Quantum Entanglement: Fundamentals, measures and application

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Week-03

Lec 12: Quantum Entanglement Measures-I

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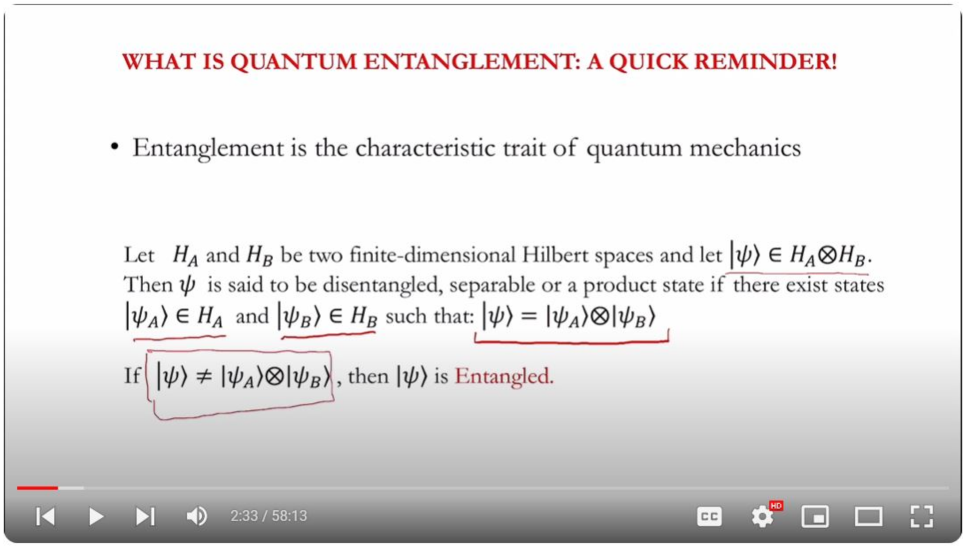

WHAT IS QUANTUM ENTANGLEMENT: A QUICK REMINDER!

- Entanglement is the characteristic trait of quantum mechanics

Let H_A and H_B be two finite-dimensional Hilbert spaces and let $\underline{|\psi\rangle \in H_A \otimes H_B}$. Then ψ is said to be disentangled, separable or a product state if there exist states $\underline{|\psi_A\rangle \in H_A}$ and $\underline{|\psi_B\rangle \in H_B}$ such that: $\underline{|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle}$

If $\boxed{|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle}$, then $|\psi\rangle$ is Entangled.

Hello, welcome to lecture 3 of module 3. This is lecture number 9 of the course. We are now prepared to discuss quantum entanglement measures. As you know, in the early days of quantum mechanics, quantum entanglement was considered to be a qualitative feature of quantum mechanics. However, after discovery of Bell's inequality, quantum entanglement got a distinctive quantitative feature. Today, there are many quantum entanglement measures and in this lecture and subsequent lectures, we are going to discuss some of the quantum entanglement measures.

However, before that, first let me tell you very briefly what is exactly quantum entanglement measure is. But even before that, let me remind you very quickly what is quantum entanglement again. So as we know that entanglement is the characteristic trait of quantum mechanics, say, H A and H B be two finite dimensional Hilbert space. And there is a state vector psi which belongs to the composite Hilbert space, H A direct product

H B. Then this state vector psi is said to be disentanglement separable or a product state if there exists a state belonging to the Hilbert space of A and Hilbert space. Another state vector psi B, ket psi B belonging to the Hilbert space H B such that I can write the composite state vector as a product state vector of the state ket A, psi A and psi B. If I can do that, then ket psi is said to be product state or separable, that means not entangled. On the other hand, if I cannot do that, as you know, then this state vector psi is said to be entangled. This was in the context of a pure state.

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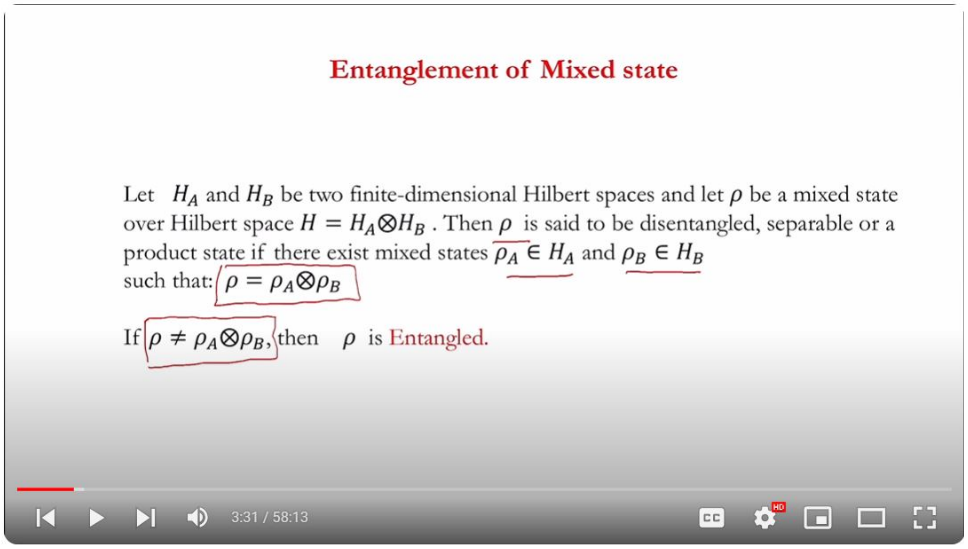

Entanglement of Mixed state

Let H_A and H_B be two finite-dimensional Hilbert spaces and let ρ be a mixed state over Hilbert space $H = H_A \otimes H_B$. Then ρ is said to be disentangled, separable or a product state if there exist mixed states $\underline{\rho_A} \in H_A$ and $\underline{\rho_B \in H_B}$ such that: $\boxed{\rho = \rho_A \otimes \rho_B}$

If $\boxed{\rho \neq \rho_A \otimes \rho_B,}$ then ρ is Entangled.

We can do the same thing with the mixed state also. Only thing is that in the case of mixed state, rather than a state vector, we will define the quantum state by a density matrix rho. And if there is a density matrix belonging to the Hilbert space of A and Hilbert space of B, then if I can write the composite density matrix of the system, composite system as a product of the density matrix corresponding to the system A and system B, then the A and B is said to be not entangled because they are now separable. On the other hand, if I cannot write the composite density matrix as a product of the respective density matrix of the system A and B, then rho is said to be entanglement. So this must basic things already by now it should be clear to all of us.

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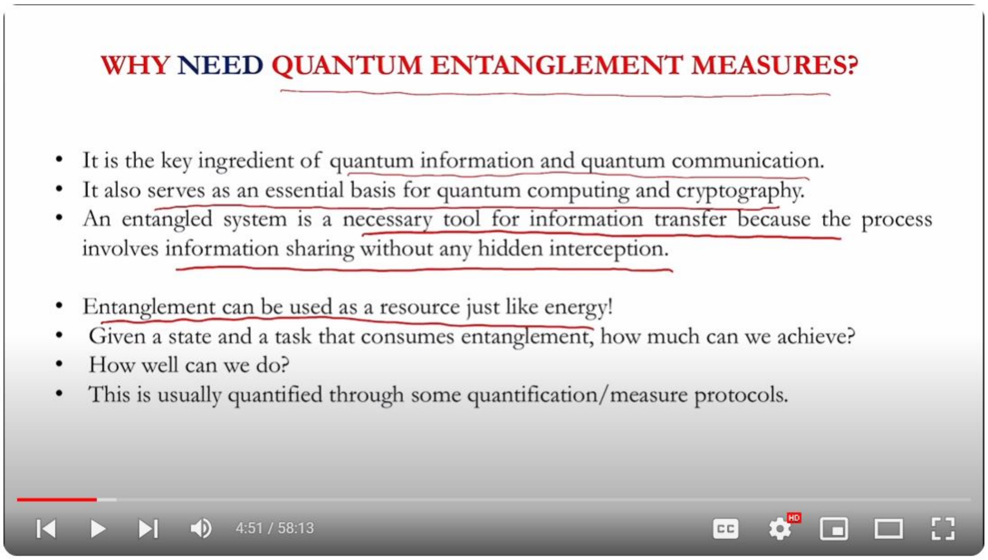

WHY NEED QUANTUM ENTANGLEMENT MEASURES?

- It is the key ingredient of quantum information and quantum communication.
- It also serves as an essential basis for quantum computing and cryptography.
- An entangled system is a necessary tool for information transfer because the process involves information sharing without any hidden interception.

- Entanglement can be used as a resource just like energy!
- Given a state and a task that consumes entanglement, how much can we achieve?
- How well can we do?
- This is usually quantified through some quantification/measure protocols.

Now the question is why we need quantum entanglement measures. First of all, it is the key ingredient of quantum information and quantum communication. It also serves as a, basically quantum entanglement measures serve as an essential basis for quantum computing and cryptography. An entangled system is a necessary tool for information transfer because the process involve information sharing without any hidden interception, which is basically the need. That is why quantum cryptography is so popular because this hidden interception is not possible if we use quantum cryptography.

Now entanglement can be used as a resource just like we use energy. So suppose you are given a state and a task that consume entanglement, the question that we can ask is that how much we can achieve, how well we can do. And this is usually, these questions can be answered usually by a quantification of entanglement or some quantum entanglement measure protocols.

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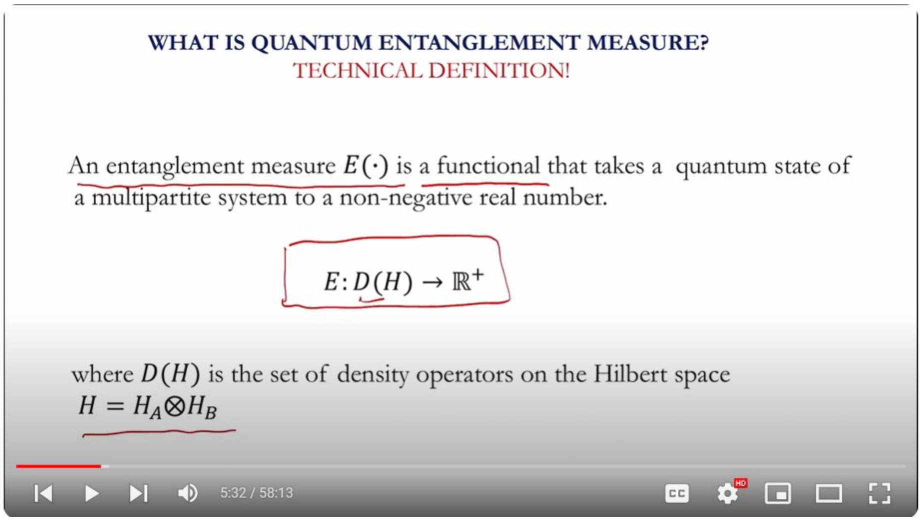

WHAT IS QUANTUM ENTANGLEMENT MEASURE?

TECHNICAL DEFINITION!

An entanglement measure $E(\cdot)$ is a functional that takes a quantum state of a multipartite system to a non-negative real number.

$$\boxed{E: D(H) \to \mathbb{R}^+}$$

where $D(H)$ is the set of density operators on the Hilbert space $H = H_A \otimes H_B$

Now what is the technical definition of quantum entanglement measure? An entanglement measure, which is generally denoted by the symbol E, is a functional that takes a quantum state of a multipartite system to a non-negative real number. I think this issue would be more clearer when I will discuss some quantum entanglement measures later on.

Now here in this expression, here this D of X is the set of density operators on the Hilbert space and we are going to discuss quantum entanglement measure mainly in the context of a bipartite system like this.

