Quantum Entanglement: Fundamentals, measures and application

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Week-03

Lec 13: Problem solving session-3

In this problem solving session number 3, we are going to solve some problems related to quantum gates or quantum circuits and entanglement sweeping and measurement.

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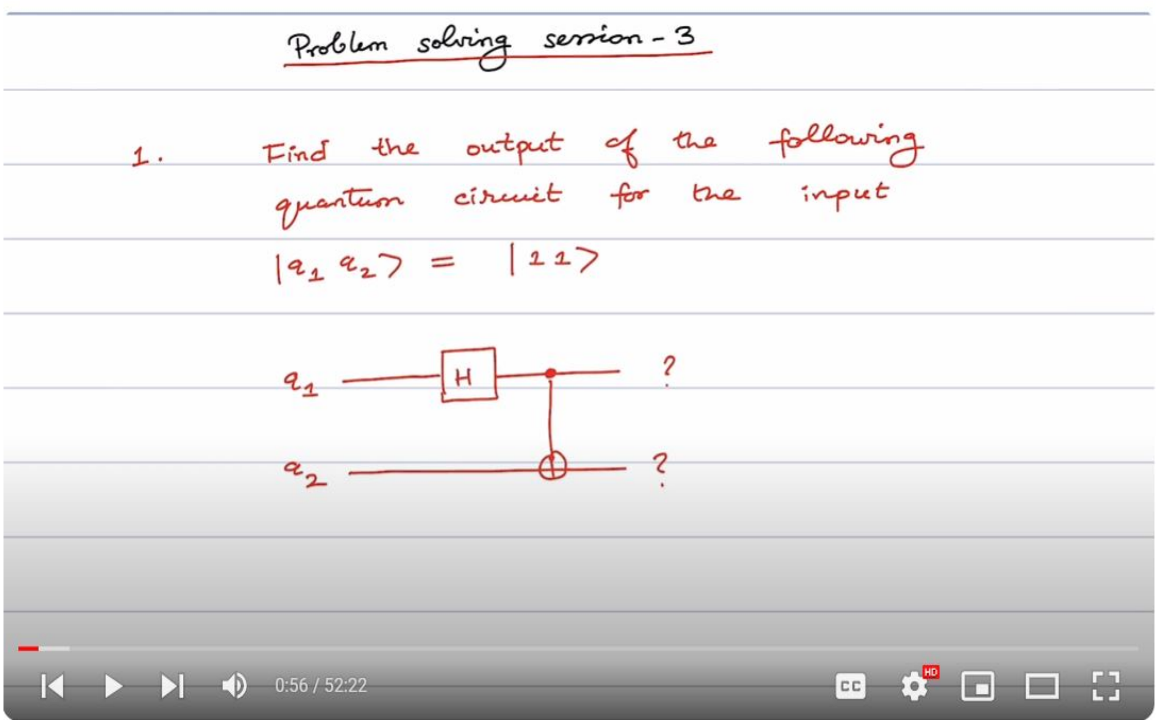

As a first problem, we are asked to find out the output of this quantum circuit and in this circuit a Hadamard gate followed by a CNOT gate is given and the inputs of the quantum circuits are Q1 is equal to KET 1 and Q2 is equal to KET 1.

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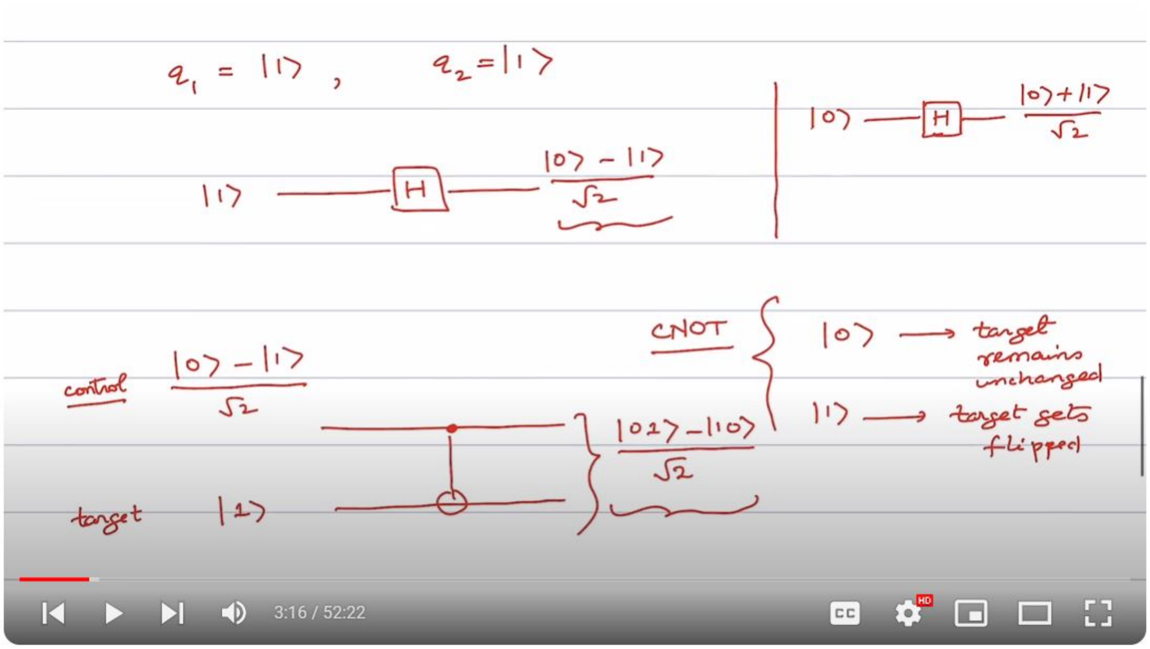

In the first channel, the input is KET 1 and in the second channel, input is again KET 1. So we know if the input to a Hadamard gate is KET 1, then at the output we get the output as KET 0 minus KET 1 by root 2. Just recall that when the input is KET 0 at the output of the Hadamard gate, we obtain KET 0 plus KET 1 by root 2. We get a superposition state. Now this particular output is going to be the input of the CNOT gate.

Now in the CNOT gate, we have in the first channel, this is our input in the first channel and in the second channel, we are having the input as KET 1. Now as per the rule of CNOT gate operation, this is my control. If the control, this is my control and this is my target.

If the control is zero, then the target remains unchanged. On the other hand, I'm talking about CNOT gate operation rule and if the control is ket 1, if control is one, then target gets flipped, right? Target gets flipped. So in this case, the output would be your, if control is zero, the target here is one. So it is going to remain unaffected. It will remain same.

If the control is one, the target is going to get flipped. So it will become zero. So therefore at the output of this circuit, we are going to get this as our state. So this is the answer.

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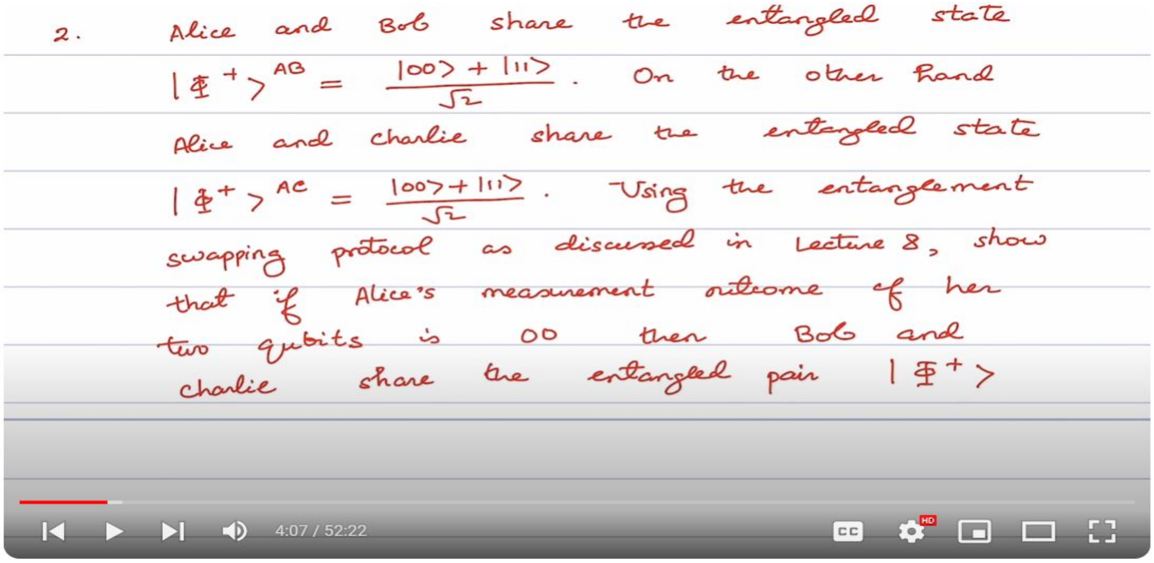

Let us now work out this problem. This problem is on entanglement sweeping and I have discussed entanglement sweeping in lecture eight. However, I have skipped certain calculation steps and intention of this particular problem is to fill up those gaps. In this problem, what is asked is this. Alice and Bob share entangled state phi plus, which is a superposition state of zero, zero plus one, one by root two. On the other hand, Alice and Charlie share the entangled state phi plus again, that is zero, zero plus one, one by root two. Using the entanglement sweeping protocol as discussed in lecture eight, you are asked to show that if Alice measurement outcome of her two qubits is zero, zero, then Bob and Charlie share the entangled pair phi plus.

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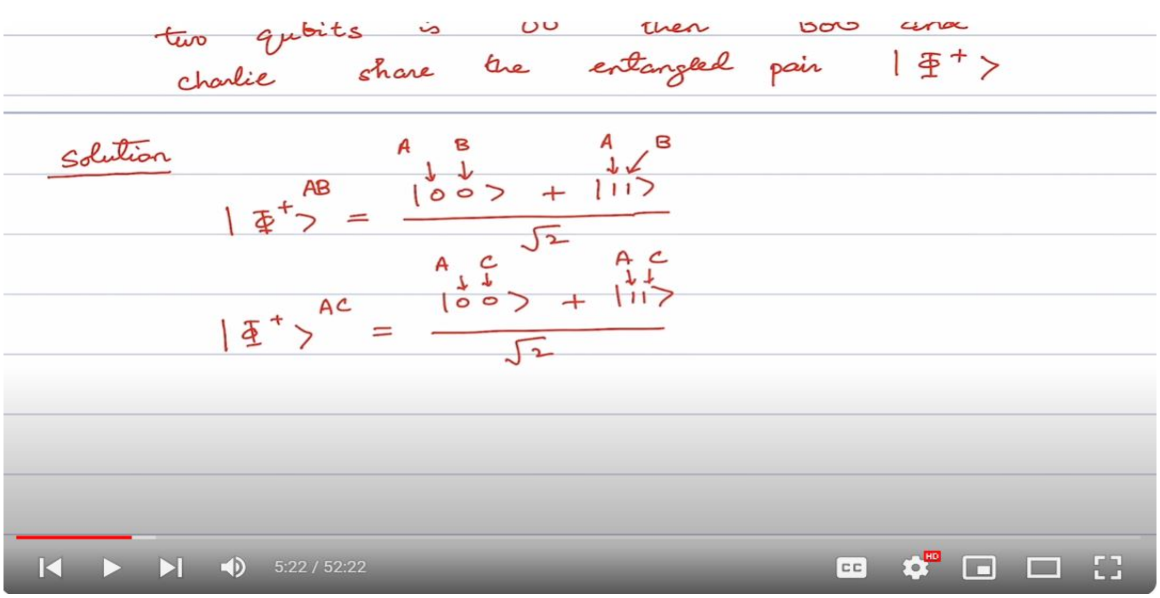

Let us work it out. So it involves very simple calculation. So Alice and Bob share this entangled state phi plus, that is equal to zero, zero plus one, one by root two. Here the first qubit belongs to Alice and the second qubit belongs to Bob. So this is phi plus shared by Alice and Bob. Similarly here, one belongs to Alice and this belongs to Bob. On the other hand, the similar entangled state is shared by Alice and Charlie as well, which is Ket 0, zero plus Ket 1, one by root two. First qubit belongs to Alice, second qubit belongs to Charlie. First qubit here belongs to Alice, second qubit belongs to Charlie.

