whether entanglement associate with state one is more than that of the entanglement in state two or vice versa that means entanglement state two is more than that of entanglement in state one this kind of question we can ask now but sometimes however there are situations where states are incomparable and we cannot go from a state say rho to another state say rho prime via locc it is not possible or we cannot go from the state rho prime to the state rho via locc so this is also a possibility and in that case we can't tell which one is having more entanglement we cannot do an ordering in this case that means we cannot say that entanglement of state one is more than that of entanglement in state two or we cannot say that e2 is you know entanglement of state state two is more or less that of entanglement in state one this kind of ordering would not be possible if we cannot create one state from another state right you i hope you get the idea.

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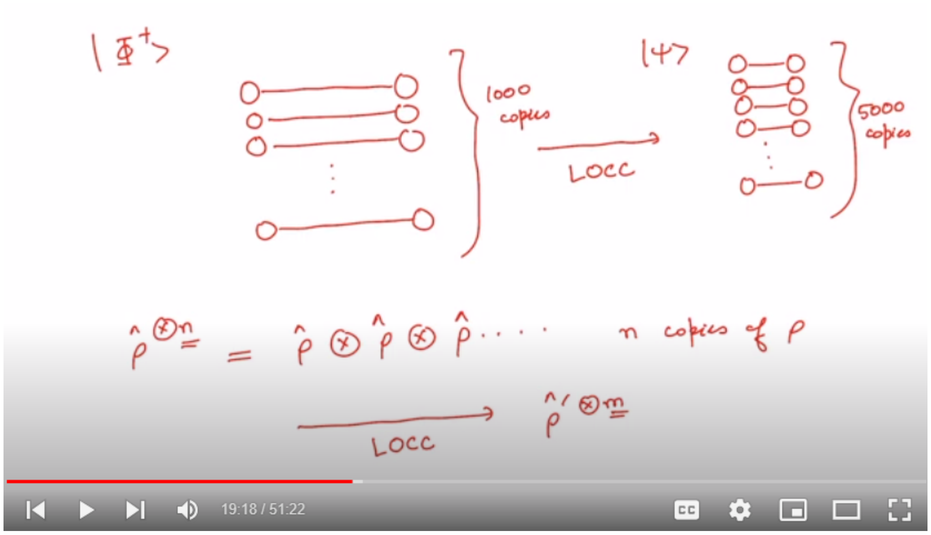

Now the question is that is there a way out in this situations yes there is there is a way instead of simply working on only two qubit states let us deal with a huge number of say thousands thousands of copies of this qubit state for example suppose we have this Bell state phi plus let us make a thousand copies or whatever number of copies you want you make a huge number of identical copies basically ensembles you you create and you have suppose this much of copies suppose thousand copies let us say you have thousand copies of phi plus and then you can ask questions such as with this thousand copies of our two qubit states are we able to produce say 50,000 copies or so of another two qubit suppose via locc can i create some other state some arbitrary state say psi ket psi some copies suppose you have started with thousand maybe you can create thousand or ten thousand or fifty thousand whatever number of copies is it possible to do that let us say we have started with thousand let us say you get another fifty five thousand copies of another arbitrary state ket psi starting with one particular Bell state okay this this particular approach turns out to be quite helpful say we take n copies of rho suppose we have n number of copies of rho that means we have rho direct product rho direct product rho like this n times we are having n copies of rho so we are having n copies of rho and via locc we want us we can get another state having say another state say rho prime having m number of copies we have started with n number of copies now we are going over to creating m number of copies using locc okay.

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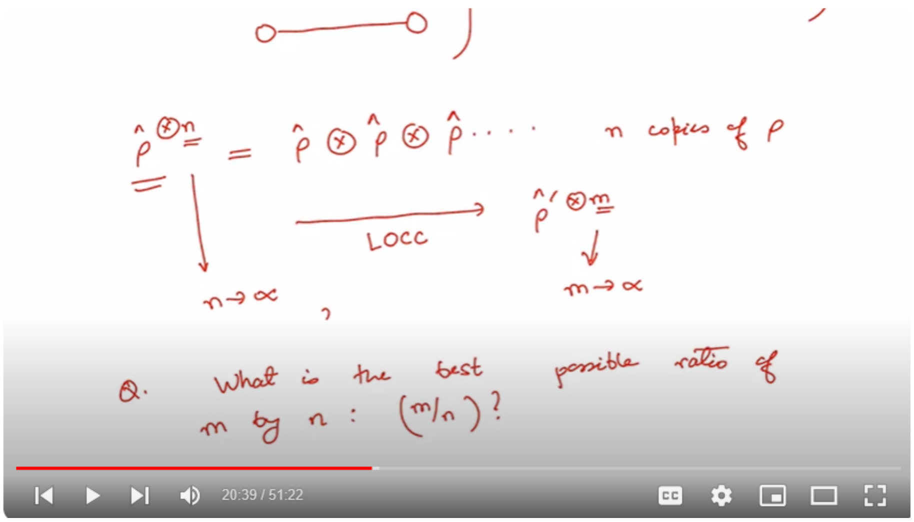

And the question here we can ask is if we allow the number of copies of rho number of copies of rho to go towards infinity say n tends to infinity this one if we allow it to go towards infinity and m also here this m also tends to infinity then what is the best possible ratio the question is what is the what is the best possible best possible ratio ratio of m by n okay that means what is the best possible ratio of m by n this is the question we can ask that means what we can achieve this has been the basis of various entanglement measure and it will be clear if i discuss some entanglement measure based on this particular idea we always want to compare states with regard to entanglement but what will you compare against now the question is well we know that Bell states are maximally entangled so we can compare against such Bell states okay.

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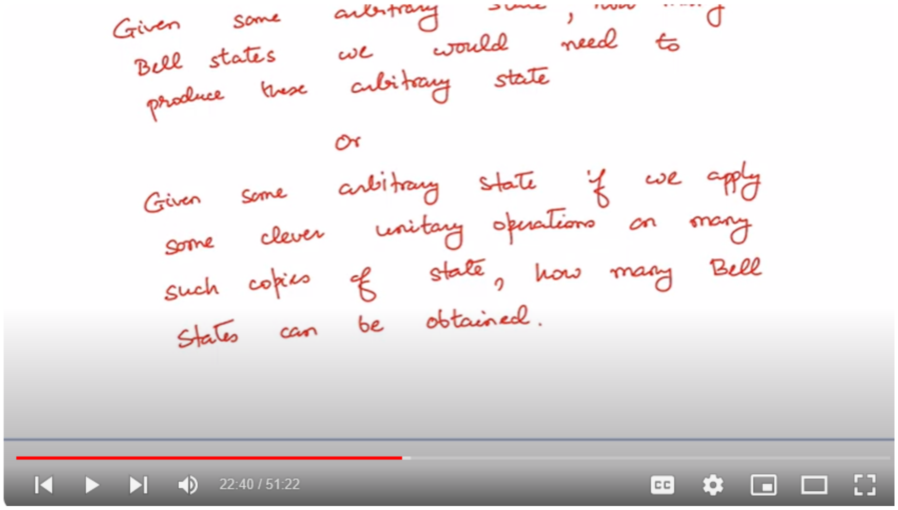

So given some say arbitrary state given some arbitrary state given some arbitrary state how many Bell state how many Bell states because Bell states are maximally entangled state how many Bell states we would need we would need to produce to produce these arbitrary states produce these arbitrary states okay starting with a Bell state or we can ask the opposite question given some arbitrary state given some arbitrary state if we apply we apply some clever some clever unitary operation unitary operation okay on many such copies on many such copies of this arbitrary state copies of state how many Bell states opposite thing we are doing how many Bell states we can produce how many Bell states can we obtain can we obtain so this is the essence of some entanglement measures that now i am going to discuss.