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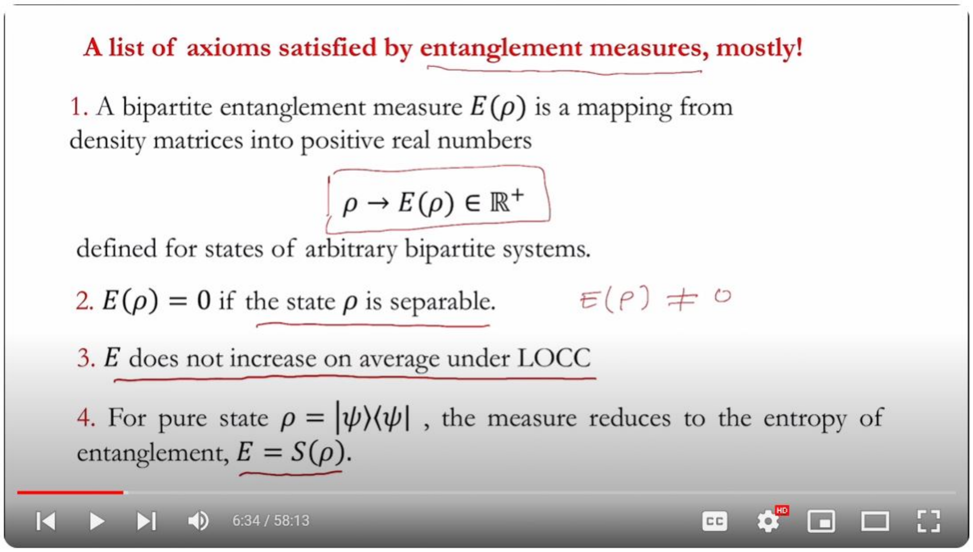

A list of axioms satisfied by entanglement measures, mostly!

1. A bipartite entanglement measure $E(\rho)$ is a mapping from density matrices into positive real numbers

$$\boxed{\rho \to E(\rho) \in \mathbb{R}^+}$$

defined for states of arbitrary bipartite systems.

2. $E(\rho) = 0$ if the state ρ is separable. $E(\rho) \neq 0$

3. E does not increase on average under LOCC

4. For pure state $\rho = |\psi\rangle\langle\psi|$, the measure reduces to the entropy of entanglement, $E = S(\rho)$.

Now a list of axioms is given here which has to be satisfied by entanglement measures mostly. And first of all, a bipartite entanglement measure is a mapping from density matrices into a positive real number as we showed in the last slide there. And if the entanglement measure is 0, E rho is equal to 0, that means the state rho is separable. So if the state is basically entanglement, a bipartite system is entanglement, you are always going to get E rho, this entanglement measure is going to be a finite number.

It is not going to be 0. And this entanglement measure E does not increase on the average under LOCC, local quantum operational classical communication. For a pure state rho, the measure reduces to the entropy of entanglement which we are going to discuss in today's class.

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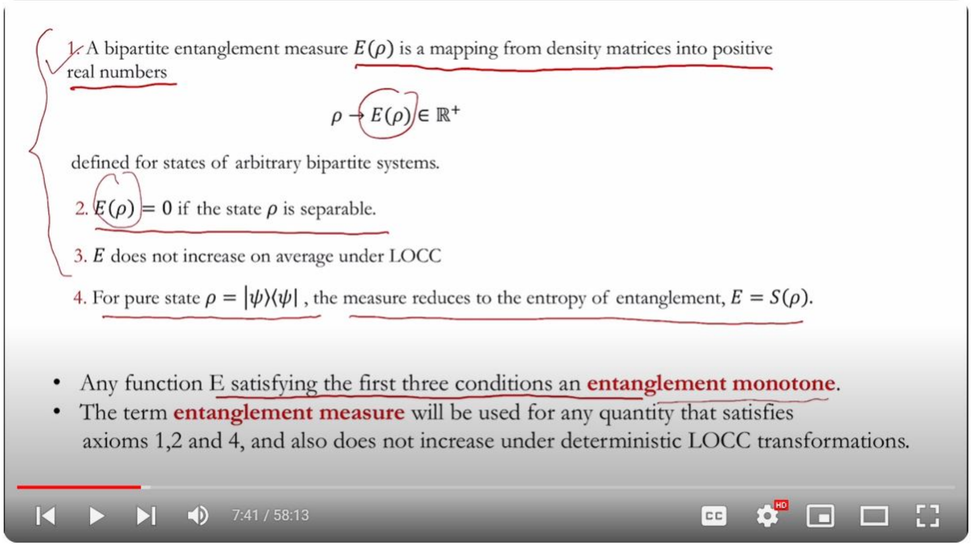

1. A bipartite entanglement measure $E(\rho)$ is a mapping from density matrices into positive real numbers

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defined for states of arbitrary bipartite systems.

2. $E(\rho) = 0$ if the state ρ is separable.
3. E does not increase on average under LOCC
4. For pure state $\rho = |\psi\rangle\langle\psi|$, the measure reduces to the entropy of entanglement, $E = S(\rho)$.

- Any function E satisfying the first three conditions an **entanglement monotone**.
- The term **entanglement measure** will be used for any quantity that satisfies axioms 1,2 and 4, and also does not increase under deterministic LOCC transformations.

Next, apart from this, you see any function E satisfying the first three conditions is known as entanglement monotone. So out of all these four that already I discussed, if these first three properties are satisfied, that entanglement measure is also called entanglement monotone.

And the term entanglement measure will be used for any quantity that satisfy the axioms. This axiom first one, that means the bipartite entanglement measure Ep is a mapping from density matrix to a positive real number. Second one is that E rho is equal to 0 if rho is separable. And the fourth one for pure state rho, the measure reduces to the entropy of entanglement. And also if the entanglement, that means the measure does not increase under deterministic LOCC transformation, then that particular E, capital E, that this particular term here is going to be called entanglement measure.

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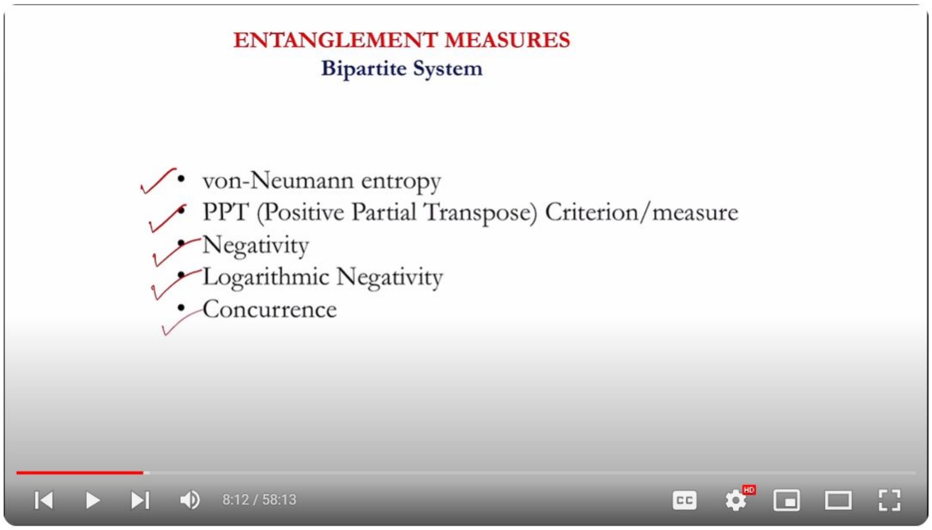

ENTANGLEMENT MEASURES

Bipartite System

- ✓ von-Neumann entropy
- ✓ PPT (Positive Partial Transpose) Criterion/measure
- ✓ Negativity
- ✓ Logarithmic Negativity
- ✓ Concurrence

Now, in this course, in particular, in this particular lecture and the next one, we are going to discuss entanglement measures like von Neumann entropy, positive partial transpose criterion or measure, which is called in short PPT. Then there is another measure called negativity, analogous, a similar one is the so-called logarithmic negativity, concurrence and so on are going to be discussed. The first quantum entanglement measure that I'm going to discuss now is the entropy of quantum entanglement, also known as von Neumann entropy. Already those who have studied physics have encountered entropy in the context of thermodynamics or maybe some of you have got that in classical statistical mechanics as well. So let me first revisit very quickly and briefly the concept of entropy in the context of classical statistical mechanics.

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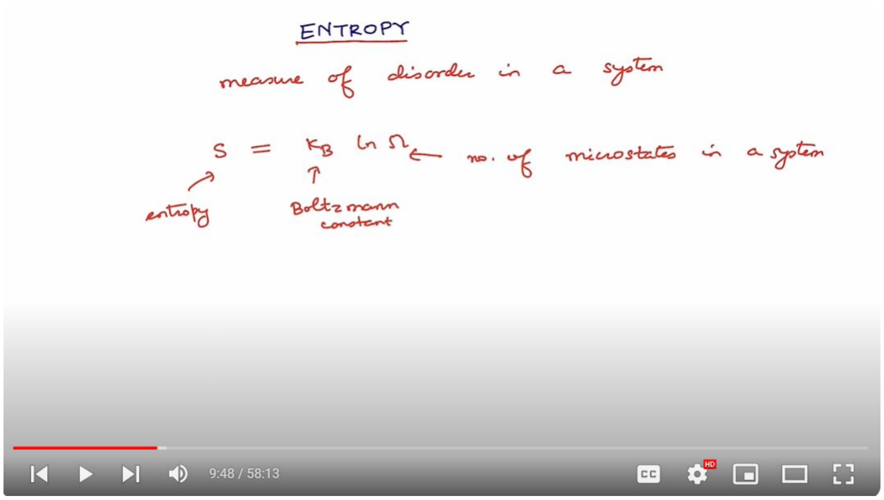

ENTROPY

measure of disorder in a system

$$S = k_B \ln \Omega \leftarrow \text{no. of microstates in a system}$$

entropy

Boltzmann constant

Entropy is generally known as a measure of disorder in a system and all of us know that entropy is a measure of disorder in a system. Entropy was first of all discovered in thermodynamics and in classical statistical mechanics, it is given by this expression or formula S is equal to K B logarithm of omega, where S is the entropy. K B is the so-called Boltzmann constant and omega is the number of microstates in a system.

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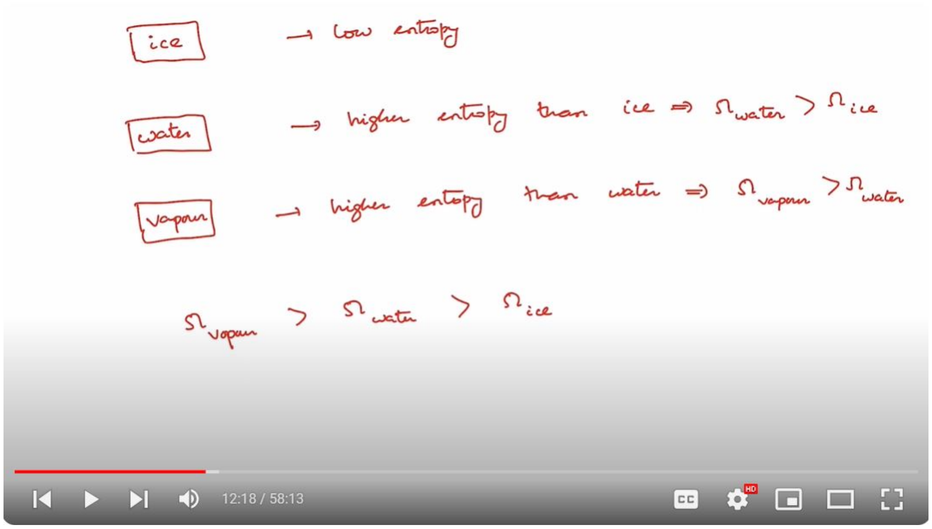

ice → low entropy

water → higher entropy than ice $\Rightarrow \Omega_{water} > \Omega_{ice}$

vapour → higher entropy than water $\Rightarrow \Omega_{vapour} > \Omega_{water}$

$\Omega_{vapour} > \Omega_{water} > \Omega_{ice}$

To give you a quick and simple example, let us consider three states of water. Water can be in three states as we know, it can be in say ice, that's the solid state and the liquid state is water and then we have a so-called vapor state.

So these are the three states of water. Now in the case of ice, you know, the molecules in ice have to stay in a lattice as it is a rigid system, so ice has a low entropy. On the other hand, molecules have more positions in the case of water, right? So I mean to say that molecules in water has more positions to move around, so molecules in water has more entropy than ice, so it has a higher entropy or more entropy than that of ice. And finally, what does it basically means? That means molecules in water has more number of microstates than that of ice because it has more space to move on. On the other hand, in the case of vapor, molecules inside the vapor can pretty much go anywhere they want, so water vapor has high entropy. It has the higher entropy than that of water, right? So then water, in other words, it means that the number of microstates in vapor is greater than the number of microstates in water.