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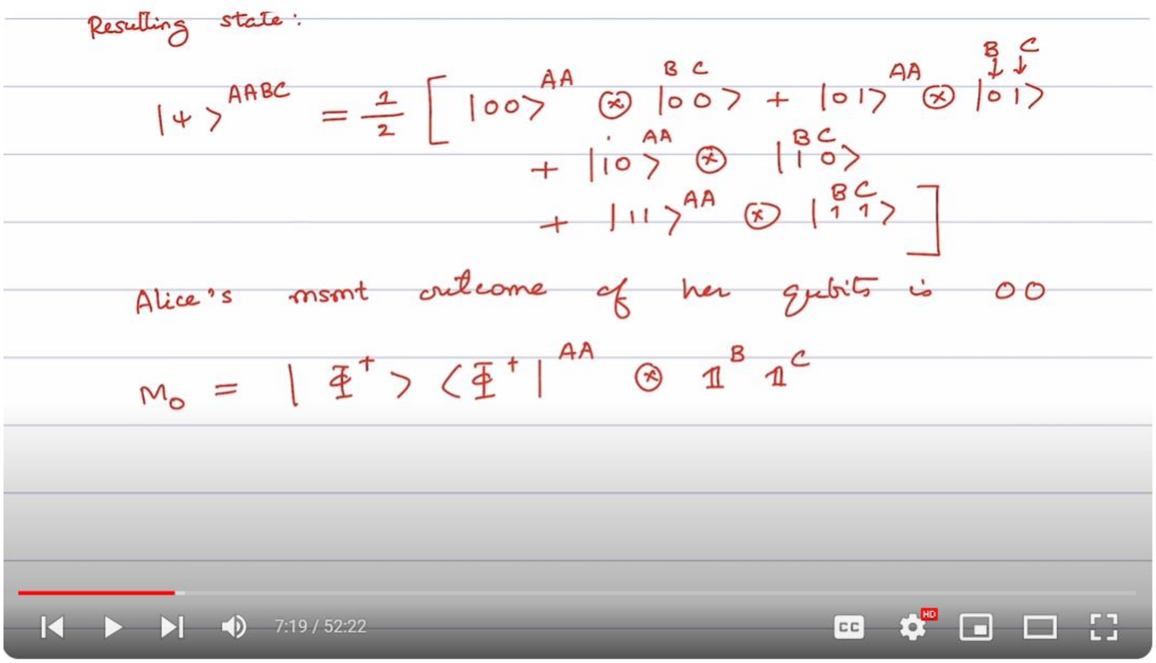

Now the resulting state, as we already discussed in lecture eight, the resulting state where the first two qubit is belonging to Alice can be written in this form, A, A, B, C. First two qubits belong to Alice and the second two qubits belong to Bob and Charlie respectively. Zero, zero belongs to Alice and then zero, zero. So this way I can write all other possibilities. Zero, one belongs to Alice, zero, one here this belongs to Bob, this belongs to Charlie and then we have one, zero belonging to Alice and one, zero belonging to Bob and Charlie and finally we have one, one belonging to Alice, first two qubits and the other two belonging to Bob and Charlie respectively.

So this is basically the resultant state. Now Alice measurement outcome as given, Alice measurement outcome is outcome of her qubits, her qubits is zero, zero. So therefore Alice is using the measurement operator M0 is phi plus, phi plus A, A. So she can make measurement on her qubits only and Bob and Charlie's qubit remaining untouched.

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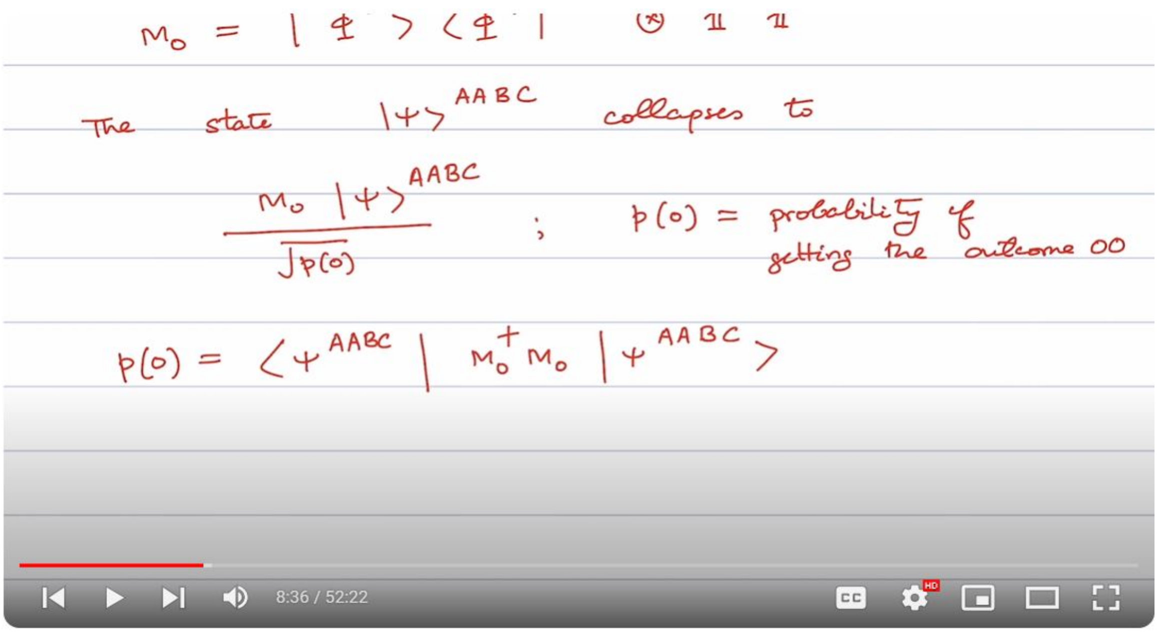

Alice cannot do anything about that. Now because of this measurement, the state psi, A, A, B, C, this state collapses to from measurement problem chapter we know that lesson we have already learned that this collapses to the state given by this expression M0 psi, A, A, B, C divided by square root of P of zero. Here P of zero as we know is the probability, probability of getting the outcome, getting the outcome zero, zero and in fact mathematically we can write P of zero is equal to expectation value of the measurement operators M0 dagger M0 psi, A, A, B, C. This already we know.

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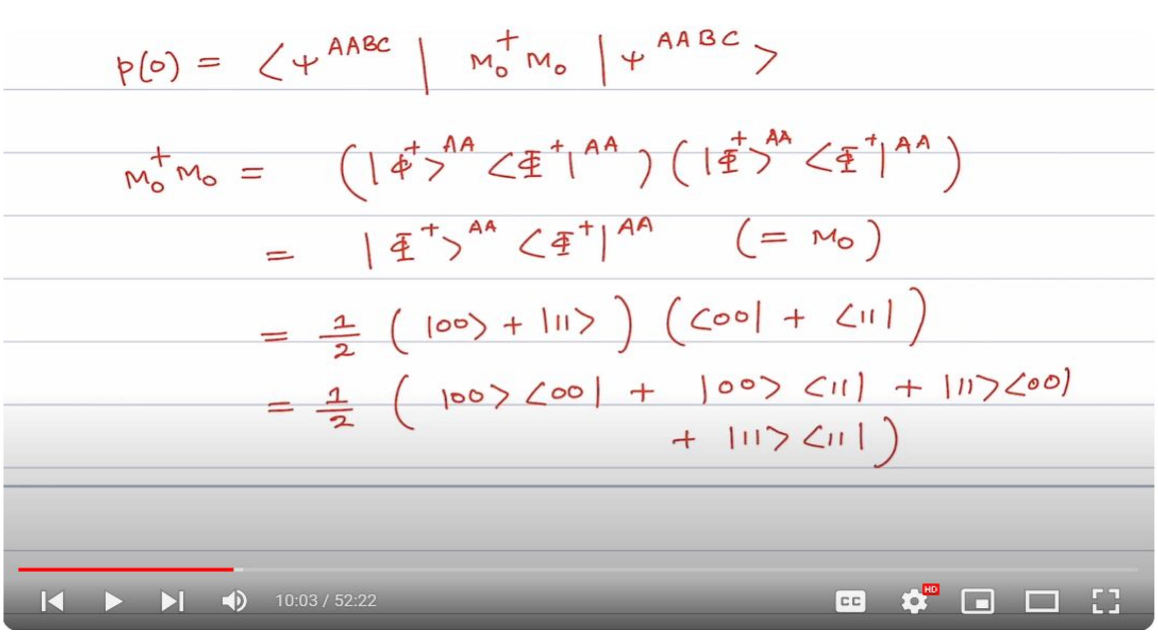

Now we have to do the detailed calculations. Firstly let us calculate M0 dagger M0 which is phi A, A, phi plus A, A bra phi plus A, A. This is M0 dagger is equal to actually M0. So this is M0 is phi plus A, A bra phi plus A, A. Let me now open it up because this is nothing but phi plus A, A, phi plus A, A. If I this is exactly equal to M0 right. So let me now open it up. I have phi zero plus is Ket 0 zero plus k two one one and the other one is again because one by root two one by root two becomes half and here I have zero zero plus one one and the total would be half if I open it up zero zero I will have zero zero one one plus one one zero zero as you can see easily one one one one right. This is what I will have.

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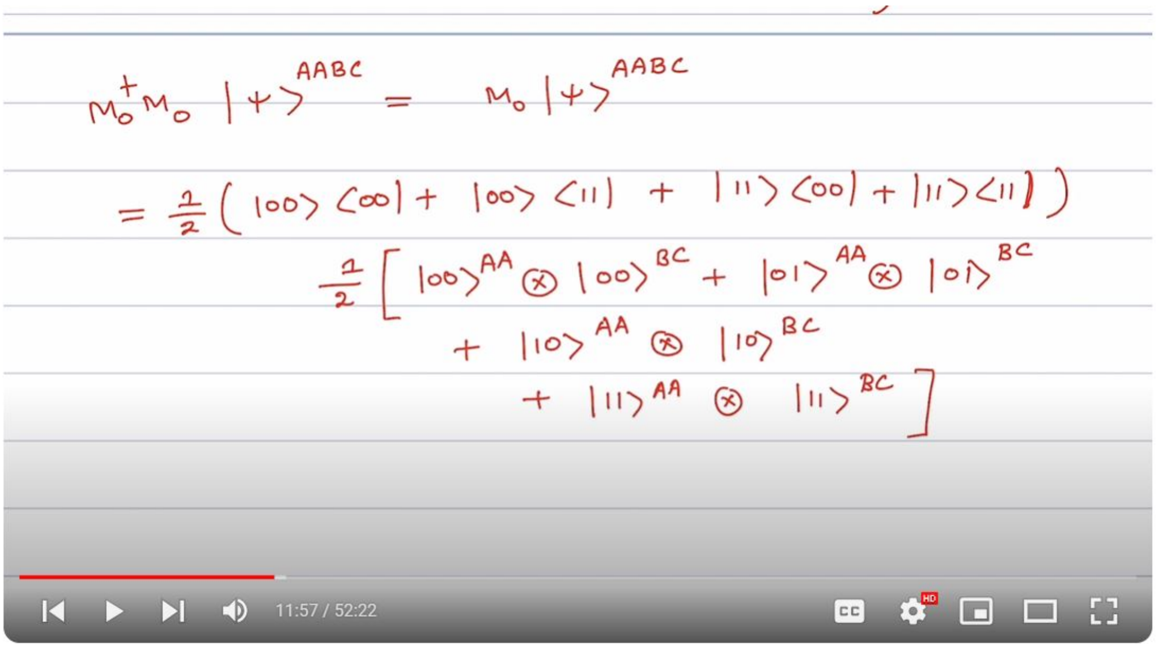

Now if this guy operates M0 dagger M0 M0 dagger M0 operates on psi A, A, B, C which is equivalent to operation of M0 on psi A, A, B, C. Let me work it out M0 this would be equal to M0 is half let me write everything zero zero plus zero zero one one plus one one zero zero plus one one one one.

Okay so this is operating on the state half psi A, A, B, B. Let me write everything so that you can easily follow it A, A this zero zero B, C plus zero one A, A. You will have zero one B, C plus one zero A, A tensor product with one zero B, C. Actually I should it's not outer product it's direct product so let me be careful here this is Ket 0 zero and similarly here this is Ket 0 one this is Ket 0 one and here also it should be Ket 1 zero plus one one A, A tensor product with one one B, C.