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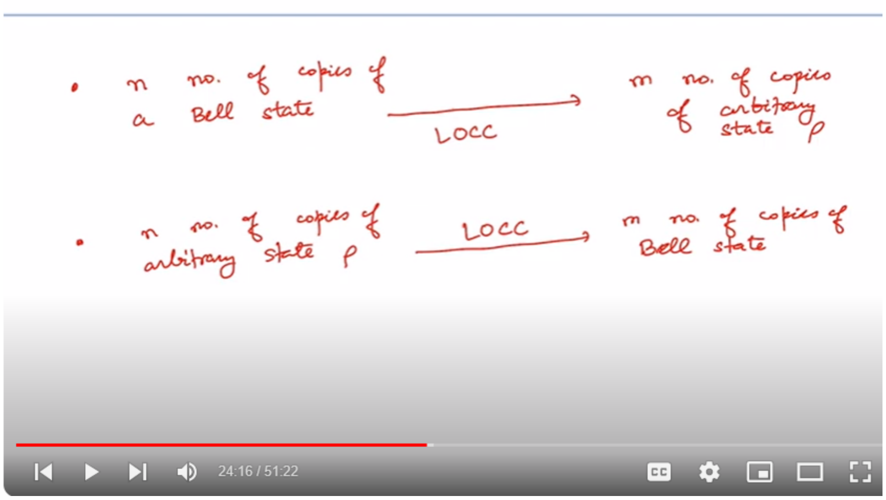

So i hope you are getting the idea let me again say again state this once again suppose you have n number of suppose you have n number of copies copies of a Bell state Bell state there are four Bell states suppose you pick up any Bell state and you take n number of copies of the Bell state and by locc operation local quantum operation and classical communication you get m number of copies of arbitrary state arbitrary state rho right and we are doing it we are doing it because as i said keep Bell state has is maximally entangled and we would like to compare our greatest state with a maximally entangled state or this is one thing or we have say n number of arbitrary state m number of copies of arbitrary arbitrary state rho and via locc local operations and classical communication we want to create m number of copies m number of copies of Bell state so this particular idea is at the root of many entanglement measures and we are now going to discuss some of them.

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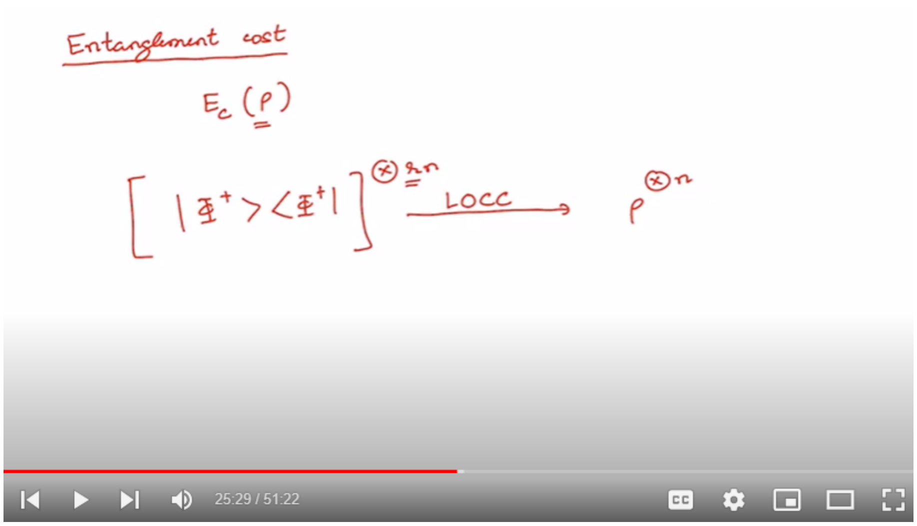

The first entanglement measure based on these ideas i want to discuss is a so-called entanglement cost entanglement cost and this is denoted by the symbol e suffix c c refers to cost and it's a function of the density operator and here rho rho represents the density operator representing the arbitrary state the concept behind this measure is simple say we have one of the maximally entangled Bell state say phi plus the density operator corresponding to this Bell state would be ket phi plus bra phi plus and let us say we have r into n number of copies of this Bell state and we hope to using local operation and classical communication we hope to create an arbitrary state rho and we want to create n number of copies of this arbitrary state and here we the goal is to find out the smallest r for which this is possible then tells us about what is the cost of producing the entangled state.

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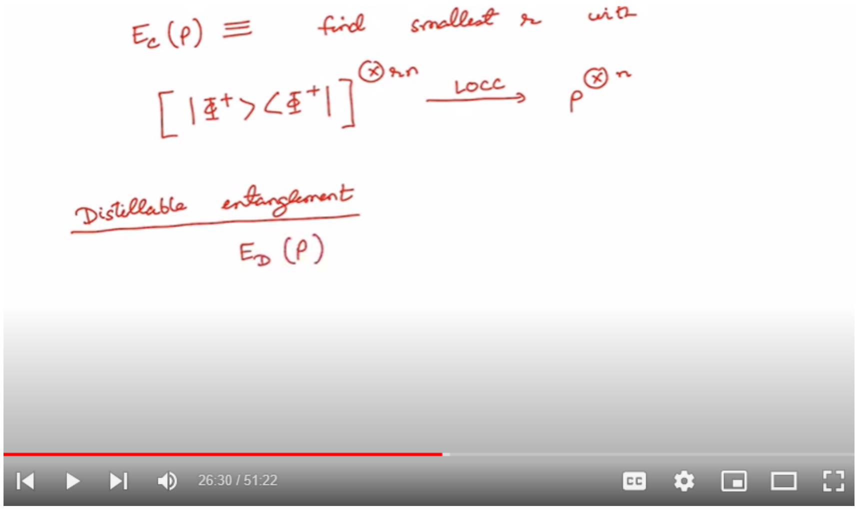

So entanglement cost boils down to basically finding out find smallest r find smallest r with you will start with i am repeating myself here you will start with say the particular maximally or any maximally entangled state and from here you create via locc local operation and classical communication you create n number of arbitrary states and there is a actually another measure related to this this is called distillable entanglement let me discuss that also and in fact i have already mentioned about it a while back distillable entanglement this is similar to entanglement cost and that's the reason it is called dual to entanglement cost and it is denoted by the symbol e suffix d rho.

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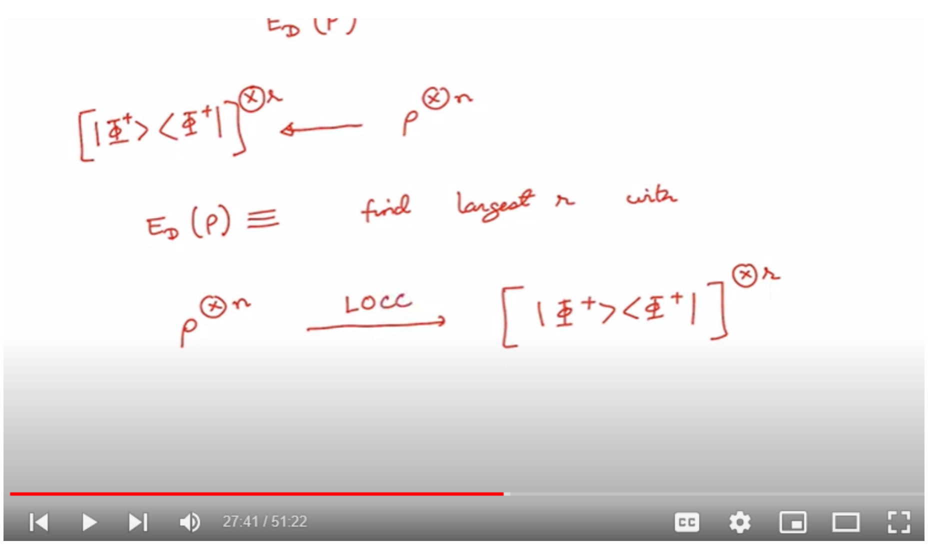

And here we you just go to other direction to that of the entanglement cost you take n number of arbitrary states you take n number of arbitrary state and from this arbitrary state you produced maximally entangled state say phi plus you produce r number of r number of maximally entangled state and here the idea is to have you know maximum number maximum r that means find largest r so entanglement of distillation or distillable entanglement boils down to find largest r largest r with you are going from an arbitrary n number of copies of the arbitrary state with maximum number of bale state you want to generate starting with some number of copies of an arbitrary state and which is opposite to that of the entanglement cost as you can see.