So therefore, out of all these three states, then we can write that as regards the microstates are concerned, we can write that the number of microstates in vapor is greater than the number of microstates in water. And the number of microstates in water is greater than that of the number of microstates in ice.

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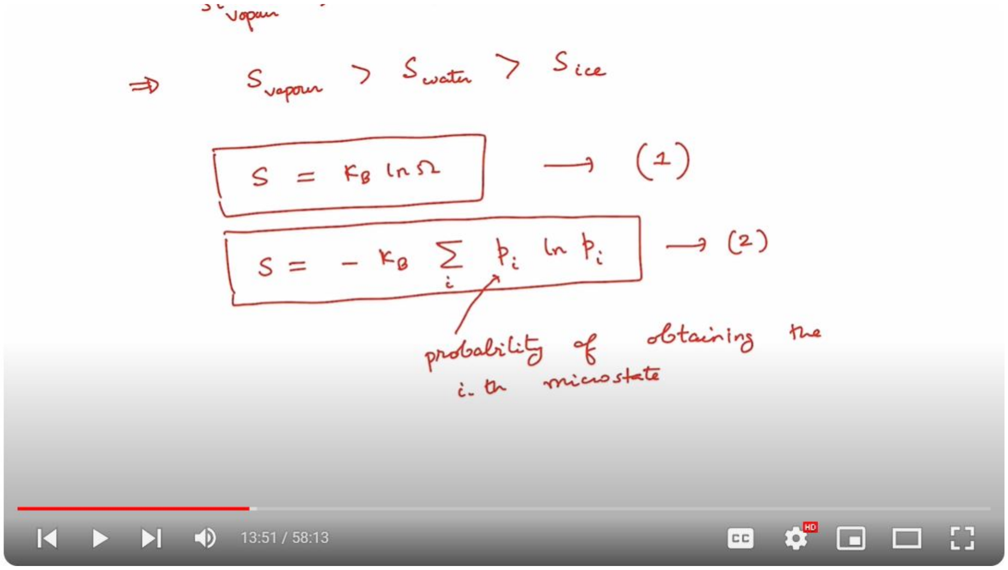

$$\Rightarrow \quad S_{vapour} > S_{water} > S_{ice}$$

$$\boxed{S = k_B \ln \Omega} \longrightarrow (1)$$

$$\boxed{S = -k_B \sum_i p_i \ln p_i} \longrightarrow (2)$$

probability of obtaining the i-th microstate

Which basically means that the entropy of vapor is greater than the entropy of water and entropy of water is greater than that of the entropy of ice. So this expression S is equal to Kb logarithm of omega. This is, as I said, this is called Gibbs entropy and this was discussed in thermodynamics and classical statistical mechanics.

It was discovered and this expression can be written in terms of probability as well. S is equal to say minus Kb summation Pi logarithm of Pi. This is another way to write, express this equation number one. So let me say this is equation number two. Here, this Pi is the probability of obtaining the i-th microstate.

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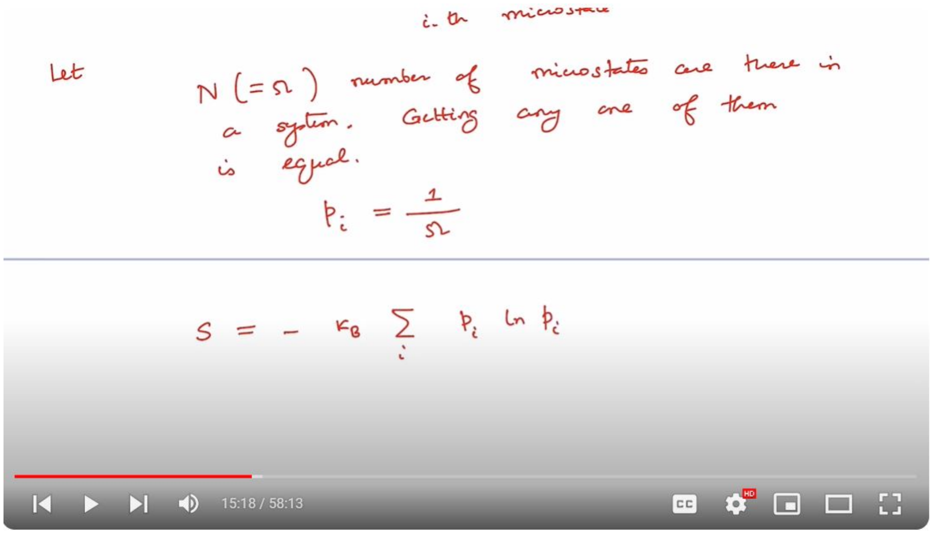

i-th microstate

Let N (=Ω) number of microstates are there in a system. Getting any one of them is equal.

$$p_i = \frac{1}{\Omega}$$

$$S = -k_B \sum_i p_i \ln p_i$$

Actually, I can obtain equation one from equation two. To do that, let me show you how to do that in a very simplified manner. It's an oversimplification. Let us say there are n number of microstates or omega number of microstates are there. I'm going to obtain equation number one from two. So for that, I'm assuming, say, let n number of microstates are there in a system. Microstates are there in a system and all of them are equally probable. And all of them are, that means getting any one of them, one of the microstates, any one of the microstates is equal. That means the corresponding probability, that means getting any microstate, any, just let us take i-th microstate. So it would be given by the probability would be one by omega, right? Kp omega, which is the number of microstates. So this means I can write this expression as S is equal to minus Kb summation i, sum over i Pi, logarithm of Pi.

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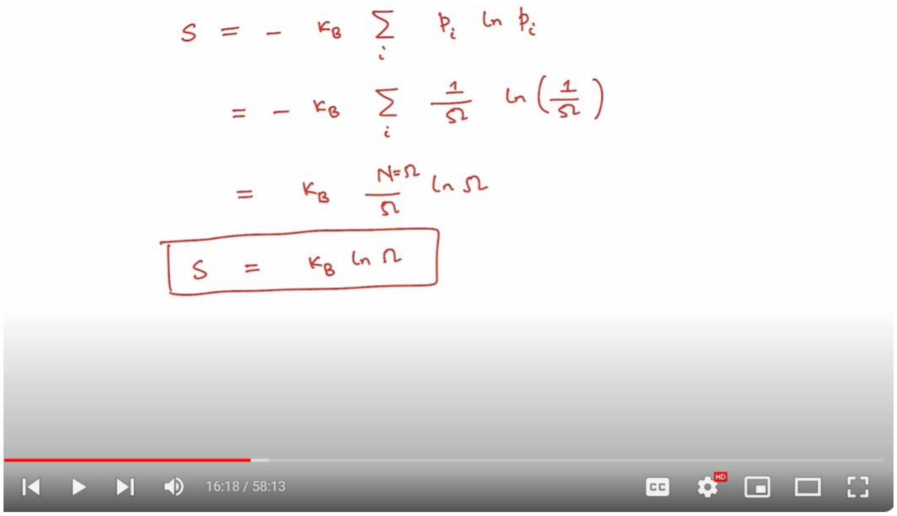

$$S = -k_B \sum_i p_i \ln p_i$$

$$= -k_B \sum_i \frac{1}{\Omega} \ln\left(\frac{1}{\Omega}\right)$$

$$= k_B \frac{N=\Omega}{\Omega} \ln \Omega$$

$$\boxed{S = k_B \ln \Omega}$$

I can write it as minus Kb summation i Pi is one by Kp omega, logarithm of one by omega. So this is going to give me Kb. You see what I'm going to get because of all these things is that this would be n by omega, logarithm of one would be zero. So therefore I will have simply logarithm of omega. Okay. And n is equal to omega by the way, right? n is equal to omega. So therefore I get Kb logarithm of omega. So this was the original expression, right? This is equation number one.

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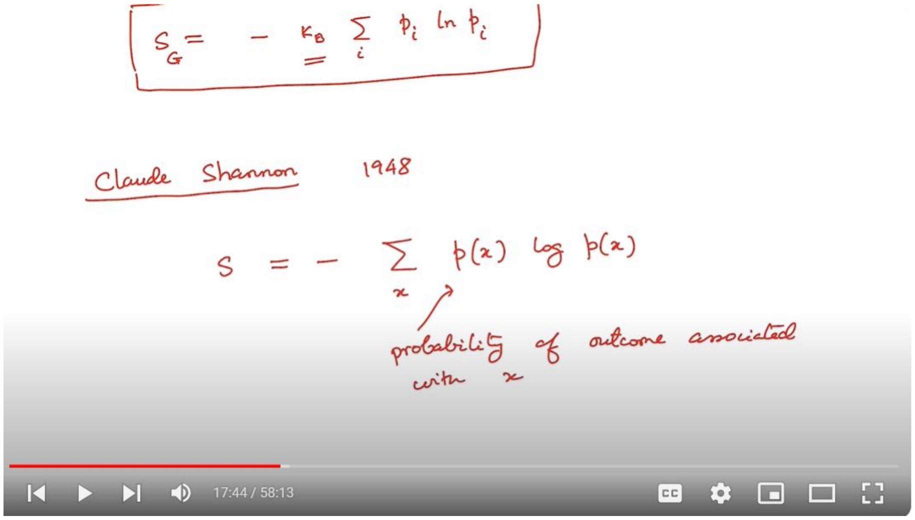

$$\boxed{S_G = -\underline{\underline{k_B}}\sum_i p_i \ln p_i}$$

Claude Shannon 1948

$$S = -\sum_x p(x)\log p(x)$$

probability of outcome associated with x

So this formula S is equal to minus Kb sum over i Pi logarithm of Pi. We write in the context of classical statistical mechanics, which is known as Gibbs entropy.

Taking clue from Gibbs entropy, Claudes Shannon, Claudes Shannon, an information theorist in 1948, introduced the following formula for entropy in the context of information theory. The formula was S is equal to minus summation P of x logarithm of P of x, where this summation denotes the sum over the possible values of the random variable X. And P of X is the probability of the outcome, probability, probability of outcome associated, associated with the random variable X.

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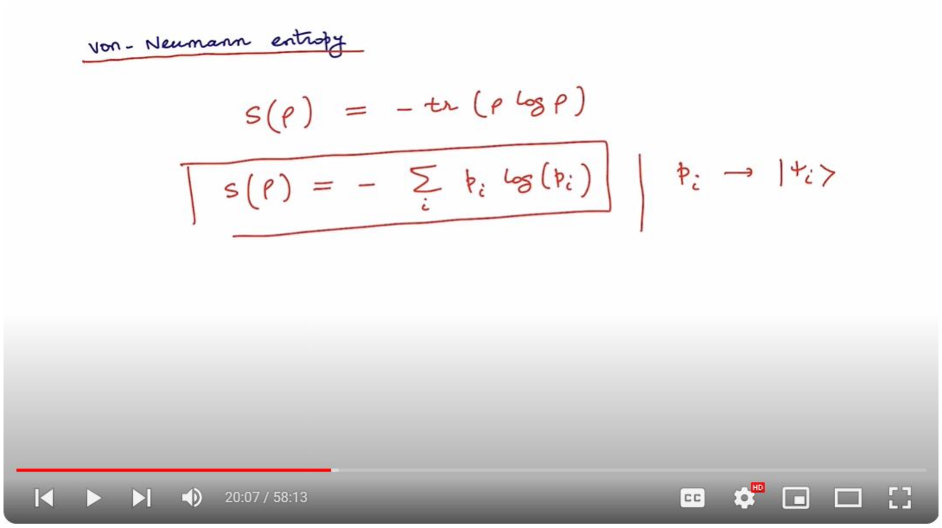

von- Neumann entropy

$$S(\rho) = -\mathrm{tr}(\rho \log \rho)$$

$$\boxed{S(\rho) = -\sum_i p_i \log(p_i)} \quad p_i \rightarrow |\psi_i\rangle$$

And as you can see in this formula, the Boltzmann constant Kb is no longer there because, you know, we are now discussing entropy in the context of information theory, not in the context of thermodynamics. And entropy here is a measure of uncertainty or surprise inherent to possible outcomes of the variables.