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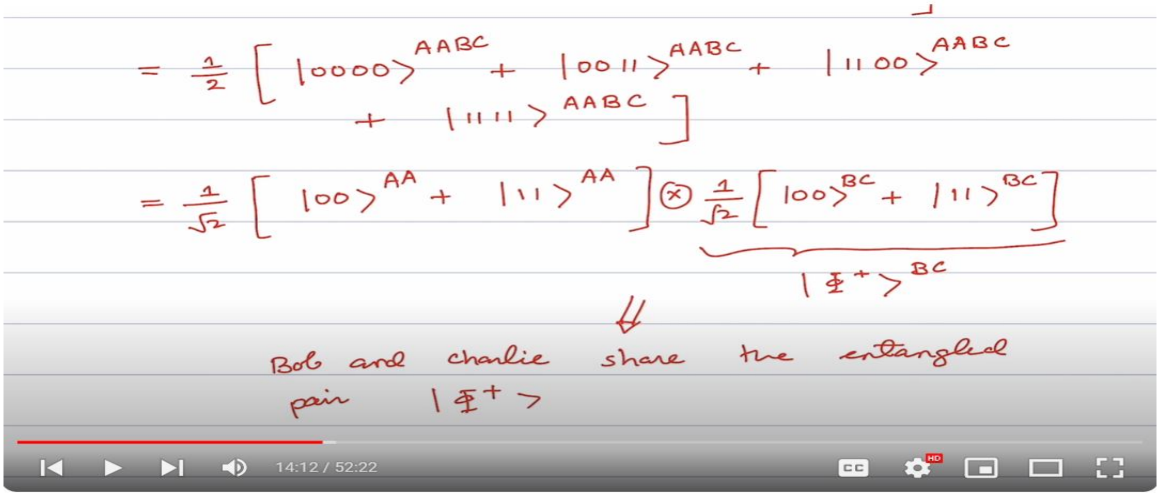

I think these calculations all of you can do it very simply and now what you will get this is going to give me a half let me write these things you will get half zero zero zero zero I am writing in certain notation A, A, B, C plus zero zero one one A, A, B, C plus one one zero zero A, A, B, C right plus one one one one A, A, B, C. Okay so this is what I will get and this I can actually write in a product form it would be one by root two I can write zero zero A, A plus zero one A, A it is not zero one it would be one one A, A one one A, A tensor product to it one by root two zero zero B, C plus one one A, A right A, A, B, C sorry it should be B, C this would be B, C as you can see this is nothing but the bell state belonging to Bob and Charlie and that's why it is clear that Bob and this basically implies that Bob and Charlie Charlie Bob and Charlie share the entangled pair entangled pair phi plus because of the measurement of Alice who got the outcome as zero zero so this is how we have proved it.

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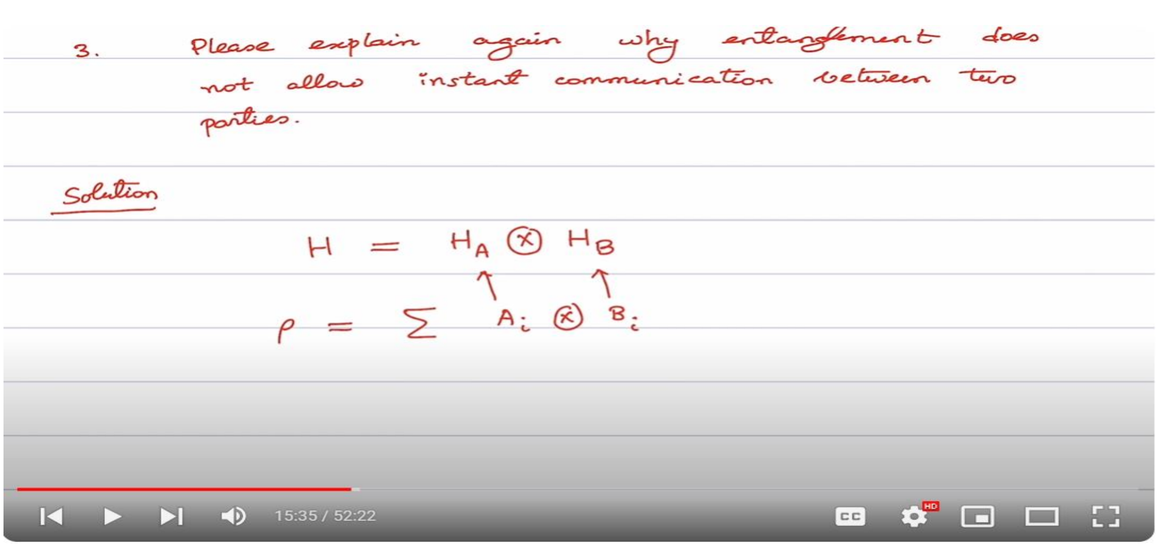

Let us now work out this problem actually here I intend to explain you again why entanglement does not allow instant communication between two parties I have already explained it in lecture number eight however let me do it again in a little bit simpler way though I cannot avoid mathematics.

So let us say though it is not solution let me write solution let us say Alice and Bob make measurement in a system H belonging to the Hilbert space H where H is a direct product or tensor product of the Hilbert space of Alice and Bob respectively now the state of the composite system is given by density operator in the Hilbert space H as we know and any density operator rho is a sum of the form say a i that is belonging to the linear operator belonging to the Hilbert space H a and b i belonging to the Hilbert space it's an operator belonging to the Hilbert space H b okay.

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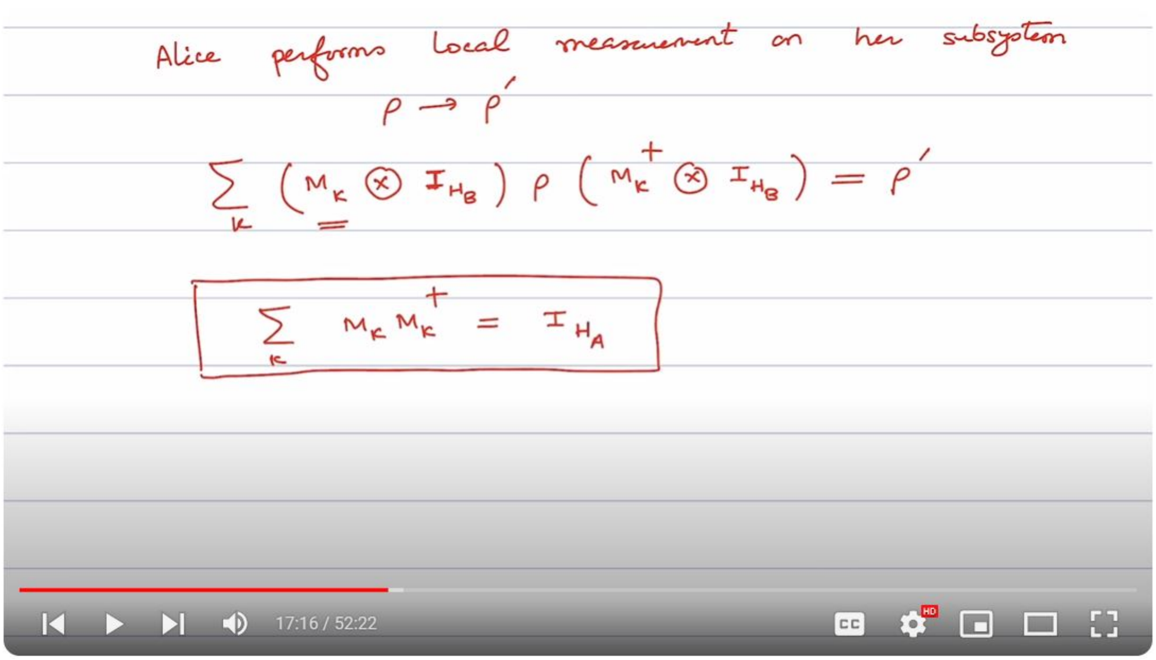

Now Alice performs local measurements so let me write here Alice performs local measurements local measurement on her subsystem on her subsystem of the following kind basically she is going to say use the measurement operator say mk and the Hilbert space of Bob is going to remain untouched so therefore that's why here I am writing identity and the state is rho and here I have mk dagger direct product with i h p this basically means that the Hilbert space of Bob is not touched and the resultant of this operator is that rho is going from because of this measurement rho is going from another state rho dash so this is going to be my rho dash now mk are the measurement operators which has to satisfy these particular properties as we know from measurement postulates mk mk dagger should be equal to i h a

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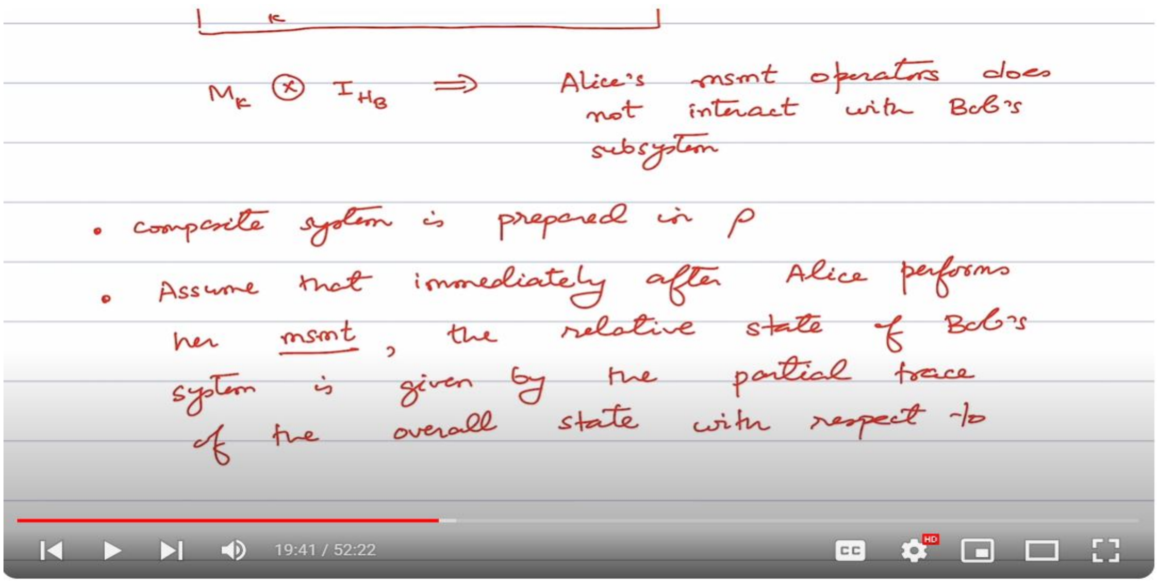

Because here mk is a measurement operator belonging to the Hilbert space of Alice and mk tensor product with i h p let me reiterate again this implies that Alice measurement operators operators does not interact does not interact with Bob's subsystem Bob's subsystem okay it's a local measurement now say the composite system is prepared in the state rho the composite system is composite system system is prepared prepared in rho so composite system is prepared in the state rho and you have to assume let us assume assume that assume that immediately after immediately after Alice performs her measurement Alice performs her measurement her measurement okay the relative state of the relative state of Bob's system Bob's system is given by the partial trace of the overall state with respect to Alice system with respect to Alice system.