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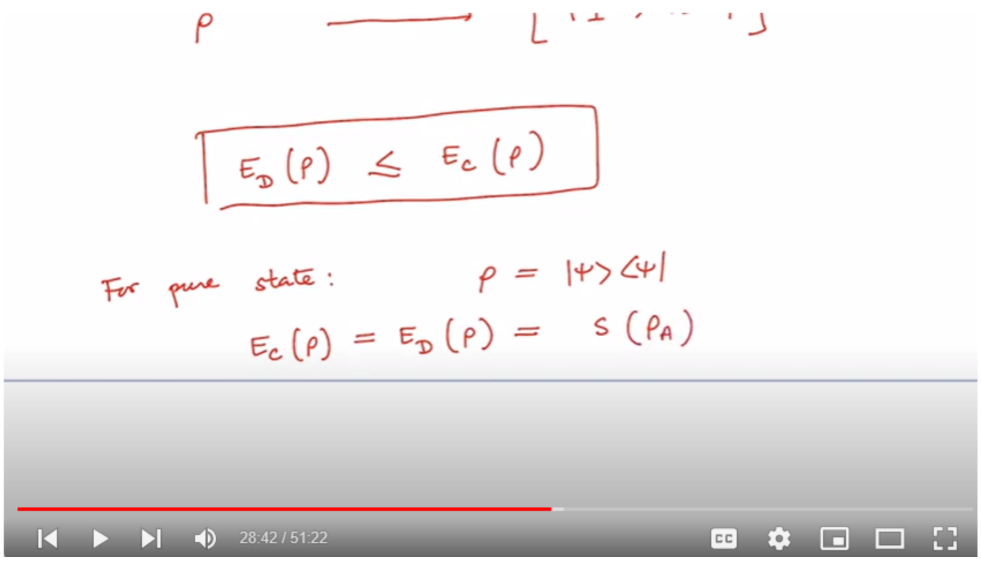

And this should also be clear to you intuitively that entanglement of distillation or distillable entanglement should be less than the cost of entanglement cost because if it is not then you see one can obtain or create entanglement by means of locc by converting bale state to a state not satisfying this particular relation this condition and converting them back again right for pure state things are quite easy for pure state for pure state you have rho is equal to ket psi bra psi and here it turns out that cost of entanglement is exactly equal to entanglement distillable entanglement and which turns out to be equal to entropy of entanglement okay and this is easy for pure state but for mixed state finding cost of entanglement or distillable entanglement is a difficult task it's in fact pretty difficult.

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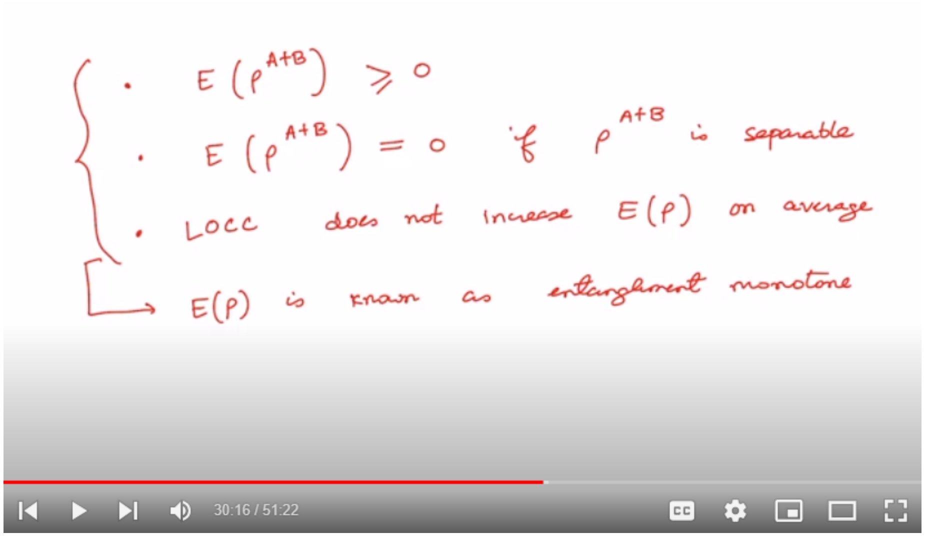

But whatever be the entanglement measure it is expected that you should get entanglement for a non-separable system to be should be greater than zero and it has to be equal to zero rho a plus b if it is this is a separable state or that means it is not entangled obviously you should get it to be zero rho a plus b is separable separable and locc local operation and classical communication does not does not increase entanglement measure e rho on the average okay this is it has to be any entanglement measure must has to satisfy this if all these three are satisfied then then e rho the entanglement measure actually e rho this quantity is known as i think i have discussed about it in a previous class e rho is known as entanglement monotone entanglement monotone.

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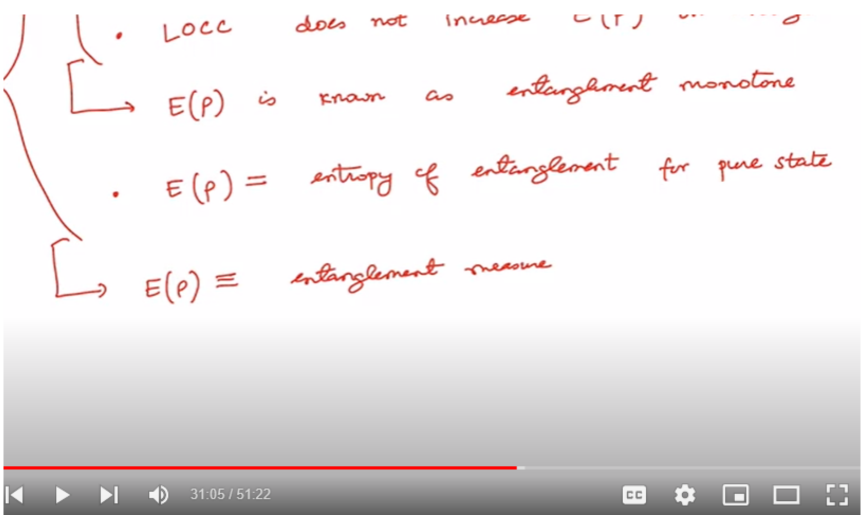

But if there is an other property a desirable property that for pure state if you have e rho is equal to entropy of entanglement entropy of entanglement whatever be the measure if boils down to entropy of entanglement for pure states this is desirable and if these four properties in fact if all these properties are satisfied then e rho is turned is called entanglement measure then e rho is qualified to be called entanglement measure entanglement measure so i just want to re-emphasize it that's why because i have already discussed it in a previous class.