In information theory, entropy is used to quantify the degree of correlation between systems. And correlation is a measure of information. When two systems, say A and B are correlated, we can ask how much information of A is contained in B or vice versa. Anyway, Shannon's entropy function is a measure of the uncertainty contained in a system described by random variables. This idea of Shannon entropy is extended to quantum information theory by von Neumann, and it is known as von Neumann entropy.

You know that the state of a quantum system is described by density matrix. So quite necessarily, the expression for von Neumann entropy also contains, is a basically is a function of density matrices, as we will now see. Von Neumann entropy is given by this formula S of rho, where rho is the density matrix, S is the entropy. Now we are discussing entropy in the context of quantum mechanics. So entropy is a function of the density matrix, as I said, it is given by this formula.

Trace over the function rho log rho, where rho is the density matrix, as I said. In fact, it can be shown that this formula can be obtained from the Shannon formula, which is minus summation over the indices i, rho, not rho, its probability Pi, log of Pi, where Pi is the probability of getting the state vector psi i. Okay, so this is the formula.

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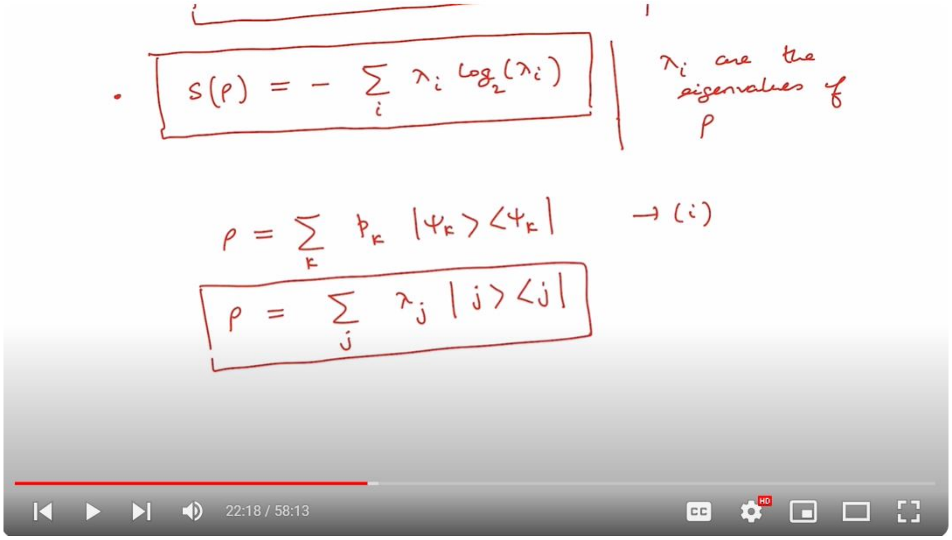

$$\boxed{S(\rho) = -\sum_i \lambda_i \log_2(\lambda_i)}$$

λ_i are the eigenvalues of ρ

$$\rho = \sum_K p_K |\psi_K\rangle\langle\psi_K| \rightarrow (i)$$

$$\boxed{\rho = \sum_j \lambda_j |j\rangle\langle j|}$$

And in fact, we can show that we can write the entropy, von Neumann entropy formula in this form also, minus sum over lambda i log lambda i, where lambda i are the eigenvalues are the eigenvalues of the density matrix rho. Okay, this is a very important and it's kind of a very useful because for many problems or quantification purpose, we will see it is we can use this formula, we just need to find out the eigenvalues of the density matrix.

And then we can just use this formula to calculate the entropy. Now, one thing one should notice that entropy in physics is defined using Neserle logarithm, while in information theory, logarithm basis used to be two. Now, let me explain the formula for von Neumann entropy a little bit more clearly. You know that the density operator is defined like this, summation pK or say Pi whatever, psi K, K psi bra psi.

This is the definition of density matrix. And if we choose the eigenbasis of the density operator, we can express this in this form, we can write it because it's a matrix density operator or density, it's a matrix. So we can use the spectral decomposition. And in using spectral decomposition, we know we just have to find out the eigenvalues and the corresponding eigenvectors. Suppose the eigenvalues are lambda J and the corresponding eigenvectors are KJ like this, then this is basically the spectral decomposition. And we can express the density matrix in this particular form.

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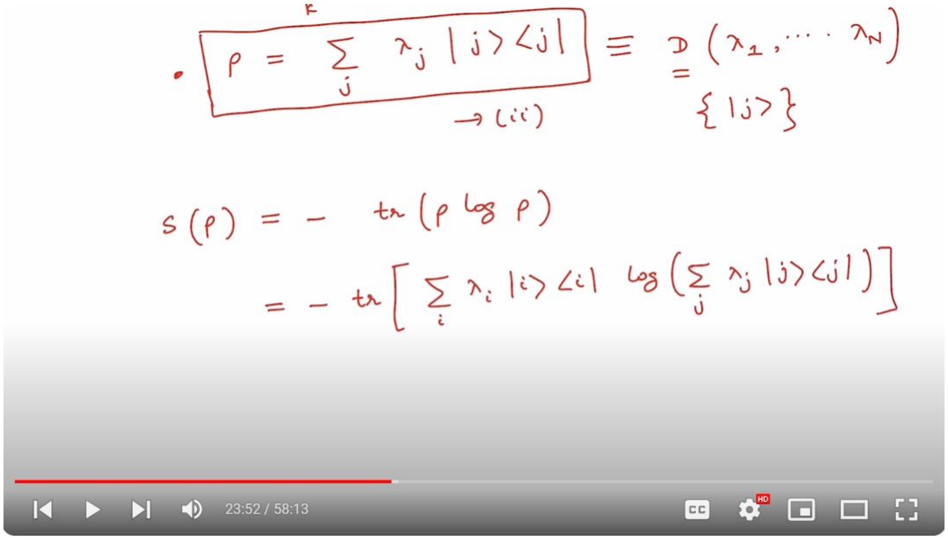

$$\bullet\ \boxed{\rho = \sum_{j}^{k} \lambda_j |j\rangle\langle j|} \equiv \underset{\underset{\{|j\rangle\}}{=}}{D}(\lambda_1, \cdots \lambda_N)$$

$\rightarrow (ii)$

$$S(\rho) = -\,\mathrm{tr}(\rho \log \rho)$$

$$= -\,\mathrm{tr}\Big[\sum_i \lambda_i |i\rangle\langle i| \log\Big(\sum_j \lambda_j |j\rangle\langle j|\Big)\Big]$$

So, in simple words, let me reiterate that we just have to find the eigenvalues and the corresponding eigenvectors of the density matrix and write it in a convenient form. This way we can express rho as a diagonal matrix, because you see we can express the density matrix as a diagonal matrix having the eigenvalues along the diagonal up to lambda N, where lambda N is the, you know, where basically capital N is the dimension of the Hilbert space we are dealing with. And D is a diagonal matrix in the basis of ket J. Let us put this expression in the von Neumann entropy formula and see what we'll get. Von Neumann entropy formula S rho is equal to minus trace rho log rho. Now, if I put the expression for rho, rho is equal to sum over I lambda I ket I bra I. And then we have here log. And again, we have say, let me use another in this J lambda J ket J bra J ket J bra J. OK, this is what I have.

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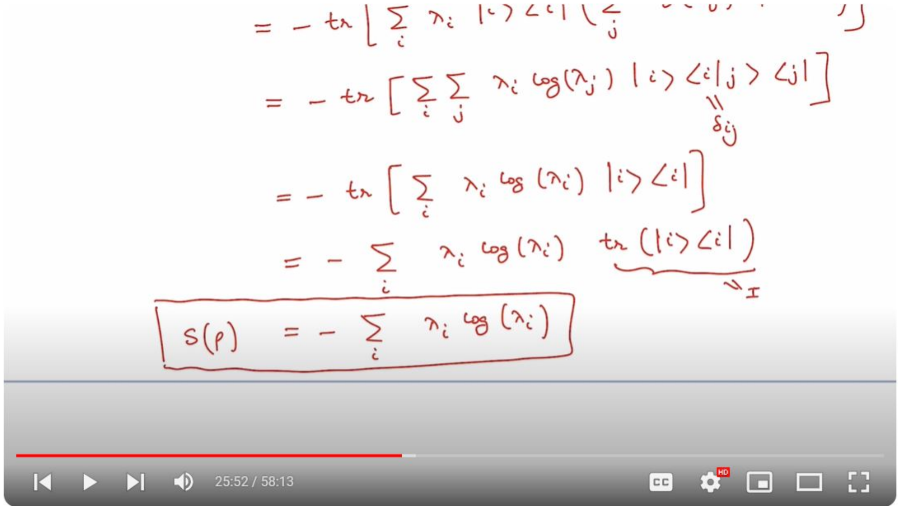

$$= -\,\mathrm{tr}\Big[\sum_i \lambda_i |i\rangle\langle i| \Big(\sum_j$$

$$= -\,\mathrm{tr}\Big[\sum_i \sum_j \lambda_i \log(\lambda_j)\, |i\rangle \underset{\delta_{ij}}{\underset{\shortparallel}{\langle i|j\rangle}} \langle j|\Big]$$

$$= -\,\mathrm{tr}\Big[\sum_i \lambda_i \log(\lambda_i)\, |i\rangle\langle i|\Big]$$

$$= -\sum_i \lambda_i \log(\lambda_i)\, \underbrace{\mathrm{tr}(|i\rangle\langle i|)}_{= I}$$

$$\boxed{S(\rho) = -\sum_i \lambda_i \log(\lambda_i)}$$

I can now write it as minus trace sum over I lambda I ket I bra I. Let me take the summation here, sum J logarithm I can take inside log of lambda J ket J bra J. From here, I can write minus trace sum over I, sum over J lambda I log of lambda J ket I bra I. This is the inner product IJ of I and J and bra J, as you can see. Now, this guy is nothing but the Kronecker delta IJ. And using this Kronecker delta property, we can now write minus trace sum over I lambda I log of lambda I ket I bra I.

And from here, finally, I can write minus summation over I lambda I log of lambda I trace over ket I bra I. And therefore, I get minus sum over I lambda I log of lambda I, because trace over this quantity is identity. So therefore, we get the entropy as von Neumann entropy is given by this expression.

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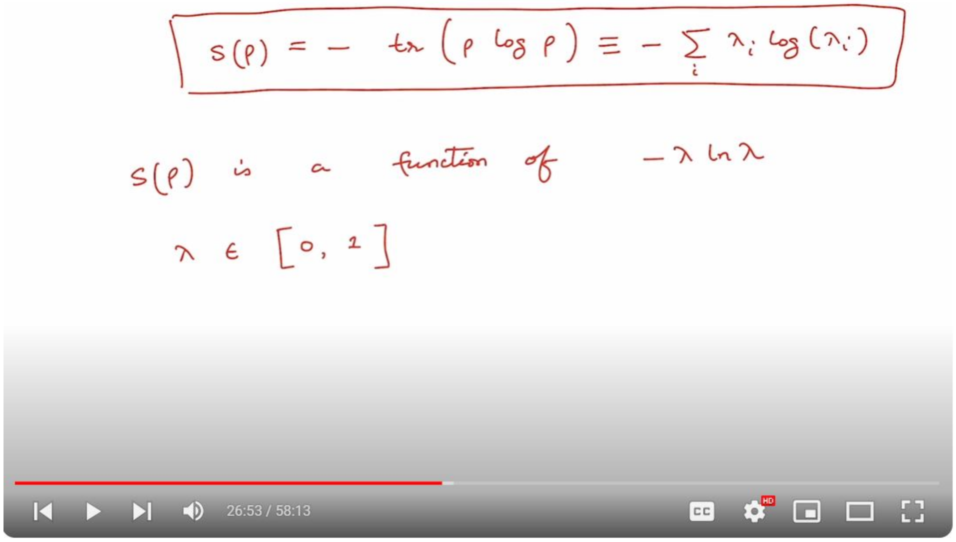

$$\boxed{s(\rho) = -\,\text{tr}\left(\rho \log \rho\right) \equiv -\sum_i \lambda_i \log(\lambda_i)}$$

$s(\rho)$ is a function of $-\lambda \ln \lambda$

$\lambda \in [0, 1]$

Now, I hope it is clear to you that von Neumann entropy formula we can now write. Let me write it once again. The von Neumann entropy formula trace over rho log rho can be written as minus sum over I lambda I log of lambda I, where lambda I's are the eigenvalues of the density matrix rho. As you can see, the von Neumann entropy is a function of, from the expression you can see, it's a function of minus lambda logarithm of lambda. Now, here lambda lies between 0 and 1, because lambda are the eigenvalues of the density matrix, and you know the diagonal elements of the density matrix gives us the probabilities, and probability cannot exceed 1. So lambda has to lie between 0 and 1.