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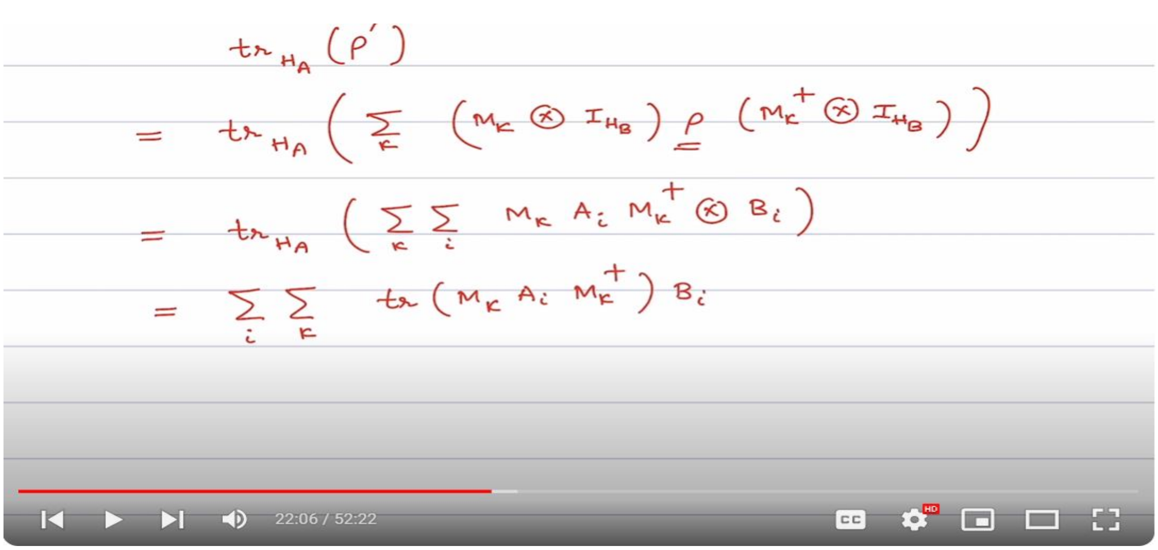

So what I mean by this is that basically to know the state of Bob I am going to trace out Alice right so I have to trace out Alice by performing this operation rho dash is basically the resultant state of of the system after Alice make the measurement so rho goes to rho dash and then I take the partial trace over Alice basically I am going to get the reduced density matrix for Bob right that's the intention this is what it mean,

so let us work it out now because we know what is rho dash so trace HA let me write what is rho dash rho dash is already I have written here okay this is what my rho dash is let me write it again here that is sum over k m k tensor product with I H P this is rho and m k dagger tensor product with I H P all right so this is what I have now because of the trace operation I can now write again let me write do it step by step I can write k sum over k and let me write sum over I here m k I'm basically replacing rho rho is a summation over ai bi right so that's what I'm going to write m k is going to act on the operator belonging to Alice ai and then I have m k dagger bob is going to remain untouched so therefore this is bi so this is what I'll have and in the next step what I can do I can take the trace inside so let me take summation I here summation k traces over Alice only so let me write m k ai m k dagger bi all right so then I can write further sum over I trace let me take the summation inside ai sum over k m k m k dagger and I have here bi and we know that measurement operator has to satisfy some properties and this would be identity.

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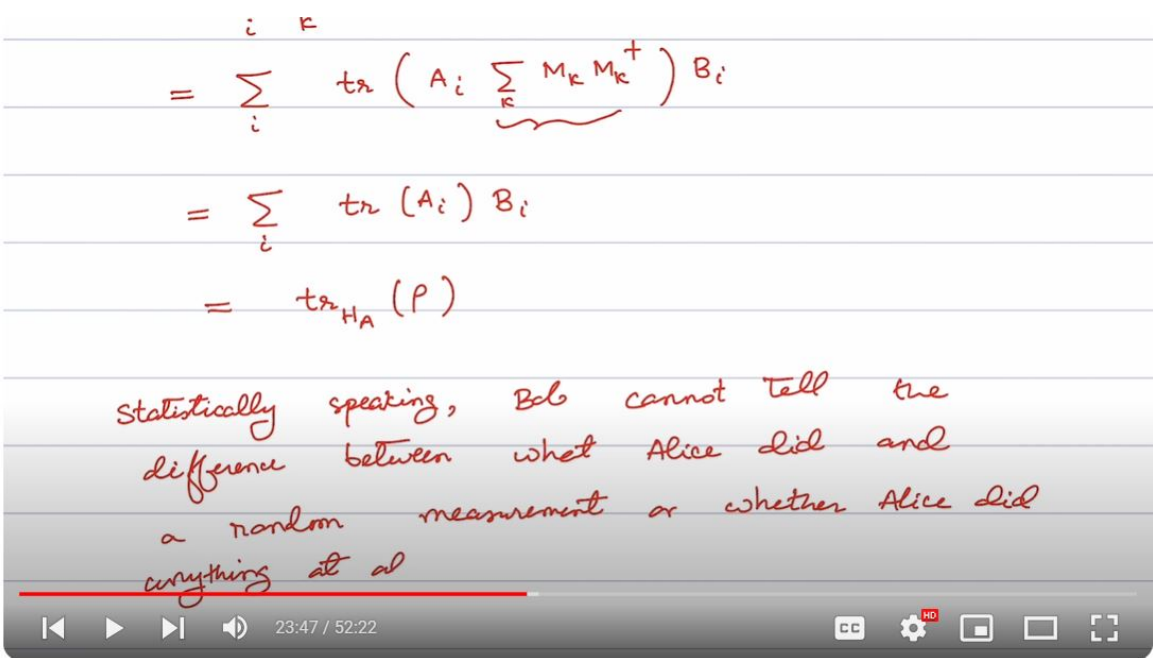

therefore I will have summation I trace over ai bi and this is basically nothing but trace I have taken over the original state rho so what does it mean it means that statistically speaking statistically speaking bob cannot tell bob cannot tell the difference difference between what Alice did and a random measurement and a random measurement or whether she did anything at all or whether Alice did anything at all so this clearly shows that entanglement does not allow instant communication between two parties.

just one thing I want to make once again clear that it may appear to some of you that I am able to write it as a kind of a product set remember that this is not a product state I'm expressing the density operator in terms of linear operators belonging to Alice Hilbert space and bob Hilbert space that I can always do irrespective of entanglement this is not state I cannot write rho is equal to state of a Alice and state of b this is what I mean by entanglement okay here I can always do that but I cannot do this when the states are entangled okay this has to be clear I think I have made it clear in lecture eight as well.

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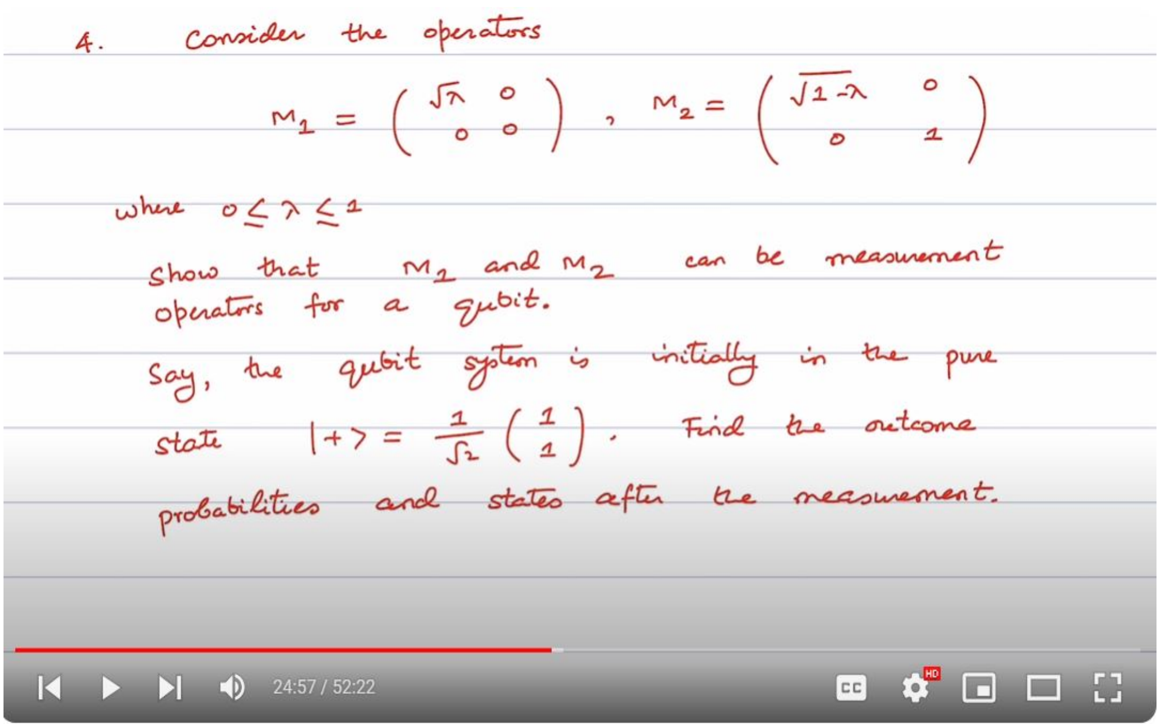

Let us now work out this problem this problem is related to measurement problem in particular povm positive operator value measures. before I do this problem let us revise povms again a little bit quickly.

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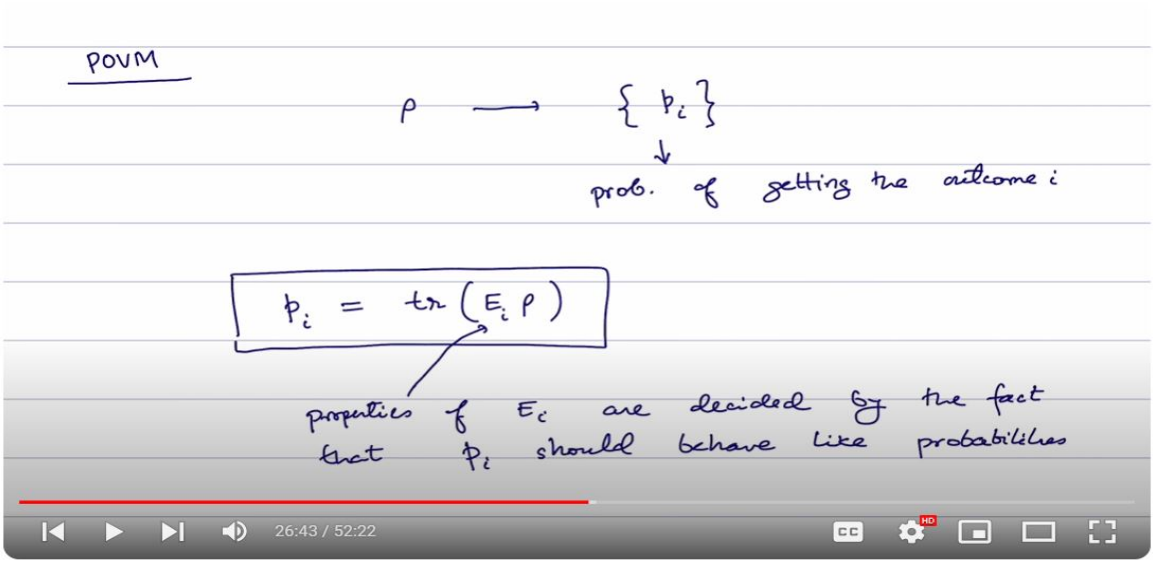

So the idea of povm was as follows say you are given a system in a state rho and we want to know a set of operations which can give us the probability say pi where pi is the probability probability of getting the outcome getting the outcome I right as we know from density matrices information is always defined by taking the expectation value of of an operator so pi must be of this form we have to take trace over some operator e n density operator is rho so our density matrix is rho then we'll this is gives us the probability of getting the outcome right now properties of ei this is the operator which is basically the measurement operator and its properties properties of ei are decided by the fact are decided by the fact by the fact that pi should behave like probabilities pi should behave like probabilities probabilities.

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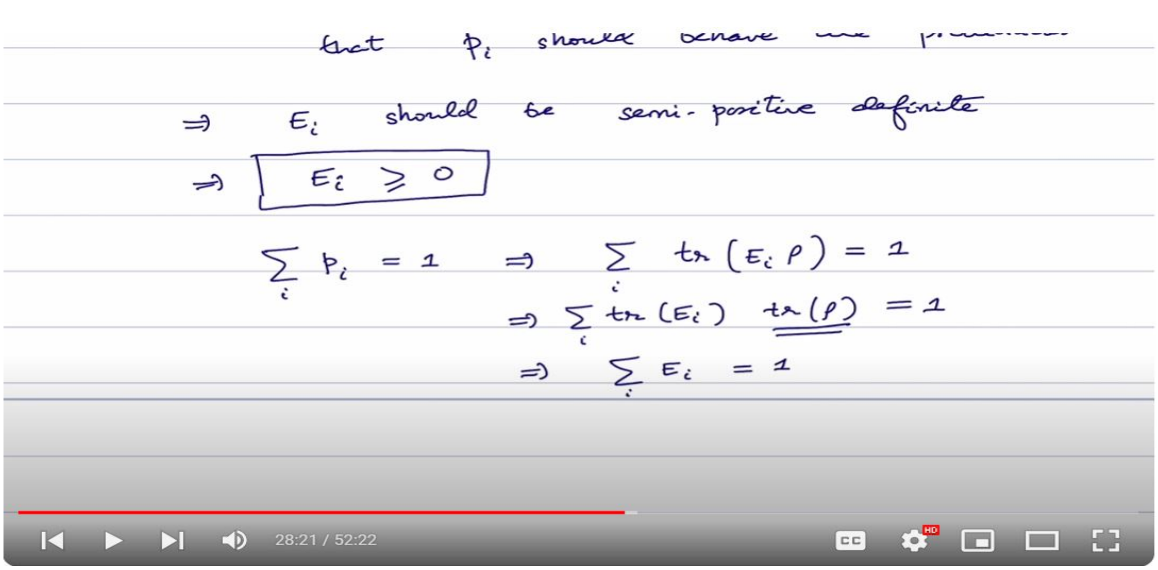

And we know that probability cannot be negative it has to be non-negative for any density operator rho or state the operators ei this implies that ei should be non-negative or should be semi-positive definite semi-positive definite

that means it eigenvalues cannot be negative it has to be always positive so this is generally denoted by this expression ei should be always greater than or equal to zero that means this operator ei should be semi-positive definite and another thing is that the total probabilities should add up to be one and this implies that summation because okay. let me write because pi is equal to trace of ei rho and because of this trace operation i can write the trace of product of two operators equal to trace of ei and trace of rho right this is what i have and trace of rho is equal to one because the density operator so this implies that i should have sum over ei is equal to one okay so this is also we must have.