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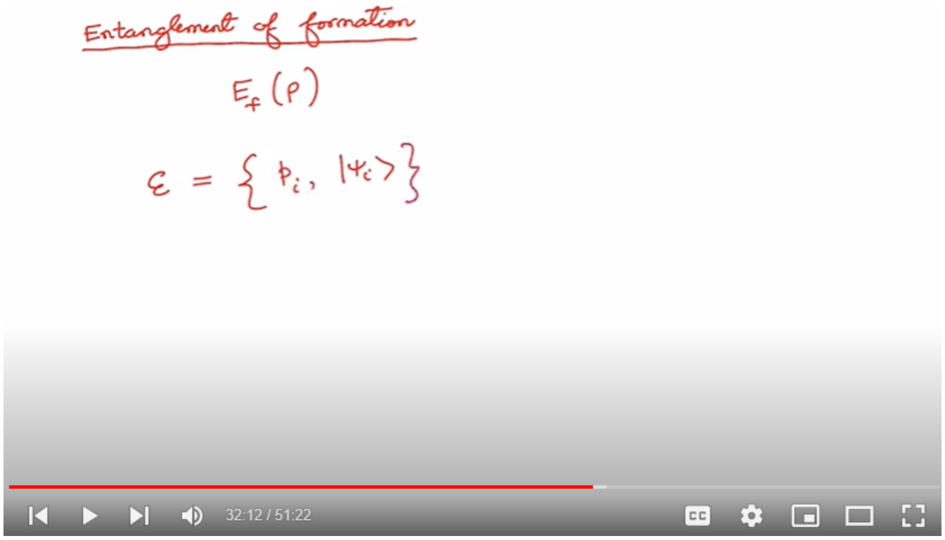

Let us now briefly discuss about the entanglement measure called entanglement of formation historically this is the first entanglement measure that appeared in literature it is also occasionally called entanglement of creation i mentioned that one desirable property of entanglement measure is that it should be equal to entropy of entanglement for pure states now the entanglement of formation is a straightforward generalization of entropy of entanglement for mixed state entanglement of formation is denoted by the symbol e suffix f and it's a function of the density operator rho now before i talk about the proper definition of entanglement of formation let me discuss briefly about the definition of entropy of entanglement in the context of a an ensemble of pure state suppose i have an ensemble of pure state e which is a collection of pure state psi i with corresponding probability pi.

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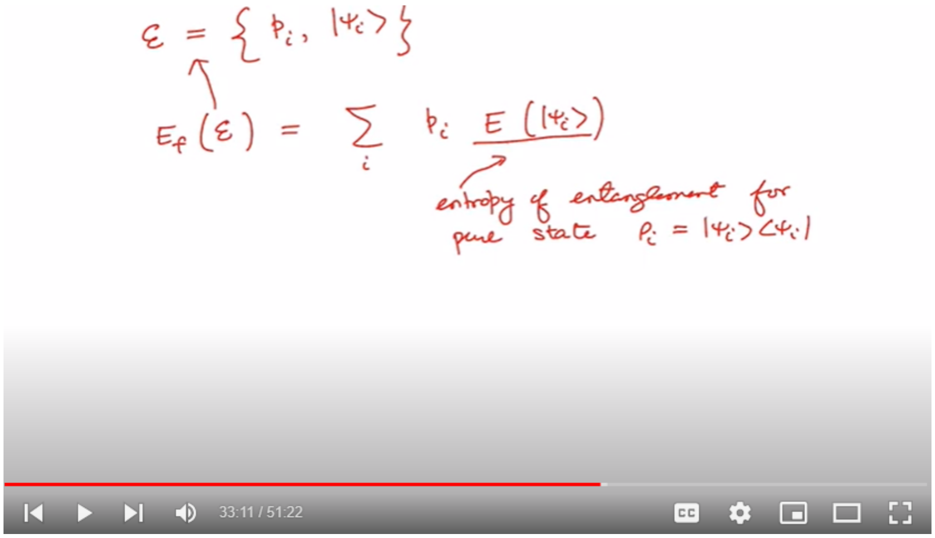

Then the entropy of entanglement or entanglement of formation is defined as the avarice of the entropy of entanglement for the states in the ensemble and it is given by this formula e f of e keep it italicized e which refers to this ensemble it is equal to sum over pi and e of psi i where e is the this is the entropy of entanglement for the pure state psi i it is entropy of entanglement entanglement for pure state if i write it in the density operator then it would be rho i is equal to ket psi i bra psi i so this is the definition for entanglement of formation for an ensemble.

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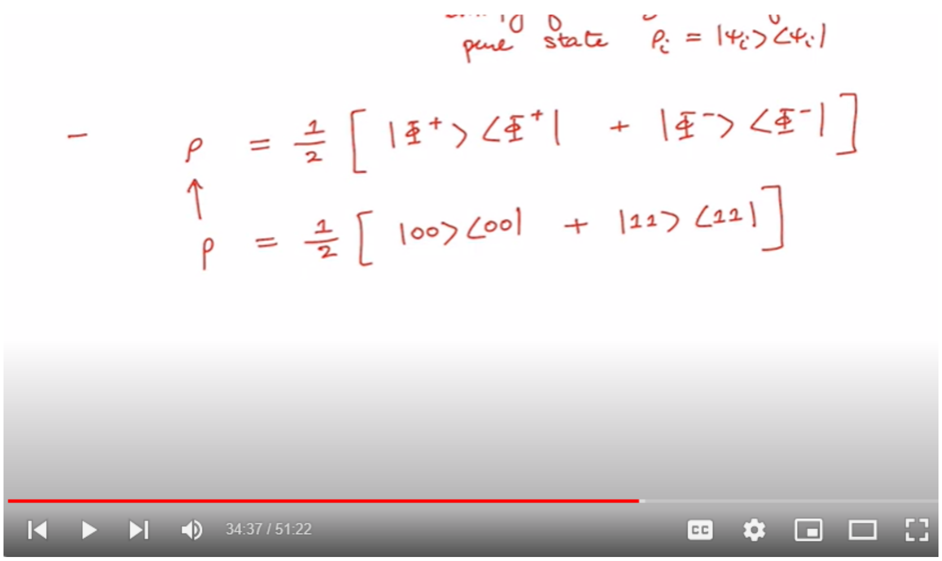

Now coming to mixed state you know a mixed state can be realized by multitude of pure state and symbols with different entanglement of formation suppose i have a you know mixed state rho given by a collection of bell states suppose i have this pure states are there phi plus phi plus this is with 50 probability then i have another collection so let me take half common there so this means that this is a mixed state you know which is formed by the pure state phi plus and phi minus and what we can get that this particular mixed state rho can also be achieved by having a mixer of ket 0 0 a pure state a mixer of pure state ket 0 0 and ket 1 1 right so this way there are many ways you can get the same rho by various mixers now as any of those ensemble realize a mixed state the natural definition for the entanglement of formation for a mixed state is the entanglement formation for the most economic ensemble that is the ensemble causing least cost of entanglement.

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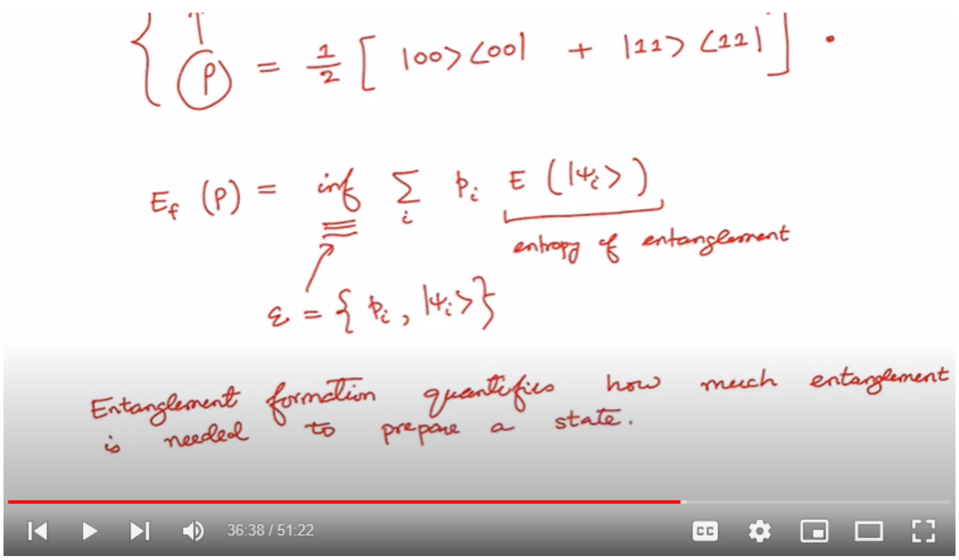