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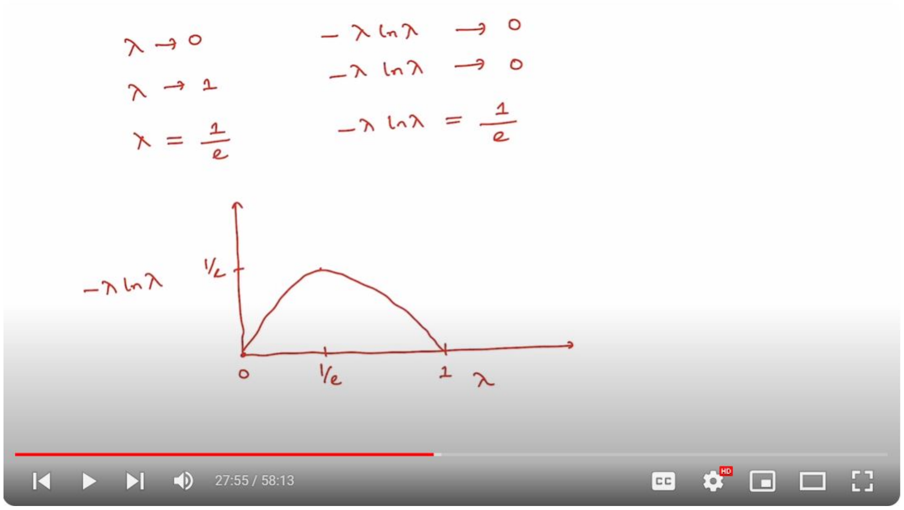

$\lambda \to 0$ $\qquad -\lambda \ln \lambda \to 0$

$\lambda \to 1$ $\qquad -\lambda \ln \lambda \to 0$

$\lambda = \frac{1}{e}$ $\qquad -\lambda \ln \lambda = \frac{1}{e}$

Now, if you look at the function, as you see, as lambda tends to 0, the function minus lambda logarithm of lambda tends to 0 and if lambda tends to 1, this function minus lambda logarithm of lambda also tends to 0. And it has a maximum value, this function has a maximum value at lambda is equal to 1 by e, and the value you can calculate, and it is also 1 by e. In fact, we can draw a plot of this function minus lambda logarithm of lambda versus the eigenvalue lambda, and eigenvalue lies between 0 and 1, and we can, we'll get a value, it says a maximum at 1 by e, and maximum value is 1 by e. So let us say this is the point, so therefore we are going to get a plot like this.

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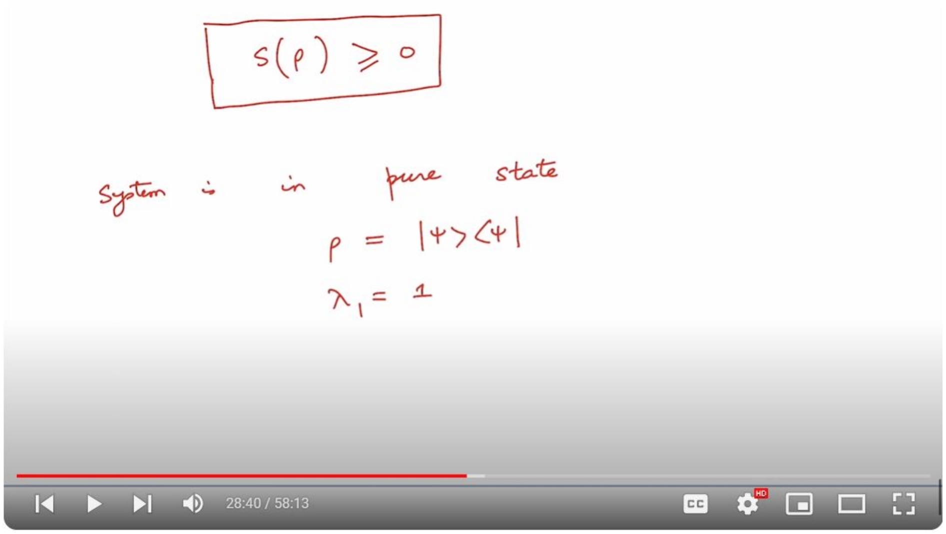

$$\boxed{S(\rho) \geqslant 0}$$

System is in pure state

$\rho = |\psi\rangle\langle\psi|$

$\lambda_1 = 1$

So this clearly shows one fact, that the entropy has a lower bound, that is, it cannot be less than 0. Right? This is what first lesson we have learned about the property of the von Neumann entropy. Now, another thing is that if the system is in pure state, say, a system is in pure state, and if it is in a pure state, that means we can have a, we can definitely describe the system by a ket psi, we have, this is the pure state, and only, we have only this particular state, and in that case, we have just one non-zero eigenvalue, and that is equal to 1.

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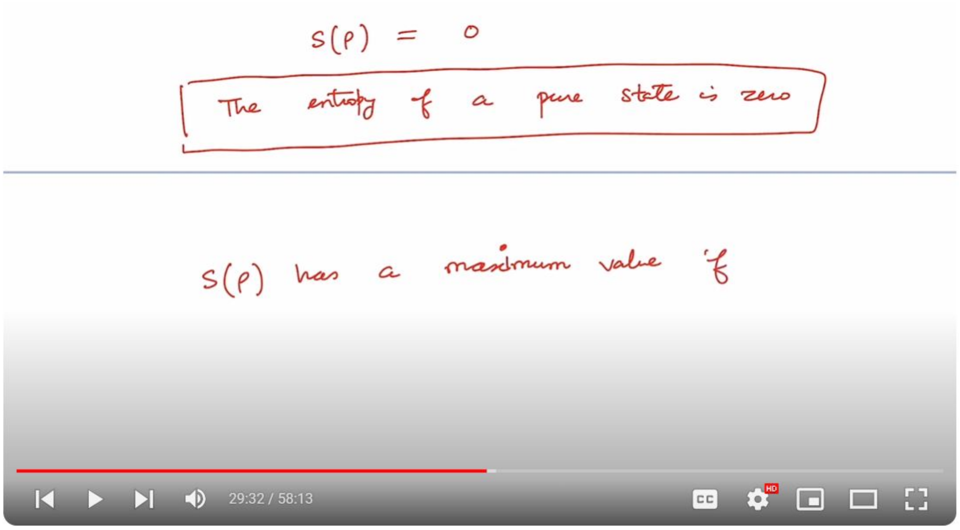

$S(\rho) = 0$

$\boxed{\text{The entropy of a pure state is zero}}$

$S(\rho)$ has a maximum value if

So therefore, immediately you see the corresponding entropy, quantum entropy would be, von Neumann entropy is going to be 0. So we can conclude that the entropy, entropy of a pure state, of a pure state is 0. So this is important, that entropy of a pure state is 0. So we have seen that the von Neumann entropy is bounded from below by 0.

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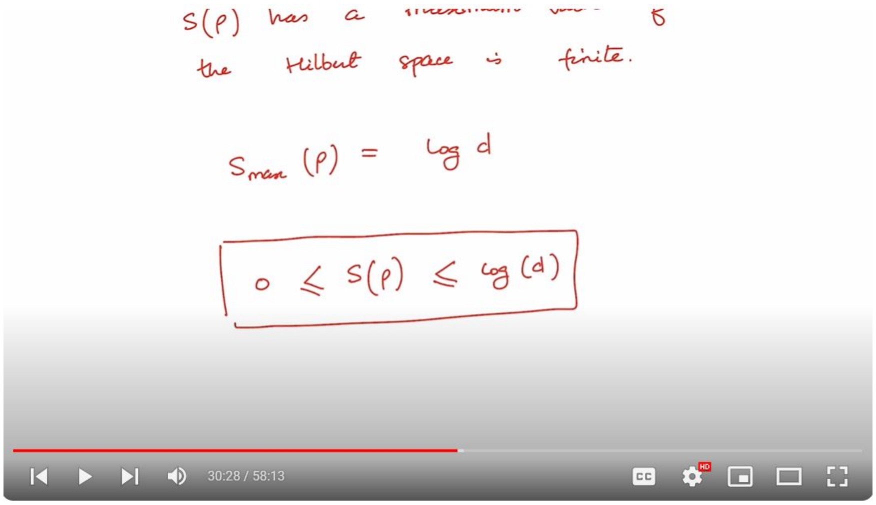

$S(\rho)$ has a maximum value if the Hilbert space is finite.

$S_{max}(\rho) = \log d$

$$\boxed{0 \leq S(\rho) \leq \log(d)}$$

It can be actually proved that von Neumann entropy has a maximum value, S rho has a maximum value, that means it has an upper bound, it has an upper bound or a maximum value, if the Hilbert space is finite. If the Hilbert space, Hilbert space is finite. Now I am just going to mention you the result here, if the Hilbert space is, has a dimension say d, then the value of the entropy, the maximum value of the entropy, S max rho is equal to logarithm of d. So therefore, what we can say about von Neumann entropy for a finite dimensional Hilbert space is, that the von Neumann entropy lies between 0 and logarithm of d, where d is the dimension of the Hilbert space.

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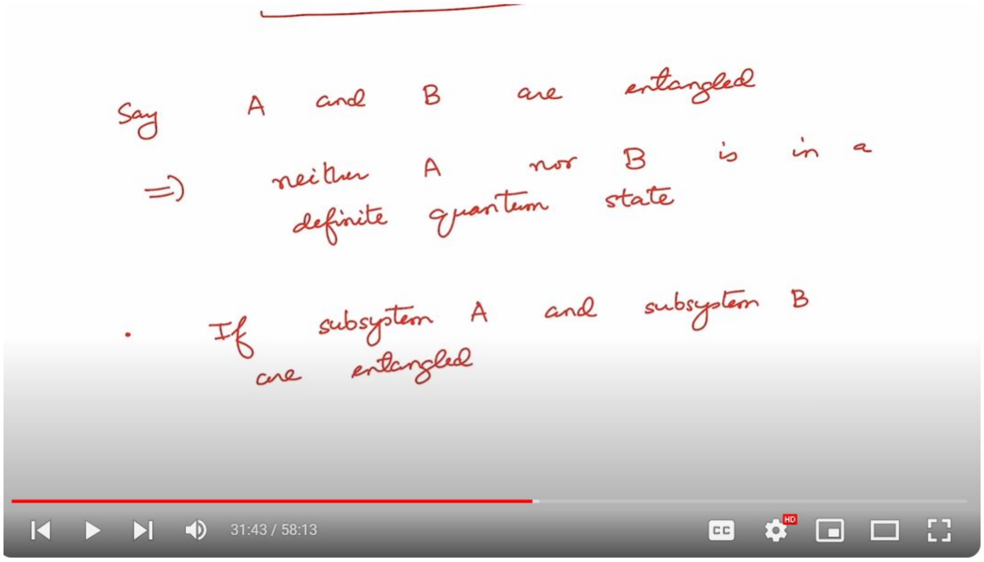

Say A and B are entangled

$\Rightarrow$ neither A nor B is in a definite quantum state

- If subsystem A and subsystem B are entangled

Von Neumann entropy can be used to quantify quantum entanglement, and reason is very simple. When two systems say A and B are entangled, neither A, let me just explain it by writing here, say A and B, the subsystem A and B are entangled. So this means, this implies that neither A nor B is in a definite quantum state, in a definite quantum state. Okay. Because we have a loss of information about the systems, we cannot know everything about A, and we cannot know everything about B without involving the other subsystems. So in this case, what we can do, say as regards the measurement is concerned, if subsystem A and B are entangled, there is going to be loss of information or uncertainty in the information.