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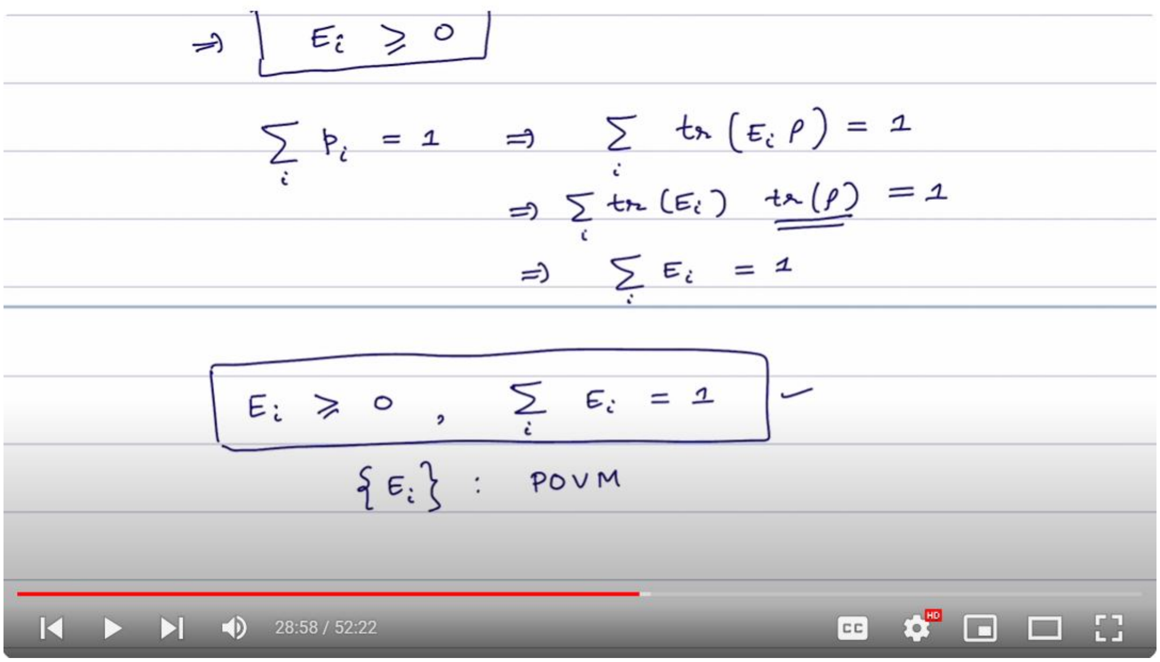

So therefore povm should obey any povm measurement operators should satisfy these two properties ei should be greater than equal to zero and summation ei is equal to one a set of operators ei satisfying this particular operation or this conditions this set of two conditions are known as povm.

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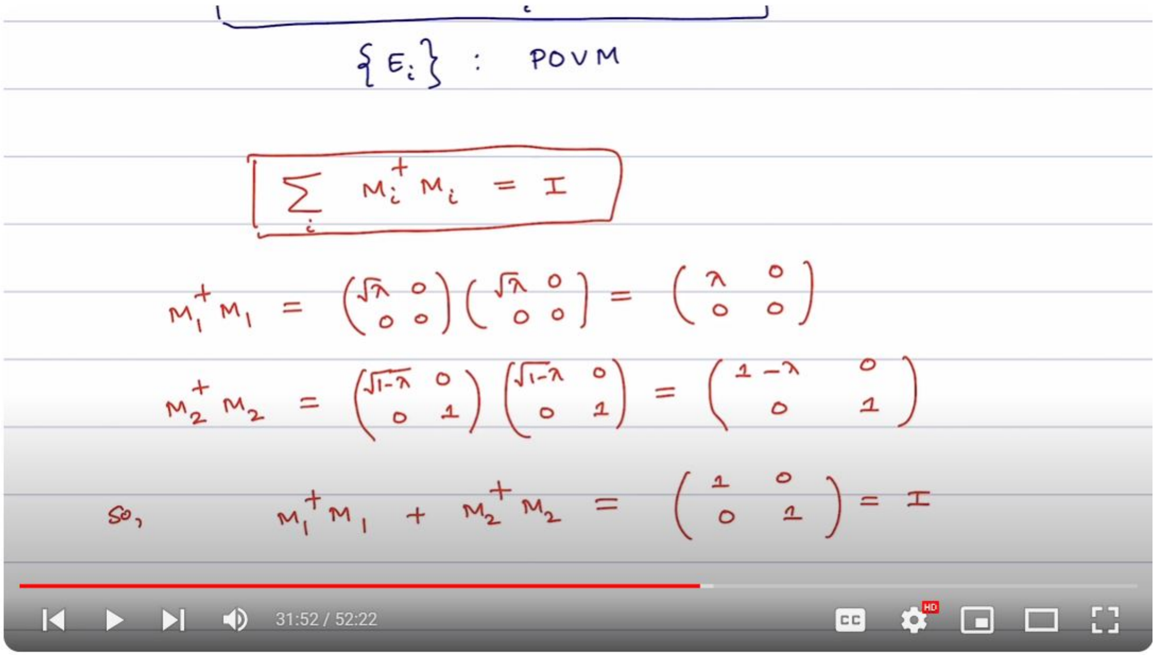

Now let us go back to the problem the problem states that considered the operators m1 is equal to root lambda 0 0 where lambda is lying between 0 and 1 and m2 another measurement operator that is square root of 1 by lambda the matrix form is given 0 0 1 right so this is what is given these two measurement operators are given these two operators are given you have to show that m1 and m2 can be measurement operators we still don't know whether they are measurement operators or not and you have to show it that they are measurement operators for a qubit because it's a two by two matrix of course it has to be it can deal with qubit and the next part of the problem is that say the qubit system is initially in the pure state plus a ket plus is equal to 1 by root 2 1 1 you have to find the outcome probabilities and states after the measurements.

okay let us work it out so first of all if m1 and m2 has to be measurement operators they have to satisfy this particular you know property they have to be equal to identity this all of us we know as regards measurements are concerned so first let us work out what is m1 dagger m1 m1 dagger m1 m1 is root over lambda 0 0 0 as given so m1 dagger would be again the same it would be root lambda 0 0 0 this is what i have if i take the matrix product i will get lambda 0 0 0 as you can see and similarly let us work out m2 dagger m2 that would be equal to m2 is given as square root of 1 minus lambda 0 0 1 and it would be m2 dagger would be 1 minus square root of 1 minus lambda 0 0 1 and if i take the product matrix product i will get 1 minus lambda here 0 0 1 now let us add them m1 dagger m1 plus m2 dagger m2 if i do the addition then i will get 1 0 0 1 which is nothing but identity so therefore clearly they satisfy this very critical property so m1 and m2 can be measurement operators they qualifies to be measurement operator for a qubit because why qubit as i said it is a two by two matrix.

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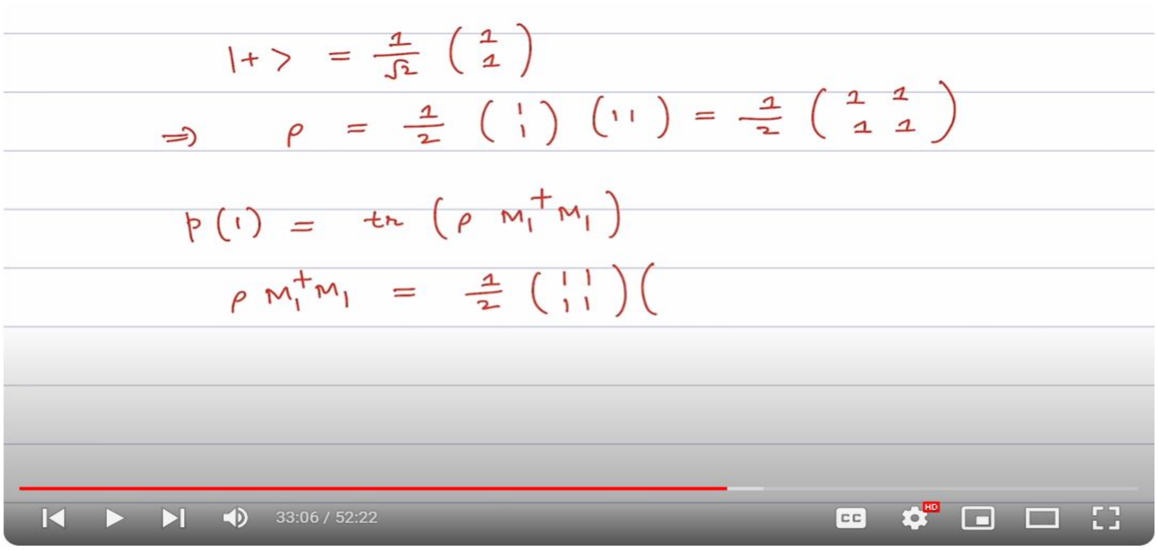

Now let us discuss the second part of the equation it is given that the system is in the state ket plus which is 1 by root 2 1 1 so let me write down the corresponding density operator for this state that would be rho is equal to half you have 1 1 1 1 so this will give you a half 1 1 1 1 that's the density operator now what would be the outcome measurement outcome for if i use m1 first outcome would be p1 is equal to trace rho m1 dagger m1 okay

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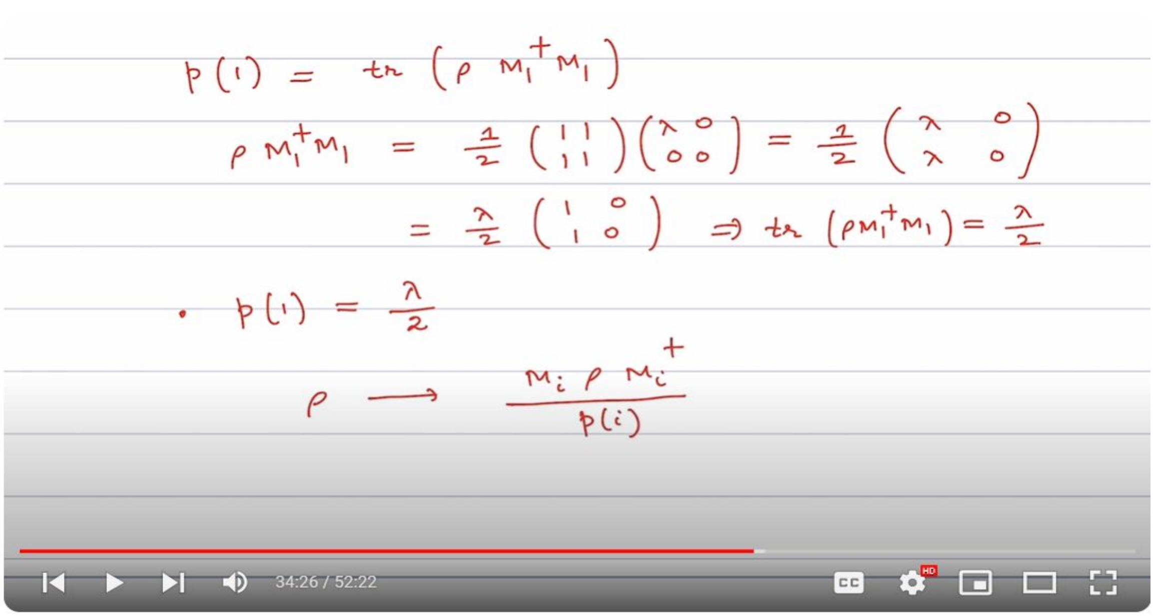

so first let me find out what is rho m1 dagger m1 and we have calculated rho that is half 1 1 1 1 we have found out rho and m1 dagger m1 is nothing but lambda this already we worked out here lambda 0 0 0 lambda 0 0 0 if i take the multiplication then i will get half lambda 0 lambda 0 or i can write it as lambda by 2 1 0 1 0 and this implies that probability of the first outcome when i have used the measurement operator m1 is lambda by 2 because i have to take the trace and trace trace of this implies actually trace of rho m1 dagger m1 is equal to lambda by 2 as you can see so the first outcome we have obtained

now what about the second outcome the second outcome we'll work out later but first let us see what would be the state immediately after the measurement of this first outcome the state would become rho would become as per the measurement postulate or may we have already known it would be mi rho mi dagger by pi