So entanglement formation for a mixed state is defined as e f of rho is equal to infimum summation sum over i p i e of psi i where this is the entropy of entanglement this is entropy of entanglement okay now you see this infimum is taken over all the ensemble it is taken over all the ensemble p i psi i or if i again take this example here entanglement formation means that out of these two possibilities suppose i have only these two mixers two possible mixers are there by which i can get this mixed state rho and out of these mixers when i am taking the infimum i will consider only one of these two for who is the entropy of entanglement is going to be minimum so i hope you get it by infimum i means that i am taking this collection of sets i will take the take that one for which i get the minimum entropy of entanglement now as you can see that entanglement of formation basically me quantifies how much entanglement is necessary on the average to prepare a state so entanglement formation entanglement formation quantifies quantifies how much entanglement is needed entanglement is needed to prepare a state okay.

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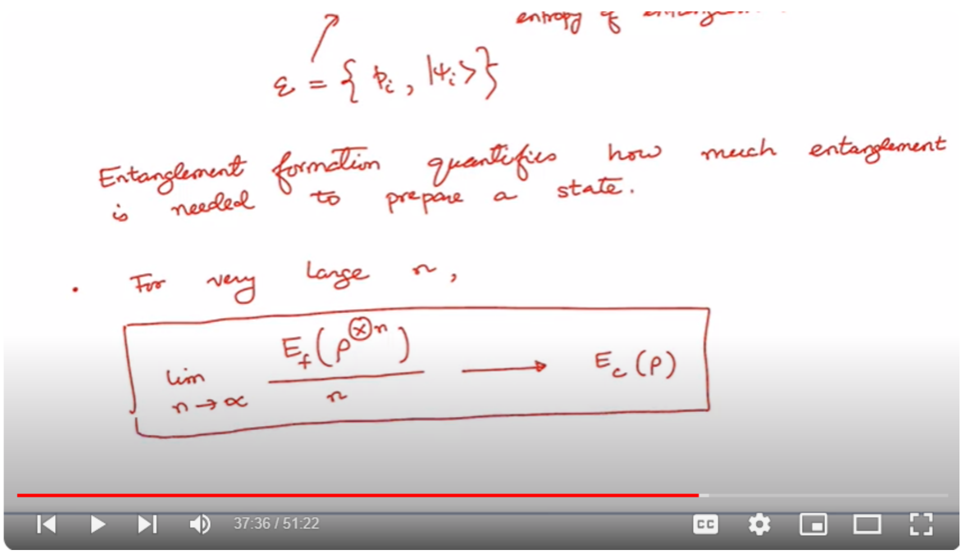

Now it turns out that for very large n for very large n the entanglement of formation for n number of say copies of a state rho represented by rho entanglement of formation for a large number of copies of a state rho if we divide it by the number of copies and in the limit n tends to infinity that means very large n then this quantity approaches or converses the cost of entanglement or entanglement cost okay this is an important point to consider this is an important point to remember.

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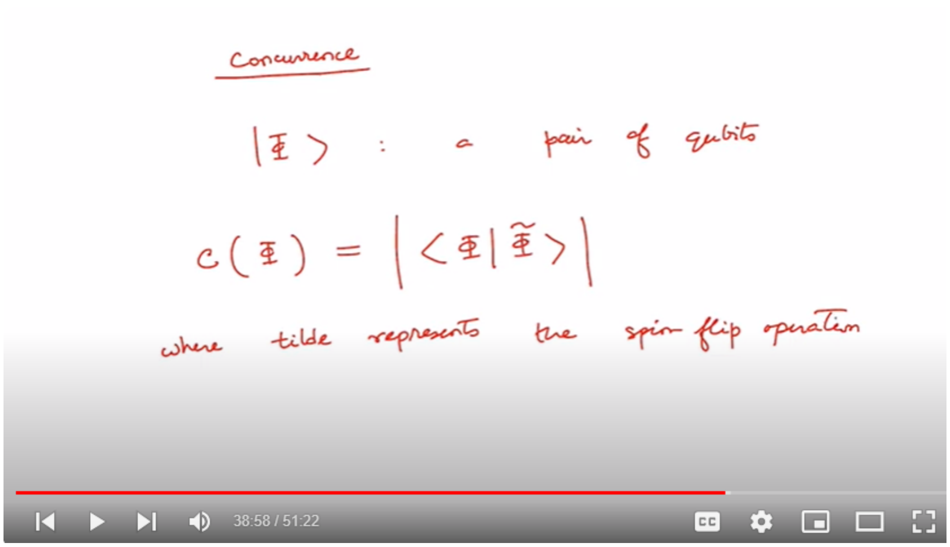

Now there is a useful entanglement measure called concurrence and this is generally defined in the context of two qubits and also associated with or related with entanglement of formation and this is we are not going to discuss that in great detail but i will just very simply discussed it for a system of two qubits and for first of all i will discuss it for pure state and suppose for a pure state we have a state kth state which represents a two qubit state this represents a pair of pair of qubits and concurrence for this state pure set phi is defined as the modulus of the scalar product of phi and phi tilde where this tilde let me write here where tilde represents the spin flip operation spin flip operation.

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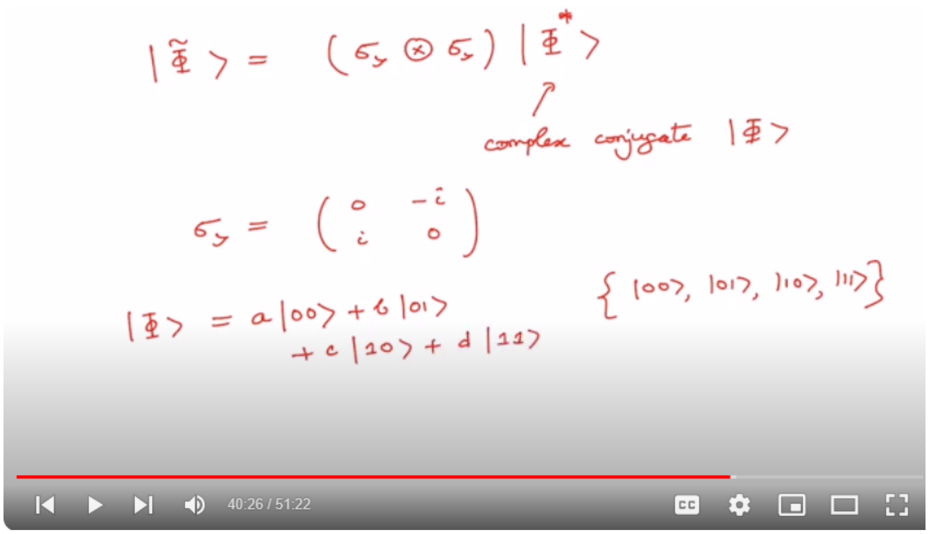

Let me elaborate this phi tilde ket phi tilde is is equal to the direct product or the tensor product of this Pauli operator sigma y over this operation is this is the spin flip operation done on the state phi star ket where phi star is this ket phi star is the complex conjugate this is complex conjugate of conjugate of phi in fact let me explain it by using an example just remember that that sigma y the Pauli matrix is given as 0 minus i i 0 suppose i have a state ket phi in the computational two qubit basis where the basis is ket 0 0 ket 0 1 ket 1 0 and ket 1 1 in this basis state i can write a two qubit state a general two qubit state maybe say superposition of this basis state a ket 0 0 plus b ket 0 1 plus c ket 1 0 plus d ket 1 1.