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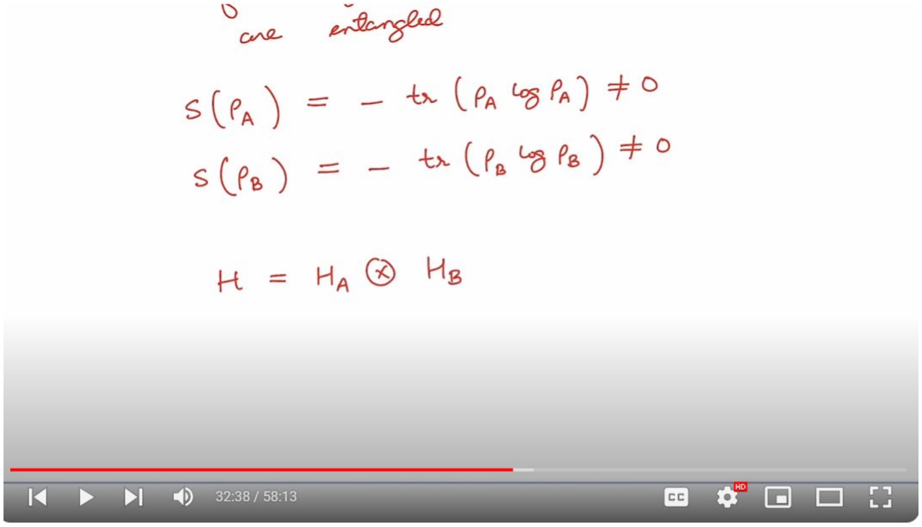

are entangled

$$S(\rho_A) = -\operatorname{tr}(\rho_A \log \rho_A) \neq 0$$

$$S(\rho_B) = -\operatorname{tr}(\rho_B \log \rho_B) \neq 0$$

$$H = H_A \otimes H_B$$

So therefore, if you calculate the density matrix or the reduced density matrix for the subsystem A and calculate its entropy, then you are definitely going to get a non-zero value of the entropy. So I think this is easy to understand because there is a loss of information or uncertainty in the information is there. Similarly, for the subsystem B, you can find out the density operator for the subsystem B and calculate the corresponding entropy, Von Neumann entropy, then you will find that it is going to give you a non-zero value. Now let me illustrate this concept by giving you an example. Let us consider again the Hilbert space of A and B .

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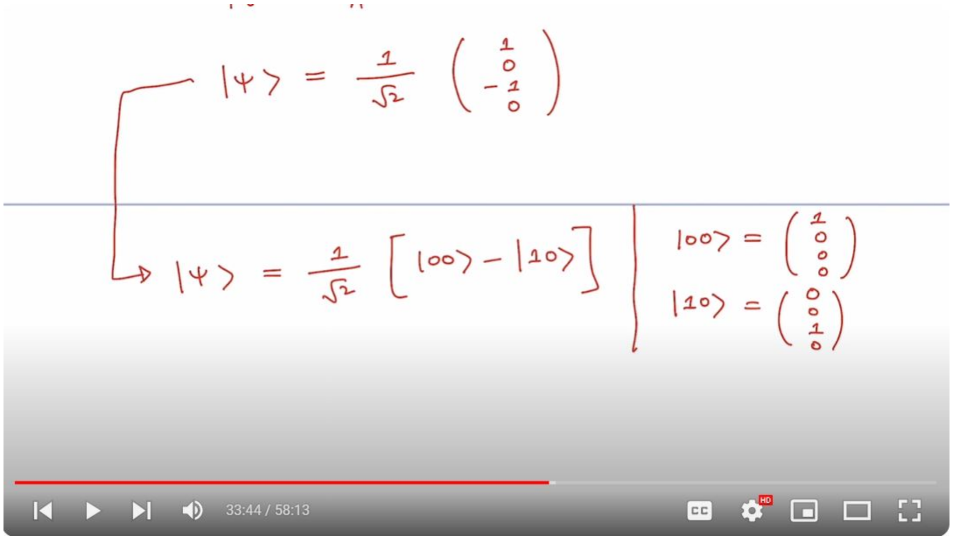

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|00\rangle - |10\rangle \right]$$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Say that a particular state is given in this Hilbert space by this ket vector or state vector 1 by root 2, 1, 0, minus 1, 0. Suppose this is the state vector we are having representing the composite system A and B and now we are asked to decide whether the subsystems A and B are entangled. And to do that, we can express this ket state, ket psi as a superposition state like this in the computational basis. We can write it as 1 by root 2 ket 0, 0, minus ket 1, 0 because if you remember ket 0, 0 in the computational basis is given by this column vector 1, 0, 0, 0. And 1, 0 is given by this column vector 0, 0, 1, 0. Therefore, we can express this state vector in a column in this particular form and this will help us to write down the corresponding density operator very quickly.

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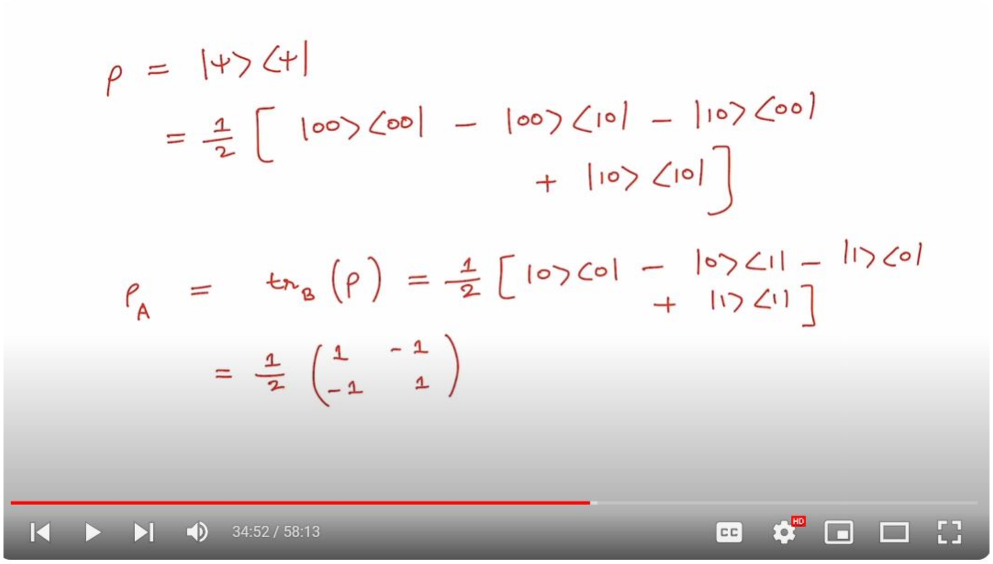

$$\rho = |\psi\rangle\langle\psi|$$

$$= \frac{1}{2}\Big[\, |00\rangle\langle 00| - |00\rangle\langle 10| - |10\rangle\langle 00| + |10\rangle\langle 10| \,\Big]$$

$$\rho_A = tr_B(\rho) = \frac{1}{2}\Big[\, |0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1| \,\Big]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Density operator for the composite system would be half ket 0, 0 bra 0, 0, minus ket 0, 0, bra 1, 0, minus ket 1, 0, bra 0, 0, plus ket 1, 0, bra 1, 0. Okay. And now just we have already done so many problems. You can find out the density operator for the subsystem A by tracing out B from here and this will give you half ket 0, bra 0, minus ket 0, bra 1, minus ket 1, bra 0, plus ket 1, bra 1. And in matrix form, I can write it as half 1, minus 1, minus 1, plus 1.

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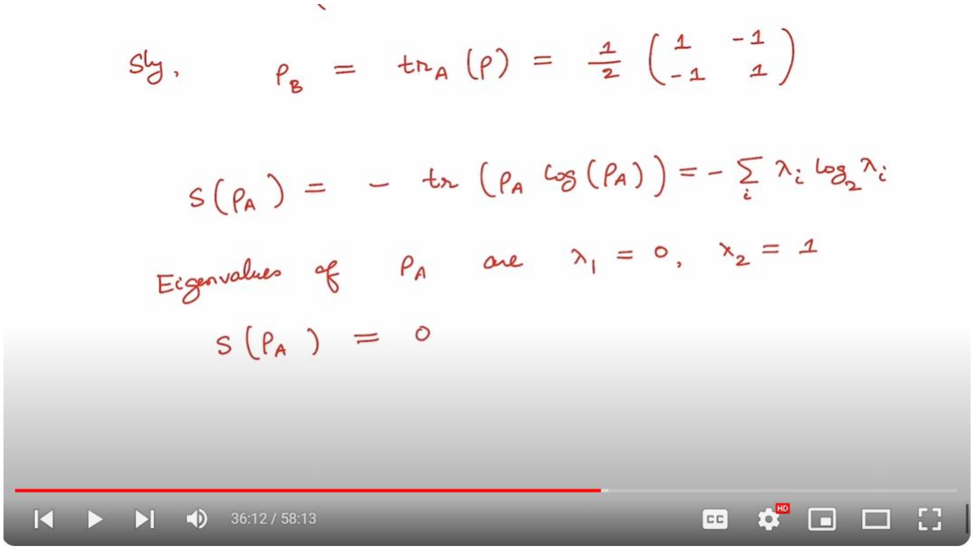

Sly, $\rho_B = tr_A(\rho) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$S(\rho_A) = -tr(\rho_A \log(\rho_A)) = -\sum_i \lambda_i \log_2 \lambda_i$

Eigenvalues of ρ_A are $\lambda_1 = 0$, $x_2 = 1$

$S(\rho_A) = 0$

Similarly, I can get the reduced density matrix for the system B would be equal to, we have to here trace out A. Then I will also get here half 1, minus 1, minus 1, plus 1. Now the entanglement entropy for the subsystem A would be, S rho of A would be equal to minus trace of rho A logarithm of rho A. And this is equal to minus sum over lambda I log lambda I with base 2. Lambda I are the eigenvalues and you can easily work out the eigenvalues of rho A.

Eigenvalues of rho A you can find as lambda 1 is equal to 0 and lambda 2 is equal to 1. And from here you can immediately get that entanglement entropy would be equal to simply 0.

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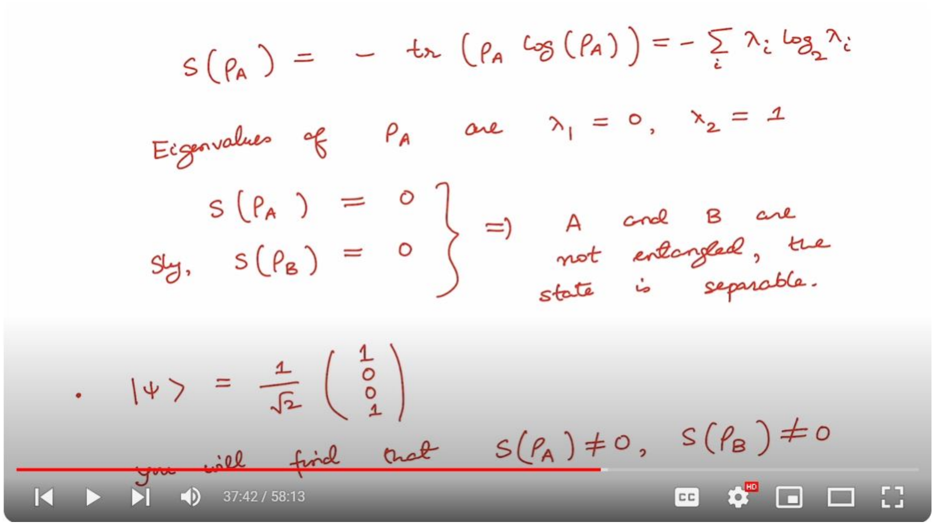

$S(\rho_A) = -tr(\rho_A \log(\rho_A)) = -\sum_i \lambda_i \log_2 \lambda_i$

Eigenvalues of ρ_A are $\lambda_1 = 0$, $x_2 = 1$

$$\left.\begin{array}{l} S(\rho_A) = 0 \\ \text{Sly, } S(\rho_B) = 0 \end{array}\right\} \Rightarrow$$ A and B are not entangled, the state is separable.