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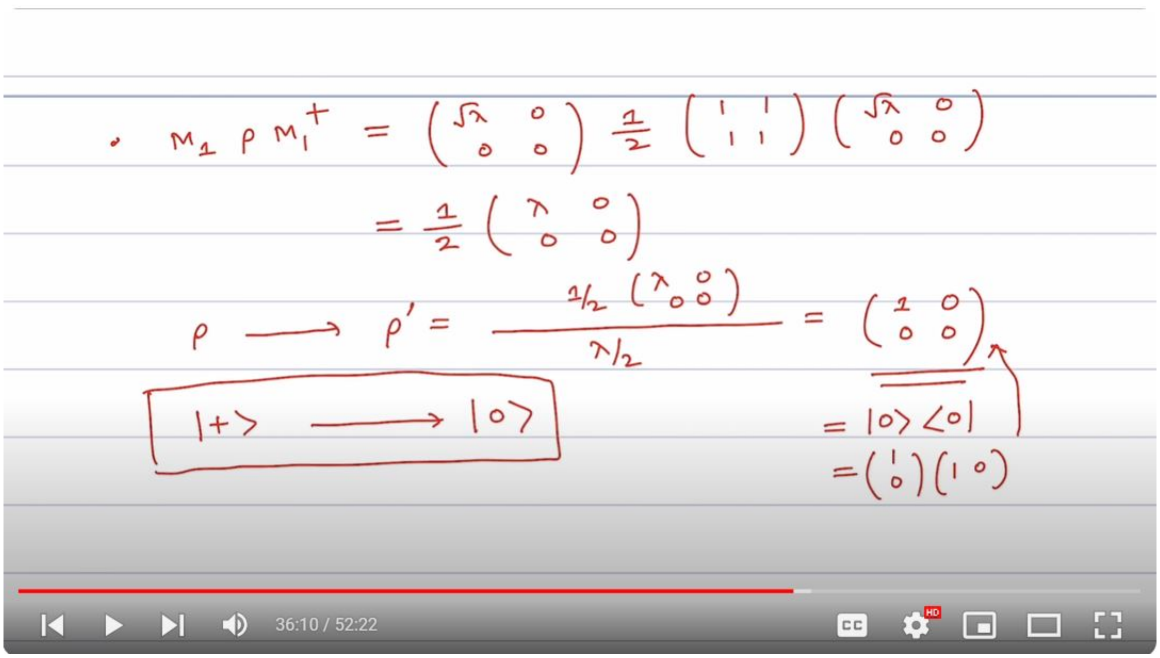

so in this case we have if i use the measurement operator m1 m1 rho m1 dagger let us work it out m1 is root lambda 0 0 0 rho is half 1 1 1 1 and this would be root m1 dagger would be root lambda 0 0 okay so if i work out the product matrix product if i take i get half lambda 0 0 0 so this is what i will get therefore your rho would become rho dash is equal to because of the measurement p1 already we got to be lambda by 2 and this we have is half lambda 0 0 0 from here i get the density operator as a result of the measurement would be 1 0 0 0 so effectively this means that i am going from the state initially the system is in the state Ket plus and after measurement m1 it will go to ket 0 right by the way this is nothing but ket 0 if i convert it to state and the density operator basically this is equivalent to this because if you remember ket 0 is 1 0 and bra 0 is 1 0 so that's why i am getting this okay so this is my state immediately after the measurement.

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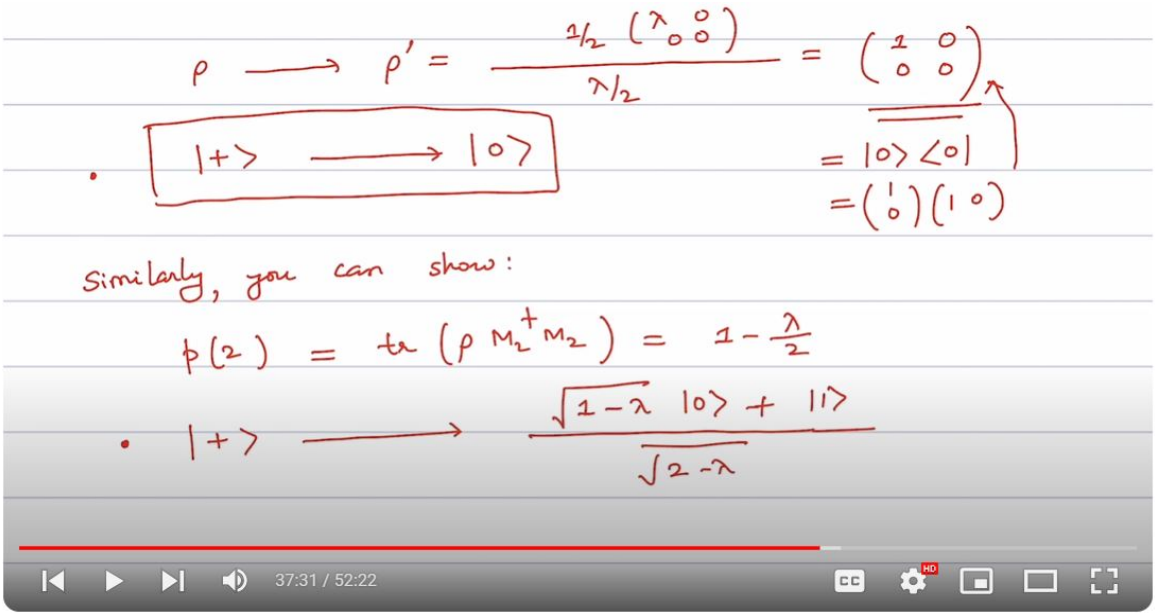

similarly you can show i leave it to you similarly you can show if you follow the same procedure that the probability of the second outcome p2 would be equal to trace of rho m2 dagger m2 and this would be 1 minus lambda by 2

if you work it out and you can also show that the state immediately after the measurement will go to this particular state that would be square root of 1 minus lambda Ket 0 plus Ket 1 divided by square root of 2 minus lambda okay.

another important point you can note here that until and unless lambda is equal to 1 the state after the measurement will not be a perfect collapse it will you see unlike here in the first measurement your plus ket plus is collapsing to Ket 0 but here after measurement when the second measurement m2 is used the Ket plus is collapsing into a superposition state but if lambda is equal to 1 it will collapse into ket 1 a perfect collapse will happen.

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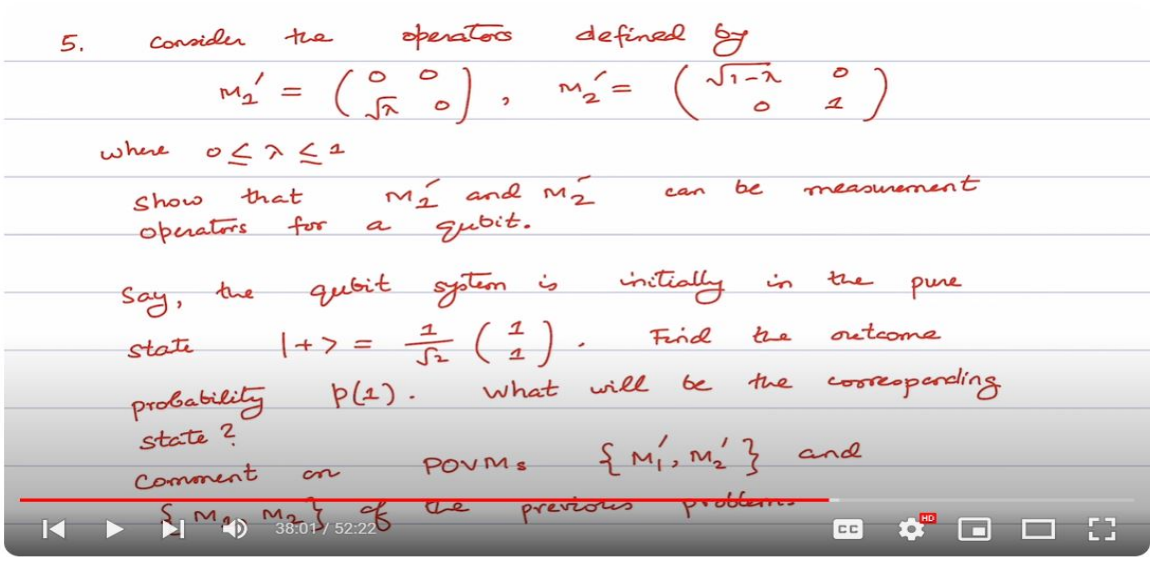

let us now work out this problem in this problem two operators are given m1 dash m2 dash and this problem is similar to the one that we have done in the previous one you have to show that m1 dash and m2 dash can be measurement operators for a qubit then in the second part of the problem the qubit system is initially is in the pure state Ket plus is equal to 1 by root 2 1 1 you have to find the outcome probability p1 and what will be the corresponding state then you have to comment on the povms m1 dash m2 dash and m1 m2 of the previous problem

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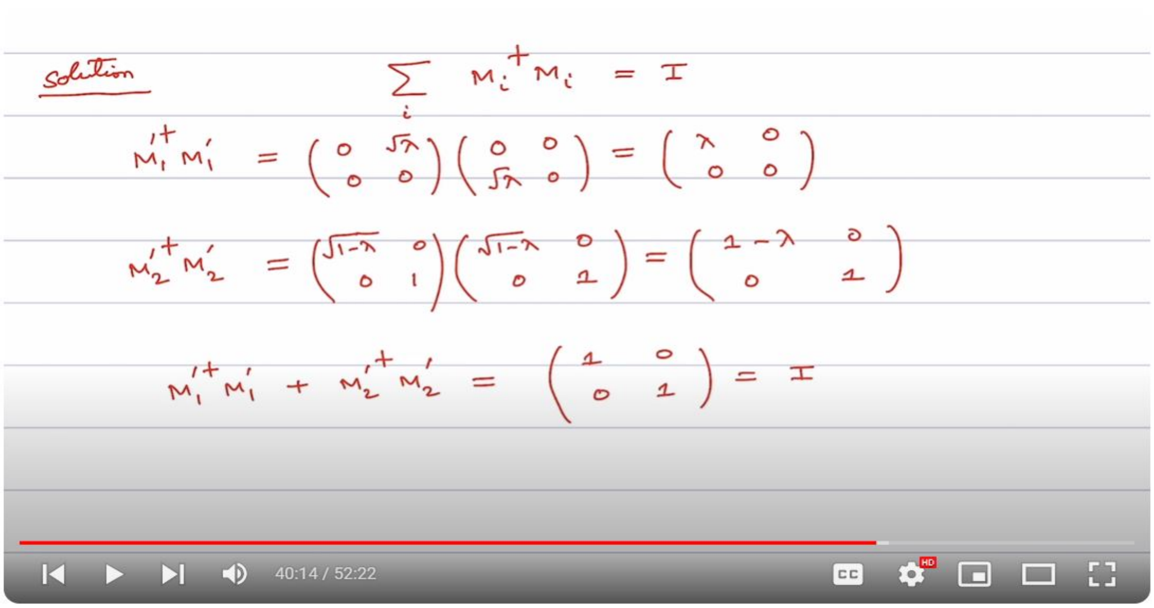

let us do it first of all we know that the measurement operators have to satisfy this condition and let us first check whether this is satisfied or not to do that let us first work out m1 dagger m1 dash dagger m1 dash that would be equal to m1 dash is 0 0 root lambda 0 and m1 dagger would be 0 root lambda 0 0 and if i take the matrix product then i am going to get lambda 0 0 0 and m2 dash is similar to m2 exactly the same as that of m2 in the previous problem still let me work it out again m2 dash is square root of 1 minus lambda 0 0 1 and m1 m2 dash dagger would be square root of 1 minus lambda 0 0 1 and if i take the matrix product then i will get 1 minus square root of 1 minus lambda actually i will get not square root of i will get 1 minus lambda 0 0 1 now if i take the sum of these two m1 dash dagger m1 dash plus m2 dash dagger m2 dash that would be equal to 1 0 0 1 unit matrix and thereby we see that m1 dash m2 dash they are measurement operators or measurement operators measurement operators.