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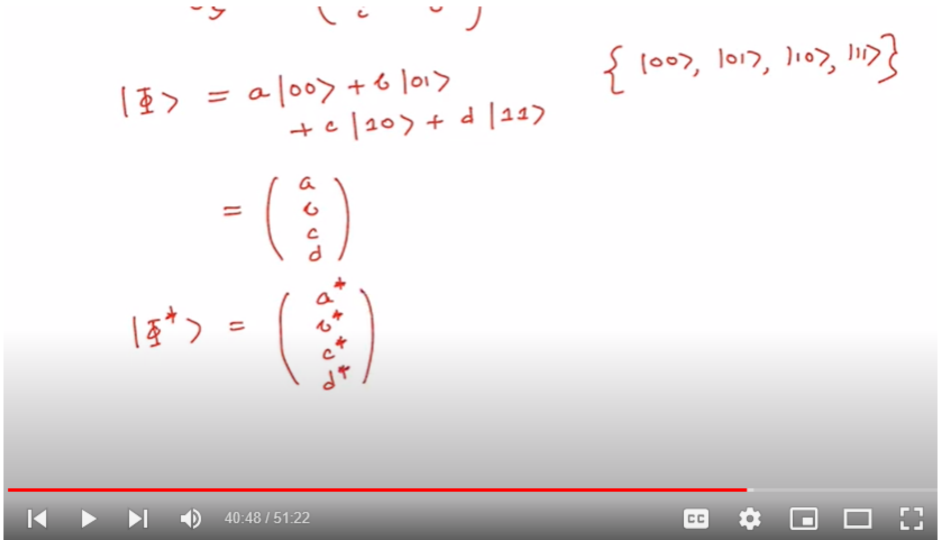

And this i can write in a column matrix of this form that would be a b c d right and phi star ket phi star would be the complex conjugate of this so that would be a star b star c star d star all right.

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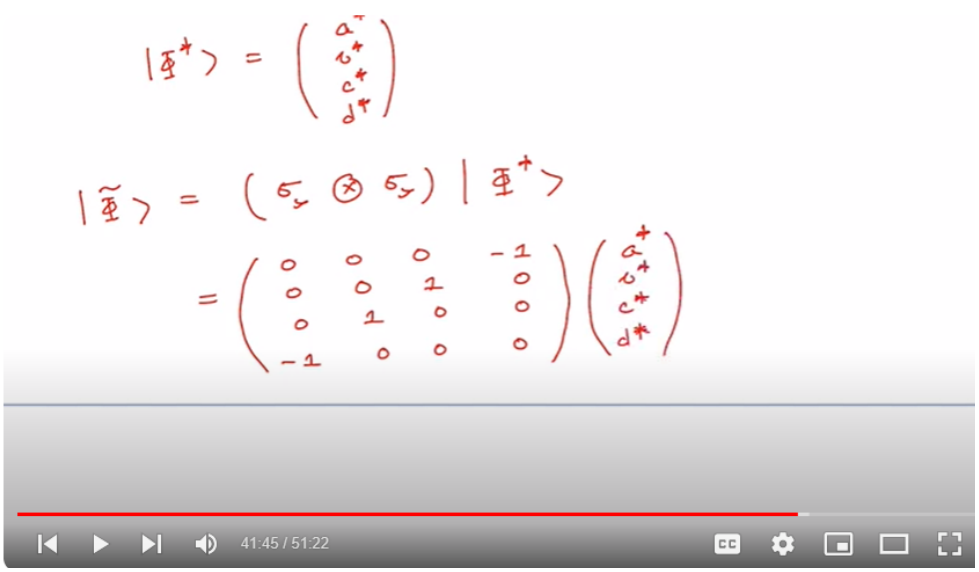

Therefore phi tilde let us work it out it's very simple you just have to work out what is the tensor product of sigma y sigma y and this operates on phi star ket phi star and sigma y tensor product sigma y if you work it out then you will get this matrix 0 0 0 minus 1 0 0 1

0 0 1 0 0 minus 1 0 0 0 this is sigma y tensor product sigma y and phi tilde ket phi tilde star is a no phi star only ket phi star would be a star b star c star d star.

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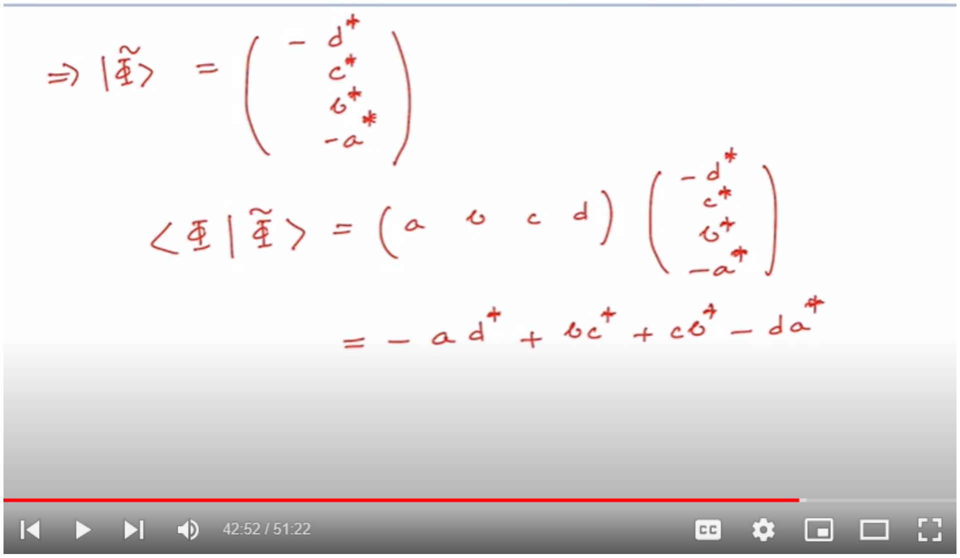

 And if you do the operation then it's easy to see that you are going to get minus d star c star b star minus a star so the inner product therefore we will get inner product of phi and phi tilde star no actually phi tilde only right this is what our phi tilde this is phi tilde so this would be equal to bra phi is the rho matrix a rho vector a b c d and phi tilde is minus d star c star b star minus a star so if you do the operation matrix multiplication if you do you will get minus a d star plus b c star plus c b star minus d a star all right .

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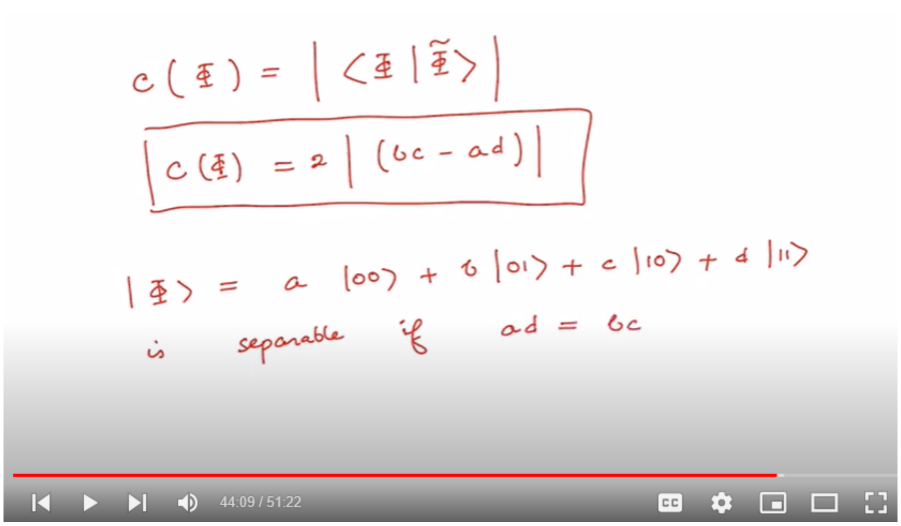

So as per the definition of concurrence for this pure state phi it is you have to take the modulus of this scalar product phi phi tilde so this is will give you you are taking the modulus so you will get twice of b c minus a d okay so this is the concurrence for a general state that we have got now you see that if in fact we know this result from our from our a very early class of this course suppose you have this general state phi is equal to a 0 0 plus b 0 1 plus c 1 0 plus d 1 1 get 1 1 and this state you can show it very easily is separable and in fact we have done it earlier it is separable if this condition is satisfied if ad is equal to bc.