- $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

you will find that $S(\rho_A) \neq 0$, $S(\rho_B) \neq 0$

Similarly, you will get entanglement entropy for subsystem B is also going to be 0. So this basically implies that A and B are not entangled and that means their system is the state, the state representing the system, subsystem A and B are separable. Okay, so this is the lesson we get. You can try this exercise for another state say ket psi is equal to 1 by, say 1 by root 2, 1 0 0 1. And I urge you to do it and in this case you will find, you will find that the subsystem A and B are not entangled. Because you will find that the entropy of entanglement with respect to system A would turn out to be non-zero and S rho B, if you calculate you will find it to be also non-zero. So the system A and B will be entangled.

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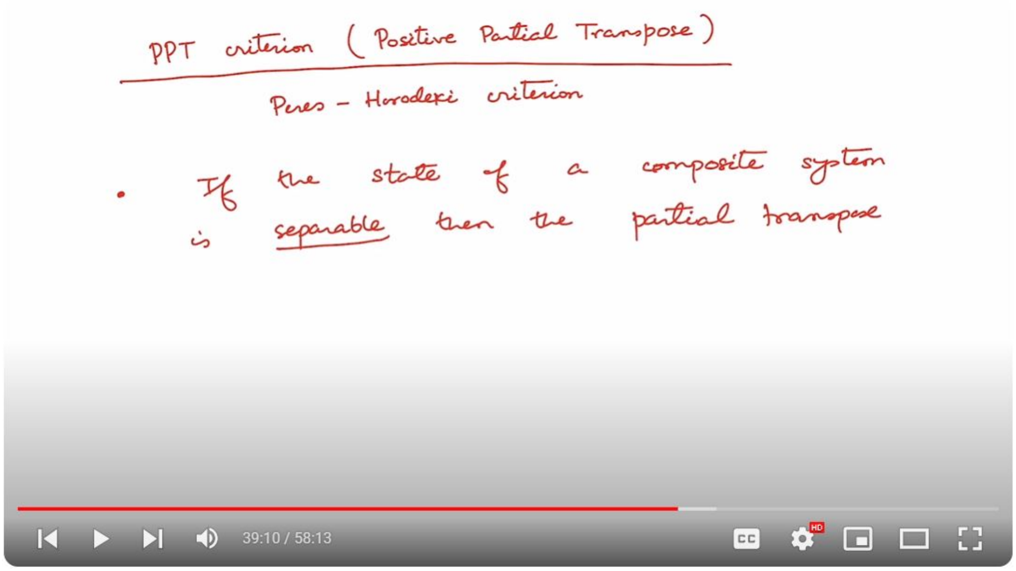

PPT criterion (Positive Partial Transpose)

Peres – Horodeki criterion

- If the state of a composite system is separable then the partial transpose

Let us now discuss about another entanglement criterion and measure. This criterion is called PPT criterion. And it's the short form of positive partial transpose criterion. It is also known as Peres-Horodecki criterion. However, I am going to refer to it as PPT criterion only.

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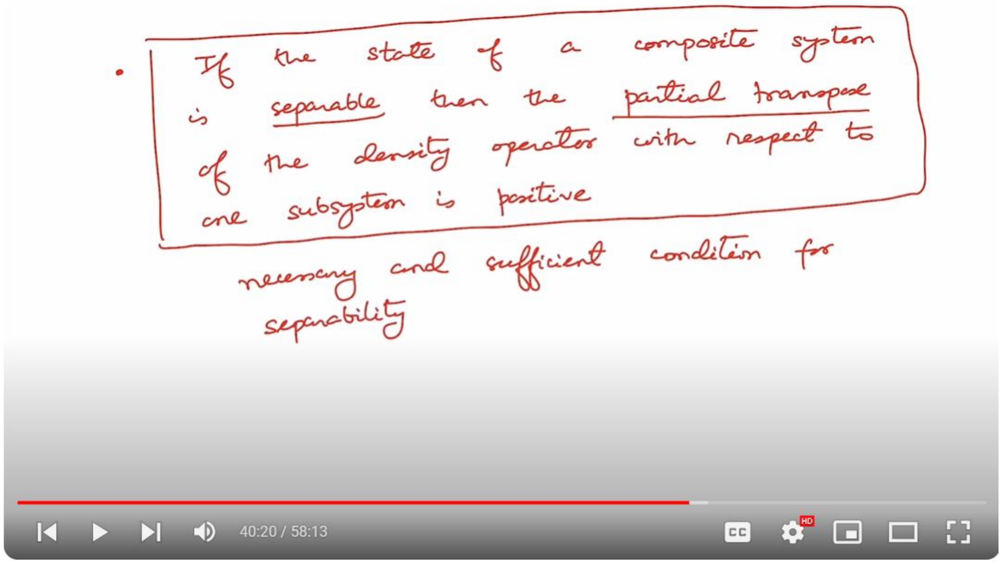

- If the state of a composite system is separable then the partial transpose of the density operator with respect to one subsystem is positive

necessary and sufficient condition for separability

This criterion states that, this criterion states that if the state of a composite system, if the state of a composite system is separable, is separable, then the partial transpose of the density operator, of the density operator with respect to, with respect to one subsystem is positive. Or basically it's a valid density matrix if I take the partial transpose. So first of all let me explain what I mean by partial transpose to give you a very trivial example.

By the way before I go there, this is a very important criterion. And this is actually a necessary and sufficient condition for separability. It is a necessary and sufficient condition, condition for separability in the context of a bipartite system in particular.

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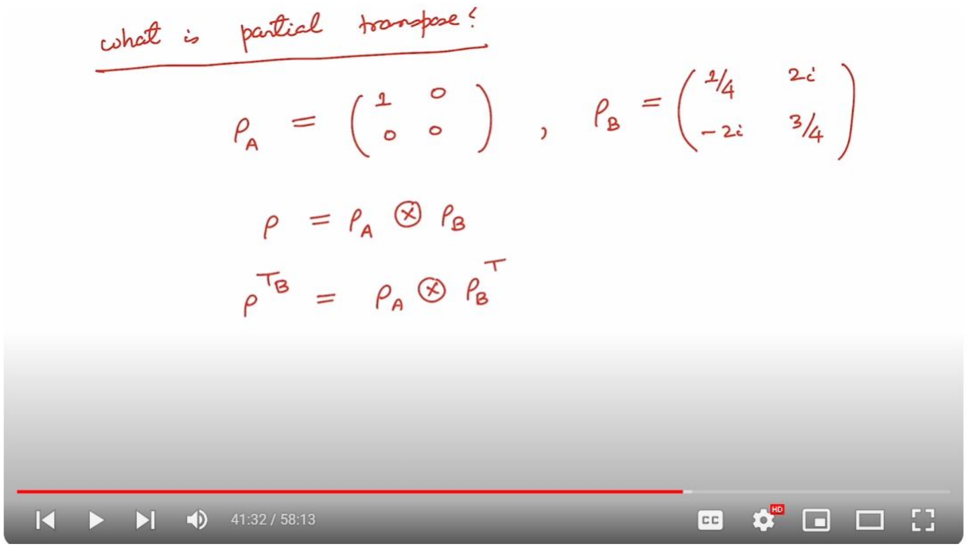

what is partial transpose?

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_B = \begin{pmatrix} 1/4 & 2i \\ -2i & 3/4 \end{pmatrix}$$

$$\rho = \rho_A \otimes \rho_B$$

$$\rho^{T_B} = \rho_A \otimes \rho_B^{T}$$

Now to explain what is partial transpose, let me give you a very trivial example. Let us say, what is partial transpose? So let us say we have a system A and its density operator is represented by rho A is equal to 1 0 0 0. And the another subsystem B is represented by the density operator rho B is equal to say 1 by 4 3 by 4 2 I minus 2 I and let us say this system A and B are separable, then I can write it in this form rho is equal to rho A tensor product rho B.

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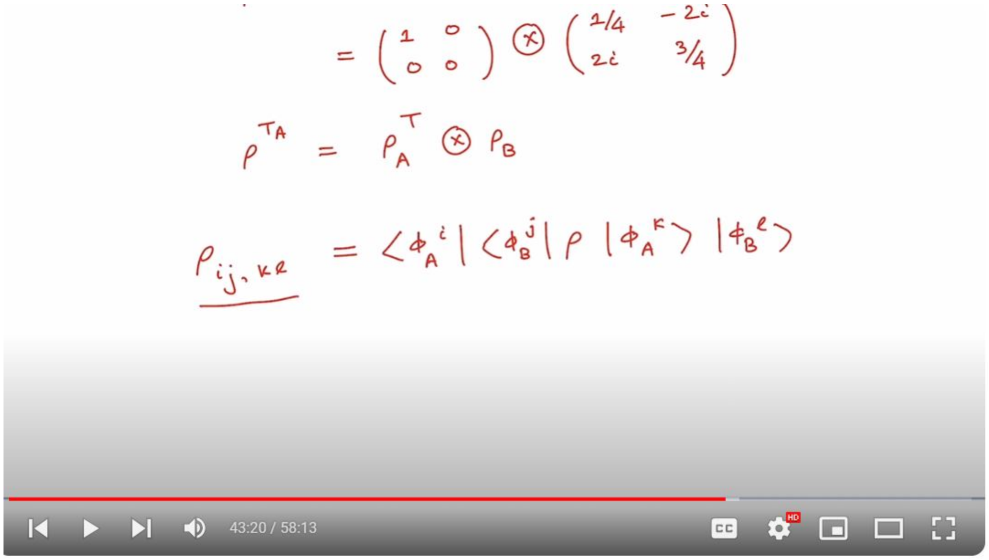

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1/4 & -2i \\ 2i & 3/4 \end{pmatrix}$$

$$\rho^{T_A} = \rho_A^{T} \otimes \rho_B$$

$$\underline{\rho_{ij,kl}} = \langle \phi_A^i | \langle \phi_B^j | \rho | \phi_A^k \rangle | \phi_B^l \rangle$$

Now the partial transpose over that of system B would be given by taking the partial transpose over the density operator representing the system B only. Right? Without touching the system A. And rho A is equal to, it remains same 1 0 0 0. But if I take the partial transpose of the matrix density matrix rho B, I just have two rows would become column 1 by 4 2 I, it would become column 1 by 4 2 I here.

And this column will become a, this row will become a column so you will have this minus 2 I 3 by 4. Okay, this is what I, what we mean by transpose, partial transpose over that of system B. We can do the similar thing over system A also rho TA would be basically taking the partial transpose over the system A, system A. And density matrix for the sub system B would remain unaffected. More technically, say the density matrix element of a composite system A plus B is given in the orthonormal basis by this, say rho I J. These are the density matrix element for the density operator or the density matrix rho. Then we can write this as in the orthonormal basis of Hilbert space of A and B would be like this. Phi A is the Hilbert space, then orthonormal basis belonging to Hilbert space of A. And phi B is the orthonormal basis belonging to the Hilbert space of B and here I have phi AK and phi BL.

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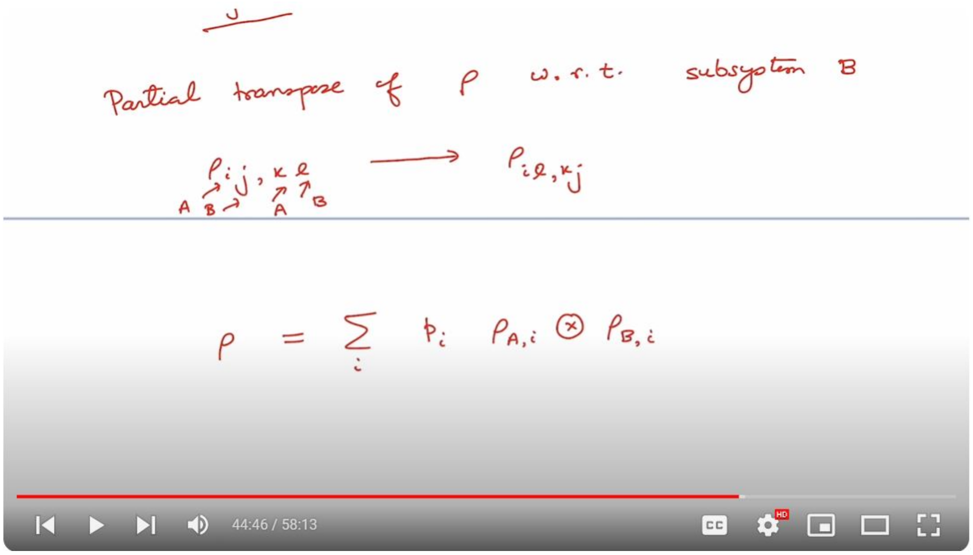

Partial transpose of ρ w.r.t. subsystem B

$$\rho_{\underset{A}{i}\underset{B}{j},\underset{A}{k}\underset{B}{l}} \longrightarrow \rho_{il,kj}$$

$$\rho = \sum_i p_i \, \rho_{A,i} \otimes \rho_{B,i}$$

Now taking partial transpose of rho, partial transpose of rho with respect to say subsystem B means that the indices that belong to the system B is getting interchanged without changing the indices belonging to the system A. Now here you see I belongs to A and K belongs to the subsystem A and J belongs to the subsystem B, L belongs to the subsystem B. And while taking transpose, we are not going to interchange the indices belonging to subsystem A, but indices belonging to subsystem B is going to change. So this is what we are going to get. So this is what we mean by taking partial transpose with respect to system B here. Now suppose the density matrix rho takes this separable form, rho is equal to sum over i, Pi, rho Ai, tensor product with rho Bi. This is a separable state.