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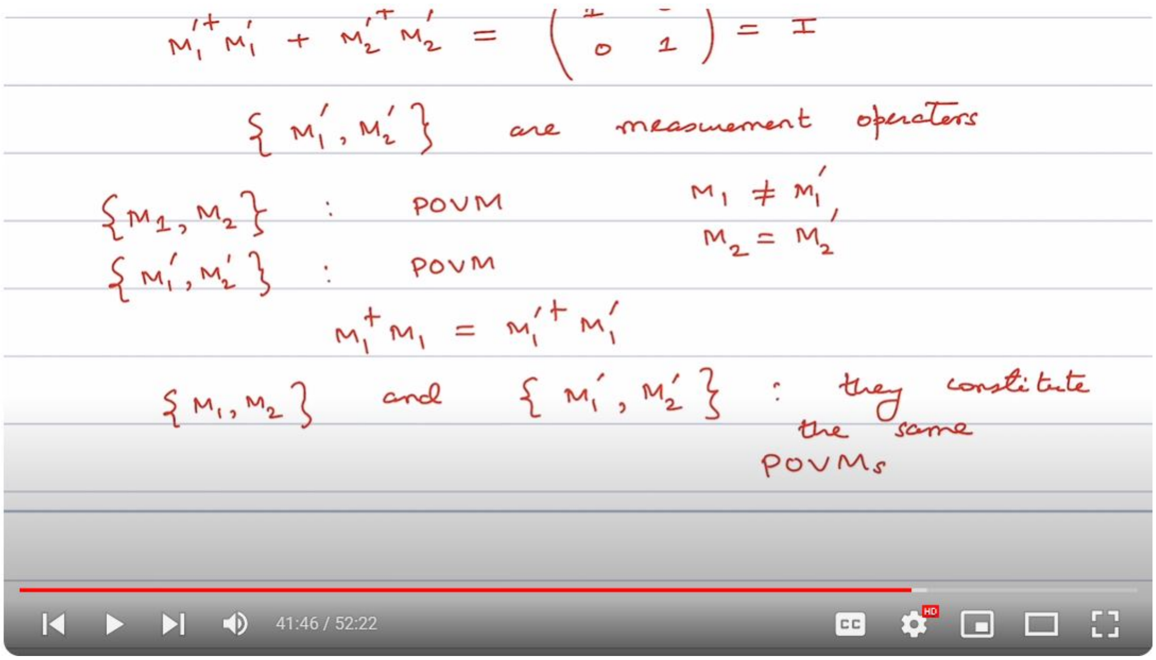

now you see one thing is that in the previous problem m1 and m2 they were povms they constitute a povm and similarly here m1 dash and m2 dash also constitute a povm where m2 is equal to m2 dash from the previous problem however in this problem m1 is not equal to m1 dash dagger but one thing you can see is that m1 dagger m1 is equal to m1 dash dagger m1 dash as you can see if you look at the previous problem as well so therefore we can conclude one thing that m1 m2 and m1 dash m2 dash they constitute the they constitute or they refer to the same povms.

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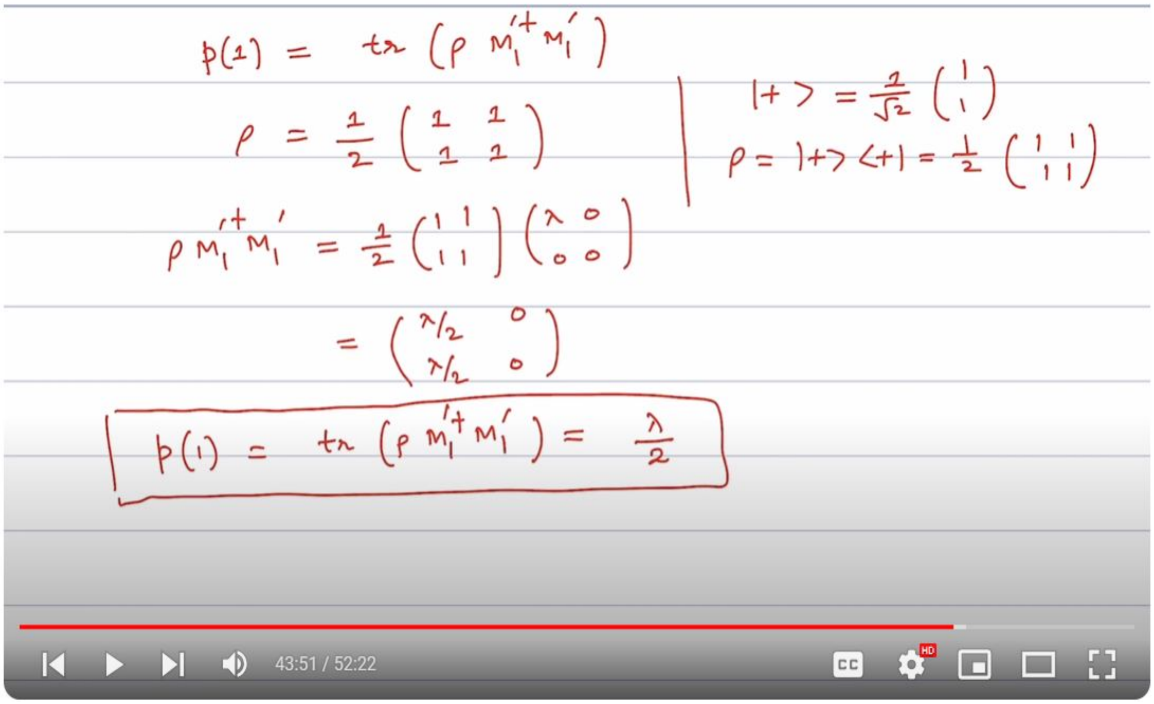

however now let us look at the second part of the problem to find out the outcome of the measurement operator m1 dash that would be equal to trace of rho m1 dash dagger m1 dash now rho is equal to it is the same as that of the previous problem this would be half one one one just to have a quick recall that we the state is in ket plus which is one by root two one one so therefore rho would be you just have to work it out and you will find that this will be half one one one one right this is what you will have now first let me work out rho m1 dash dagger m1 dash this is going to be equal to if you work it out m1 dash dagger m m1 dash already i have it this is this one right

so let me just work it out half one one one one and this is lambda zero zero zero and you will get the okay you will get what we got in the last problem so this would be equal to lambda by two so this would be lambda by two and this would be lambda by two zero zero so therefore p1 that is equal to trace of rho m1 dash dagger m1 dash this is going to be equal to simply lambda by two this is exactly the same that we have obtained with m1 as well so what do we see from this is that the outcome of the measurement is the same with the povms that's the reason i said that m1 m2 from the previous problem and in this problem m1 dash m2 dash they refer to the same povm.

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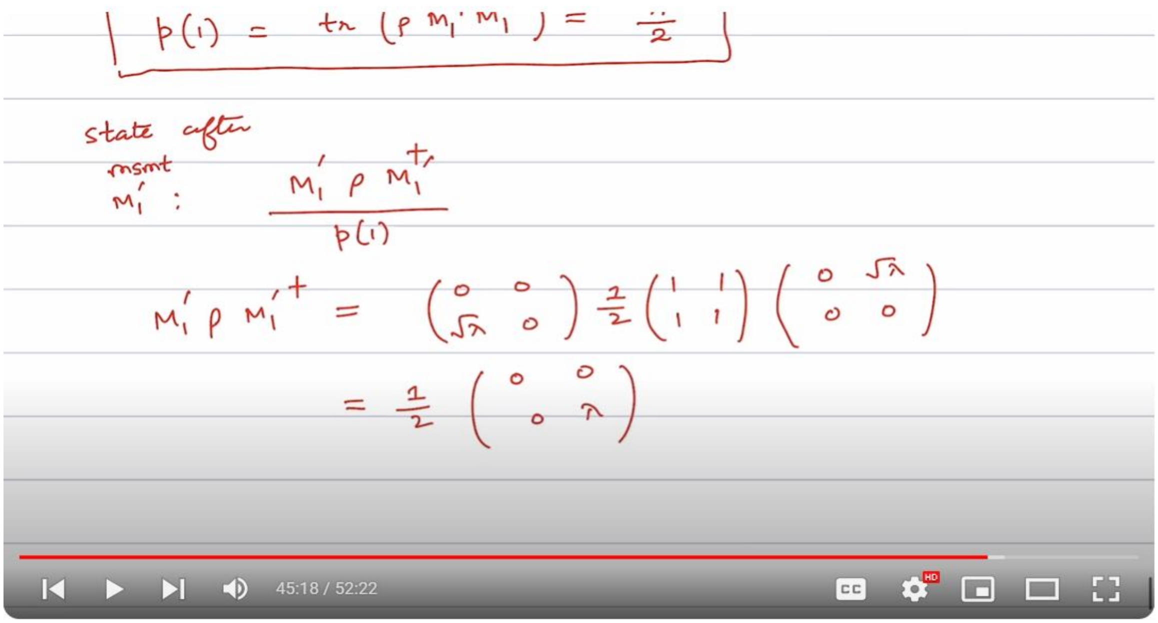

however if we talk about the state the state after this post measurement state would be given by m1 dash rho m1 dash dagger and divided by p of one and let us work out m1 dash dagger this is state after measurement m1 dash so now first let me work out m1 dash rho m1 dash dagger that would be equal to i have m1 dash the m1 is equal to zeros m1 dash is equal to zero zero root lambda zero and rho is half one one one one and m1 dagger is equal to zero root lambda zero zero and if i work it out i am going to get a half zero zero zero lambda

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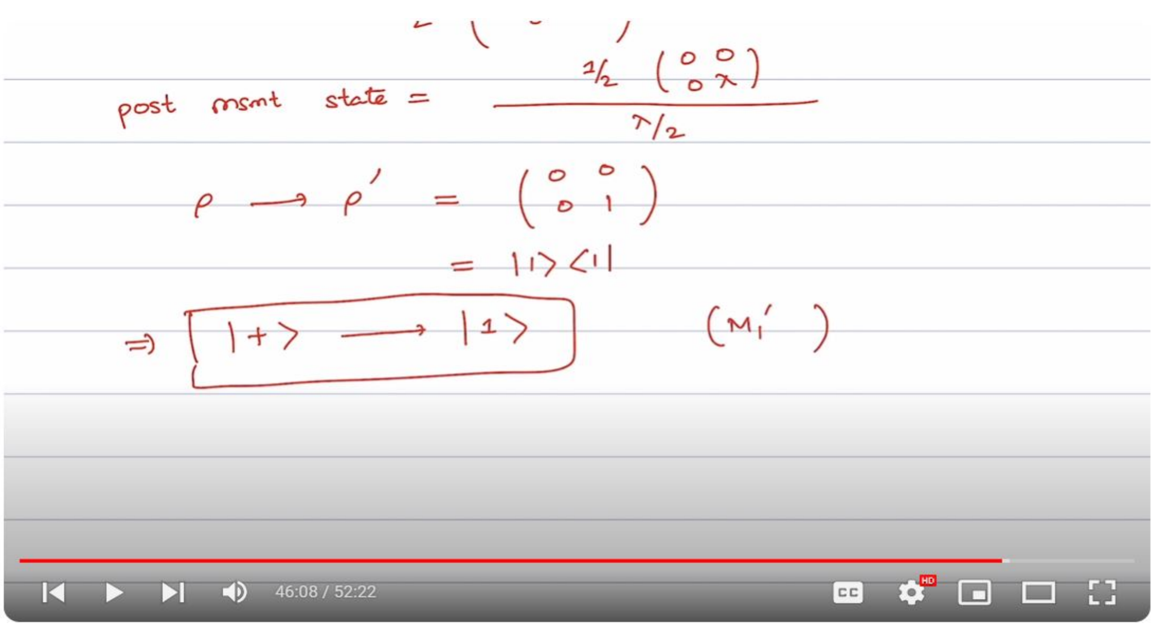

so post measurement state is post measurement state is going to be equal to half zero zero zero lambda divided by p of one and which is lambda by two so this is going to give me this is basically your rho dash and rho is going to rho this and rho this is equal to zero zero zero one and you see that this guy is nothing but ket 1 bra one so this implies that initially my state was at ket plus and now after measurement we are getting it to be at ket 1 this is because of the measurement m1 dash and by the way in the previous problem we had the initially the state was also in ket plus but measurement the state goes to ket 0 this was with the m1