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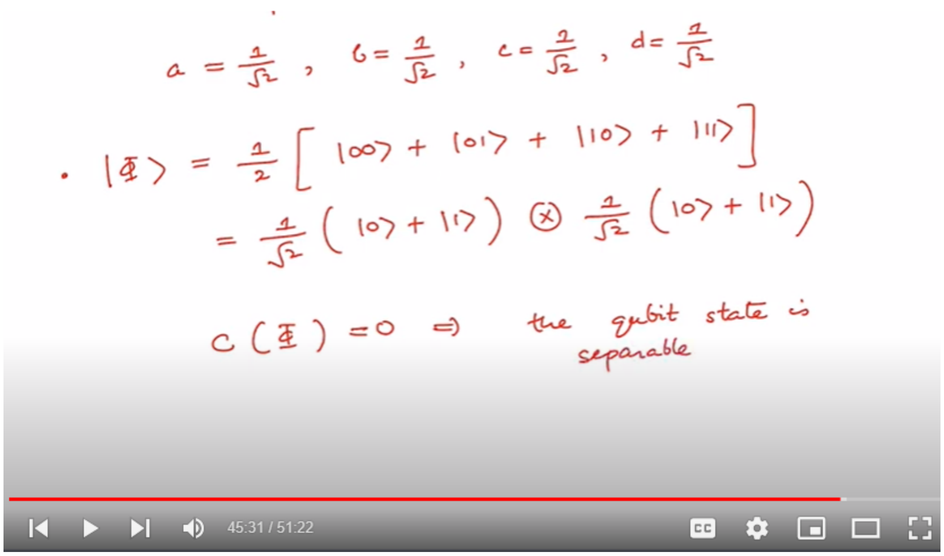

Okay what does that mean in that case you will see from here the concurrence would turn out to be 0 now let me give a one example suppose a is equal to 1 by root 2 and say b is also equal to 1 by root 2 c is also equal to 1 by root 2 and d is equal to 1 by root 2 for this simple example in that case we will have this ad is equal to bc if that is so that's straightforward to show that phi is equal to half and you will have ket 0 0 plus ket 0 1 plus ket 2 1 0 plus ket 2 1 1 and this you can express as a product state it would be 1 by root 2 ket 0 plus ket 2 1 tensor product or direct product to it 1 by root 2 ket 0 plus ket 1 right this means that when ad is equal to bc that this general state is separable or in the language of concurrence this means that if concurrence is equal to 0 that means the qubits the qubit qubit state the two qubit state is separable.

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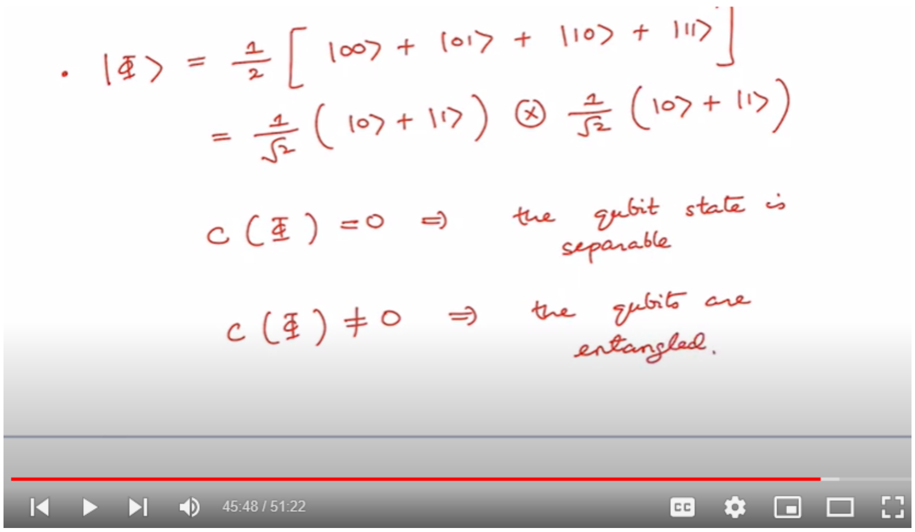

On the other hand if this concurrence c phi for the pure state if it is not equal to 0 this means the qubits the qubits are entangled okay so this is a very simple measure.

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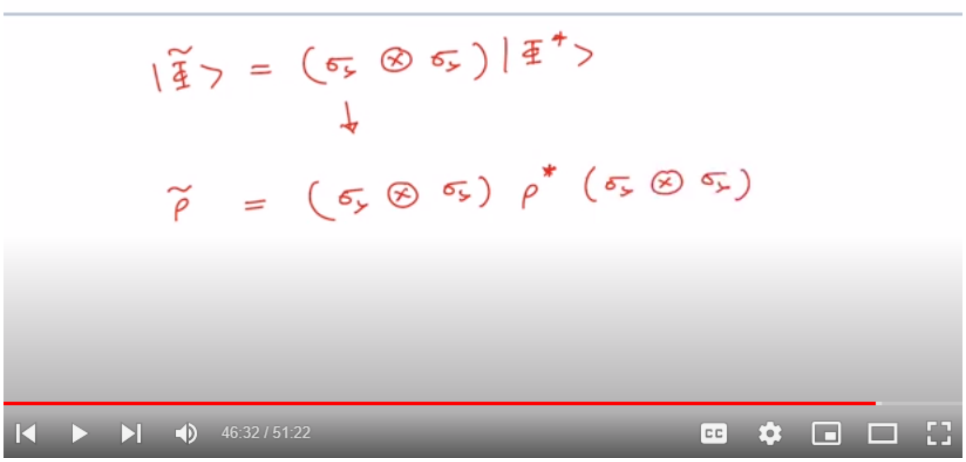

Please note that the splint flip operation phi tilde is equal to sigma y tensor product sigma y phi star ket phi star can be easily expressed in terms of density corresponding density operator as well and in terms of density operator you can easily get that this can be written in this form rho tilde is equal to sigma y tensor product sigma y rho star sigma y tensor product sigma y.