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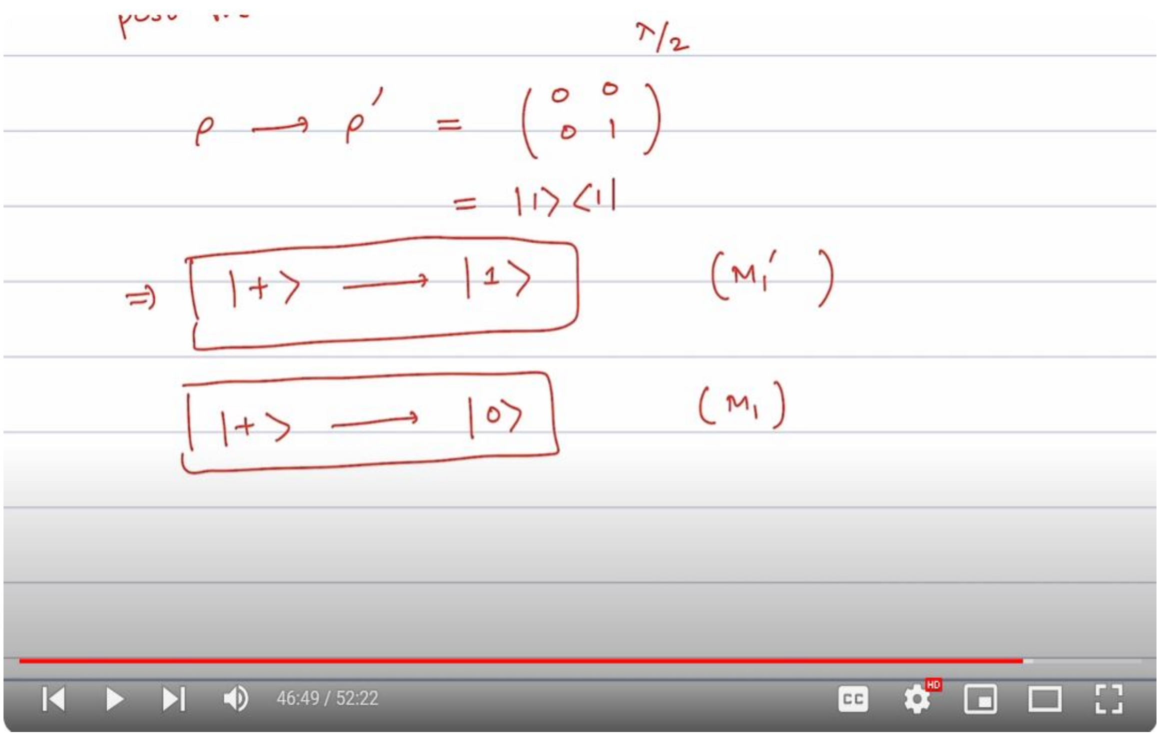

now you see the post measurement though m1 m2 and m1 dash m2 dash refer to the same POVM but they give different post measurement results so this is one of the peculiarity of POVMs in POVMs we are generally interested in knowing the outcome probability of the outcome but not interested in the post measurement state.

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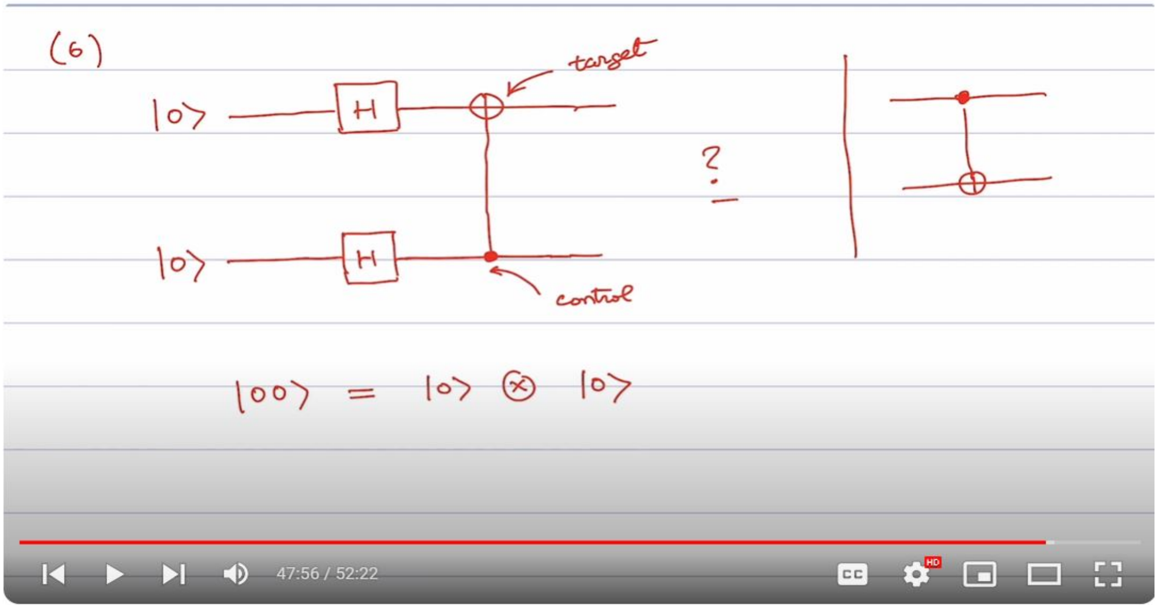

As a final problem in this session let us work out this problem related to quantum circuits we have to work out the output of this circuit when the input is given as uh zero zero this is the input given here so what is going to be the output right just notice one thing that now we are having a inverted CNOT gate after the two hadamards where in the first channel this is our target and at the second channel this is the control if you remember that usually the CNOT gate is denoted by this symbol where in the first channel we have the control and in the second channel we are having the target however here the opposite thing is there.

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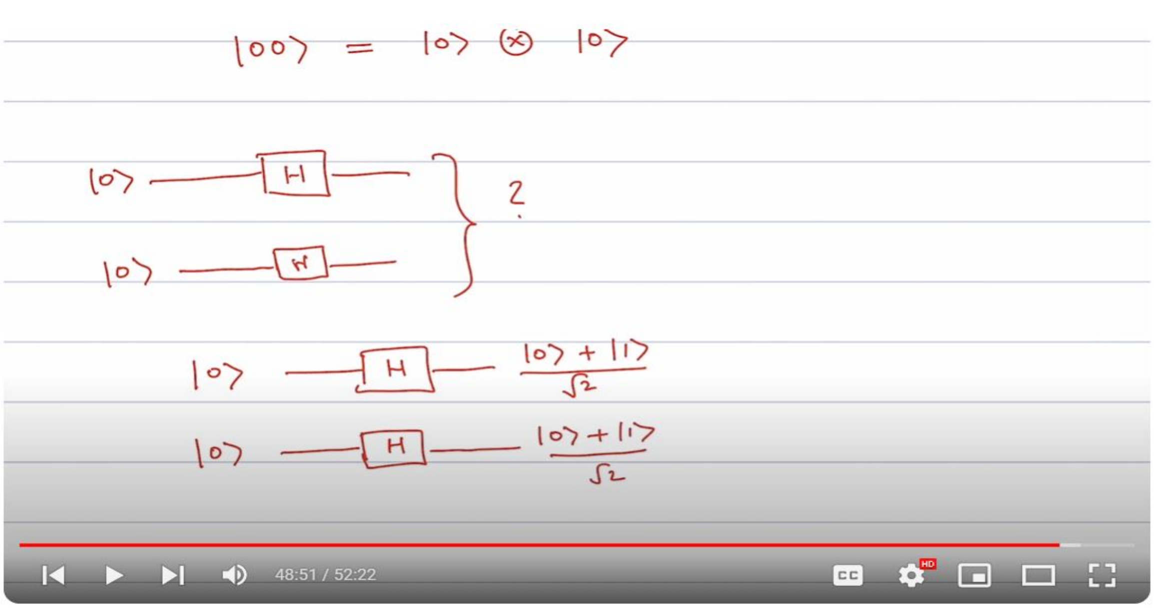

so let us work it out first of all let us see what we are going to get at the output of the two hadamard gate so we have this hadamard gates are there and input is Ket 0 Ket 0 let us find out what is going to be the output at the uh two hadamards because they are going to act this output is going to act like a input for the inverted CNOT gate we know that if ket 0 is the input for the hadamard gate at the output we are going to get the superposition state ket 0 plus ket 1 by root 2 similarly here we will get ket 0 for the second channel the if the input is ket 0 the output would be ket 0 plus ket 1 by root 2

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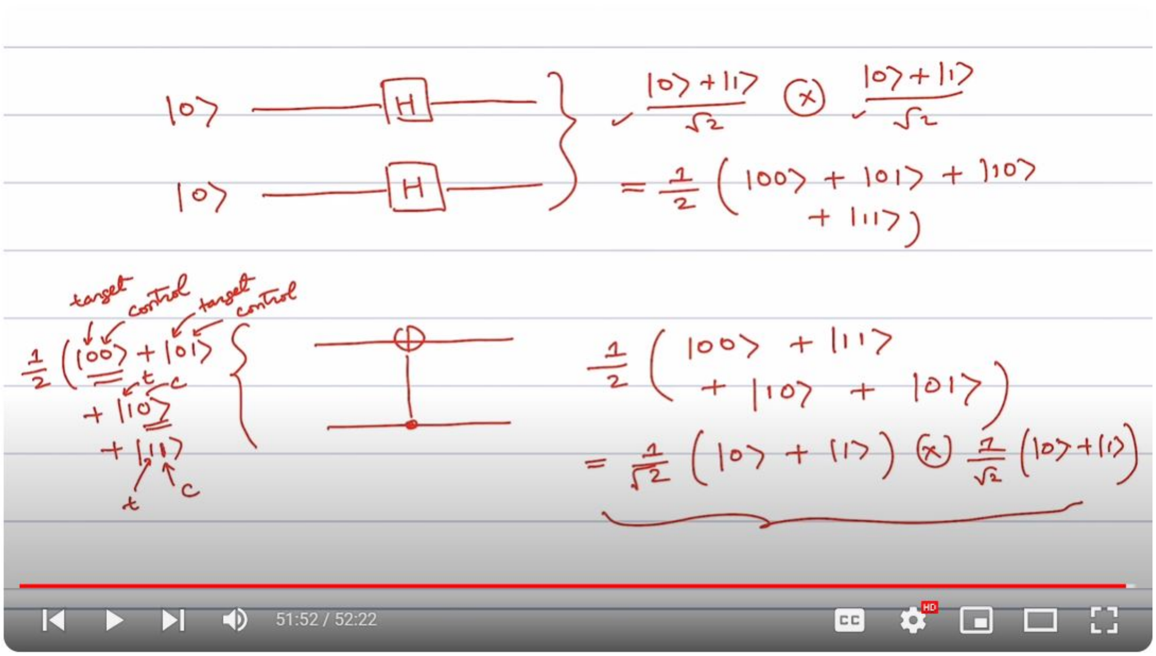

so therefore you can easily see that at the output of the two hadamard gate we are going to get at the at the output we are going to get the state as ket 0 plus ket 1 by root 2 inner product with ket 0 plus ket 1 by root 2 this one refer to the first channel this one refer to the second channel so overall we are going to get this Bell state that would be 0 0 plus 0 1 plus 1 0 plus 1 1 so this is going to be my output

now this output of the hadamard gate will act like a input for the inverted CNOT gate so here my input is half Ket 0 0 plus Ket 0 1 plus ket 1 0 plus ket 1 1 right so this is my input to the inverted CNOT gate one thing you have to just remember here is that now this first qubit ket 0 this is is going to act like a target this is my target and this other second qubit is going to act like a control

so we know that if the control is 0 the target is going to remain unchanged and if the control is 1 the target is going to get you know flipped at the output so here also you have this is your target this is your control this is your target this is your control so obviously at the output you can easily make it out that from this we are going to have

Ket 0 0 but in this case the control is 1 the target is going to get flipped so we'll get 1 1 and for from this we'll get the target is control is 0 so target is not going to get flipped so we'll have 1 0 and finally if the in the last case if the target is control is 1 the target is going to get flipped so we'll get 0 1 so this is going to be my output at the inverted CNOT gate this also we can write as 1 by root Ket 0 plus Ket 1 and 1 by root 2 Ket 0 plus Ket 1 so this is going to be the final output of the problem so i hope you get it so we can actually design various quantum circuit this way and you can play with it you.