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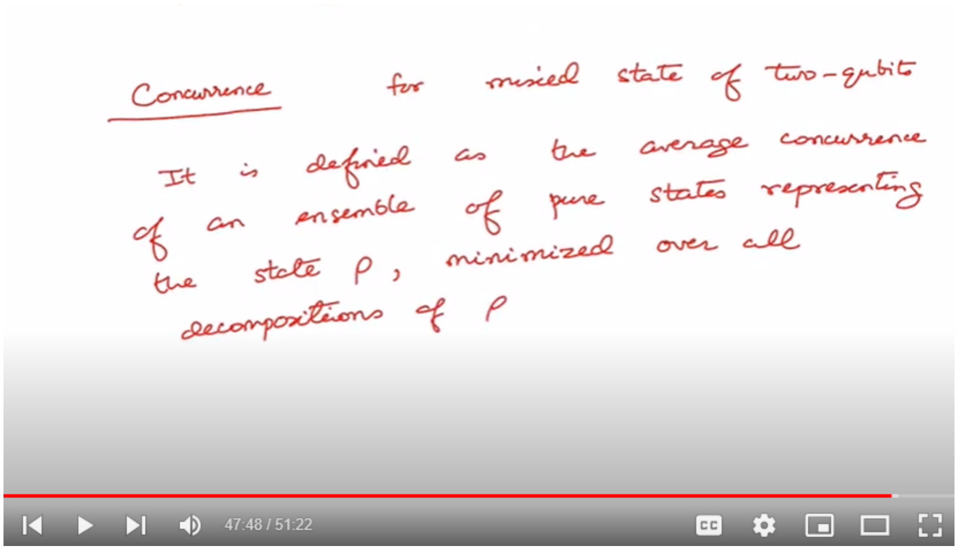

Now the concurrence can be defined in the context of a mixed state of two qubits as well so let me discuss concurrence for mixed state for mixed state of two qubits i will discuss it very briefly and it is defined as in the context of mixed state it is defined as the average concurrence of an ensemble of an ensemble of pure states representing the state rho minimized over all minimized over all decomposition of rho all decompositions of rho.

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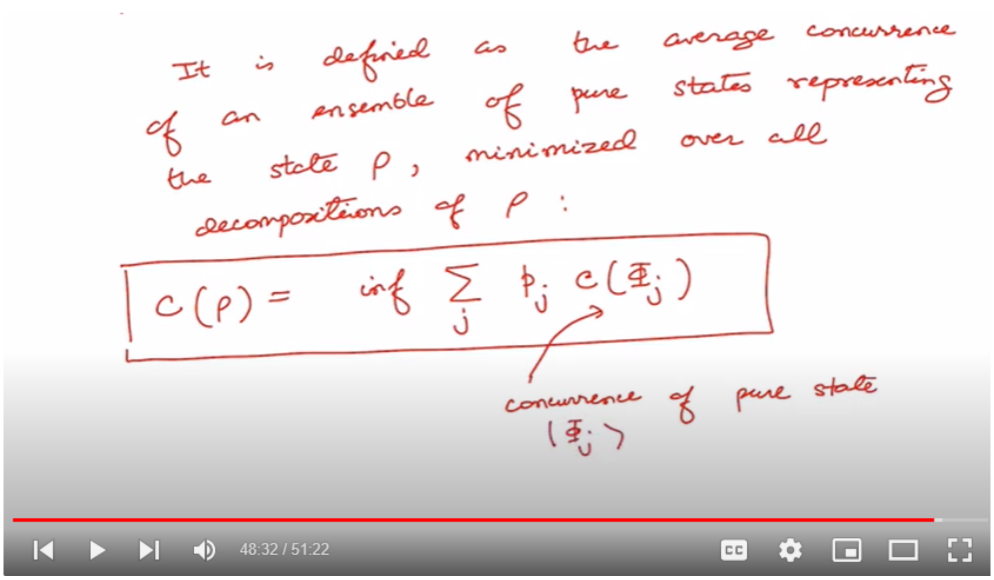

Mathematically i can write it as c rho is equal to infimum which is in other words taking minimum of this set of you know collection or set of this pure states over which i am calculating the concurrence so this is what the formula or the definition and this is concurrence of pure state phi j okay.

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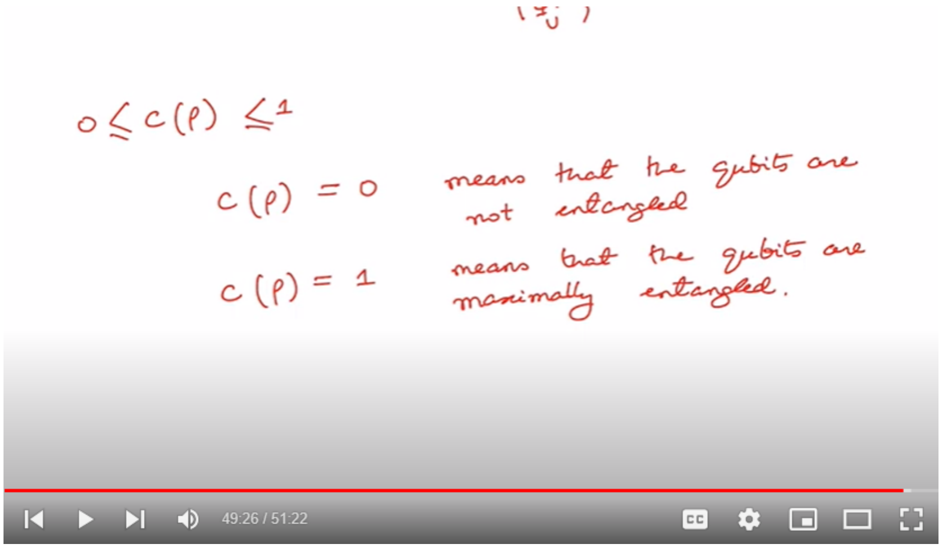

Now this concurrence 0 lies between 0 and 1 and 0 is equal to 0 you can easily guess that 0 is equal to 0 means that the qubits are not entangled qubits are separable and qubits are not entangled and 0 is equal to 1 means that the qubits are maximally entangled are maximally entangled okay.

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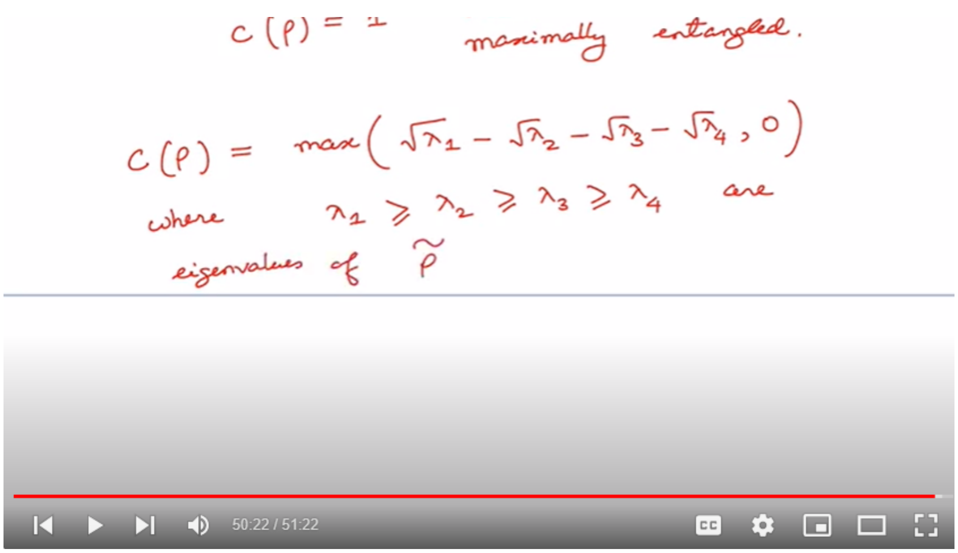

Now there is a formula to calculate concurrence for a mixed state of two qubits and it is given by i am not going to discuss how this formula has come it's a little bit involved but this formula just let me mention it is going to be maximum maximum of this see lambda square root of lambda 1 minus square root of lambda 2 minus square root of lambda 3 minus square root of lambda 4 0 where lambda 1 is greater than or equal to lambda 2 lambda 2 is greater than or equal to lambda 3 and lambda 3 is greater than or equal to lambda 4 and these are actually eigenvalues these are eigenvalues of rho tilde which we have defined earlier let me stop here our discussion on quantum entanglement measures with regard to discrete quantum system in problem solving session number four we'll discuss some problems related to quantum entanglement measures and other issues in the next class we'll start discussing some applications of quantum entanglement and also i will very briefly touch upon the topic of quantum entanglement measure in the context of continuous variable system so see you in the next class thank you.