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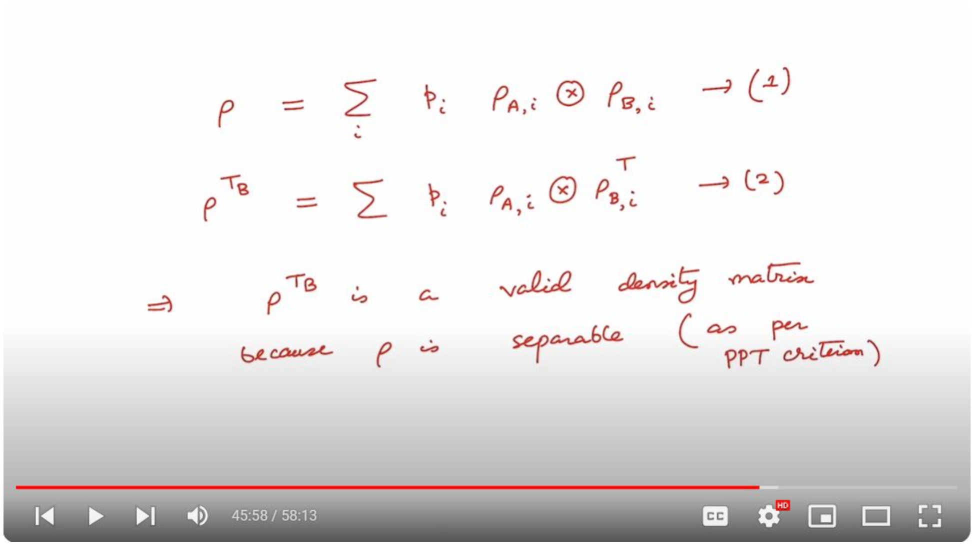

$$\rho = \sum_i p_i \, \rho_{A,i} \otimes \rho_{B,i} \rightarrow (1)$$

$$\rho^{T_B} = \sum p_i \, \rho_{A,i} \otimes \rho_{B,i}^{T} \rightarrow (2)$$

$\Rightarrow$ ρ^{T_B} is a valid density matrix because ρ is separable (as per PPT criterion)

And here when we say partial trace is taken over system B, this means that the density operator corresponding to the system A remains unaffected and uses take the partial transpose over the density matrix belonging to the system B. Right, this is what we mean.

So let's say this is my equation 1 and this is my equation number 2. Now you have to note one thing that if the system is or the density operator rho is separable, this implies as per the PPT criterion that rho TB is a valid density matrix. This is valid density matrix because rho is separable. As per PPT criterion, this we must have. PPT criterion.

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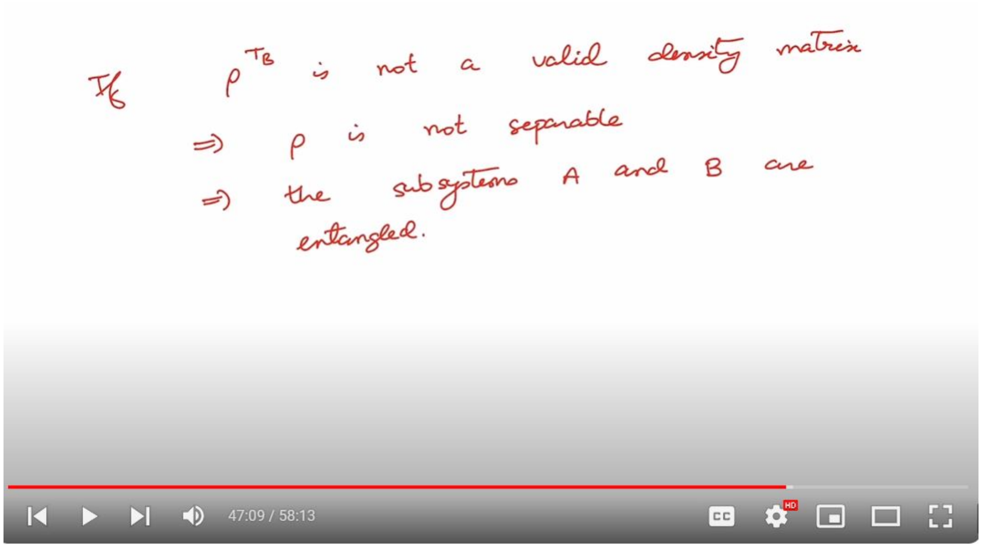

If ρ^{T_B} is not a valid density matrix

$\Rightarrow$ ρ is not separable

$\Rightarrow$ the subsystems A and B are entangled.

And what I mean by valid density matrix? That means that rho TB is semi-positive definite. It is semi-positive definite or simply a positive matrix or that means that the eigenvalues, all the eigenvalues of rho TB is non-negative. Non-negative, it cannot be negative. If rho TB is not a valid density matrix, it implies that rho is not separable. Or it implies that the subsystems A and B are entangled.

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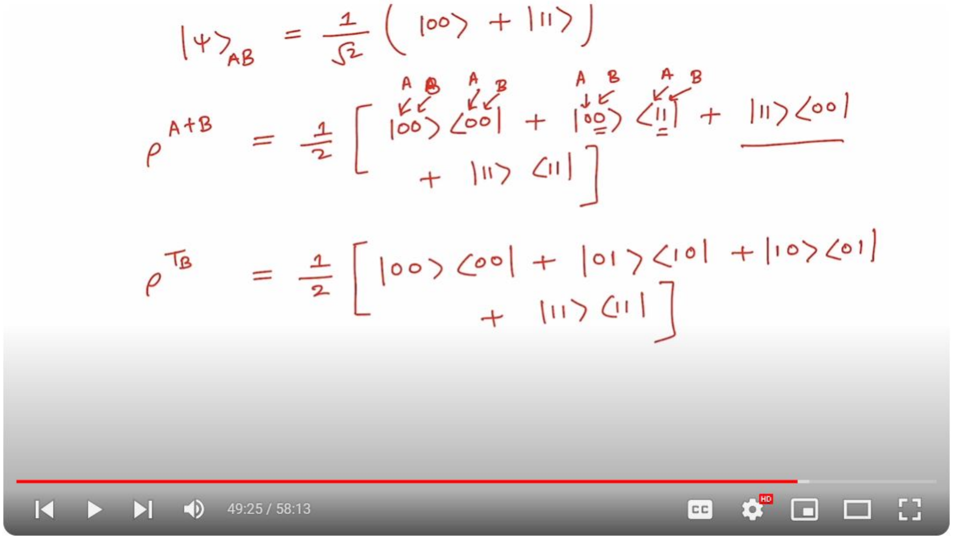

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\rho^{A+B} = \frac{1}{2}\Big[|\overset{A}{0}\overset{B}{0}\rangle\langle\overset{A}{0}\overset{B}{0}| + |\overset{A}{\underline{0}}\overset{B}{\underline{0}}\rangle\langle\overset{A}{\underline{1}}\overset{B}{\underline{1}}| + \underline{|11\rangle\langle 00|} + |11\rangle\langle 11|\Big]$$

$$\rho^{T_B} = \frac{1}{2}\Big[|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\Big]$$

Let me explain this by a simple example. Consider this composite system of A and B represented by this ket. Say we have 1 by root 2 ket 00 plus ket 11. We already know this is an entangled state but let us see what the PPT criterion tells us about it. The density operator corresponding to this composite system would be equal to half ket 00 bra 00 plus ket 00 bra 11 plus ket 11 bra 00 plus ket 11 bra 11. The important point you have to take note here is that this 0 belongs to A, this 0 belongs to A.

And similarly here this 0 belongs to A, here it belongs to, sorry this belongs to B and this belongs to B. Here this 0 belongs to, in the second term this belongs to A, this belongs to B and this belongs to A, this belongs to B and so on. So if you keep track of this, then taking the partial trace over that of B would simply interchanging the stuff belonging to A, B only without disturbing the stuff belonging to A. So if I take the partial transpose, if I take the partial transpose over that of B, we will have, this first term would be easy because it is same so it would remain 00 like this. But if you look at the second term, taking the partial transpose over B would interchange the stuff belonging to bits basically, qubits belonging to B.

So you will get 01 here and here you will get 10 and in the third term you will get 1001 and in the fourth term it will remain the same because similar 1111.

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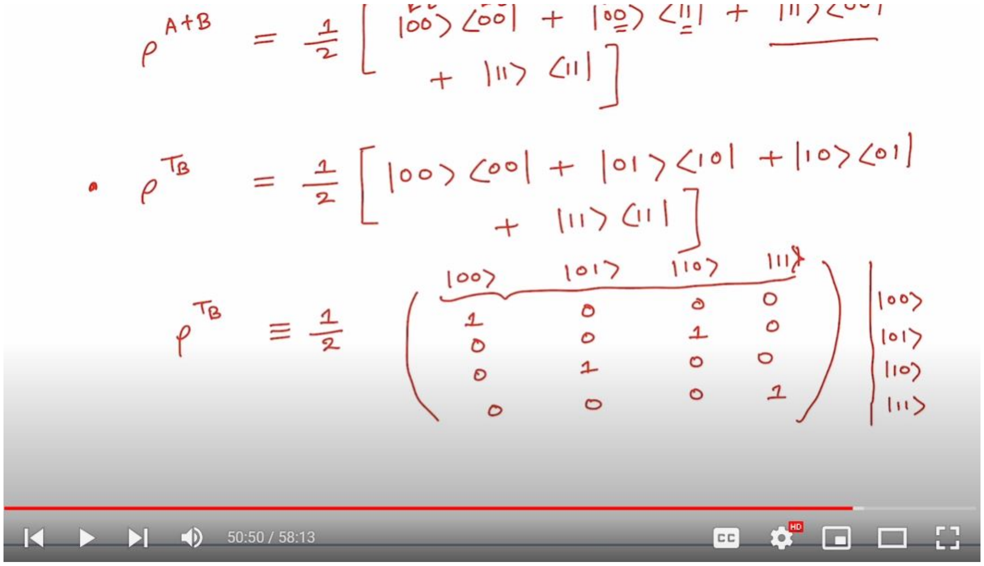

$$\rho^{A+B} = \frac{1}{2}\left[|00\rangle\langle 00| + |00\rangle\langle 11| + \underline{|11\rangle\langle 00|} + |11\rangle\langle 11| \right]$$

$$\rho^{T_B} = \frac{1}{2}\left[|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11| \right]$$

$$\rho^{T_B} \equiv \frac{1}{2} \begin{array}{c|c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

Like this you will have this and this you can express in a density matrix form in this way, you can write it, the trick to write it in this form is this, say the row, let me arrange the row this way, 00011011. Okay, this is my row side and then column also I will arrange accordingly say 00, I will write it, this is just a trick to remember 00011011. So now from this expression here, from here I can write the matrix, density matrix for the partially transposed system would become, that matrix would become here, the first term would become 1000, as you can see second row 0010 and the third row would become 0100, fourth row would be 0001. Now the question is, is this partially transposed matrix is a valid density matrix?

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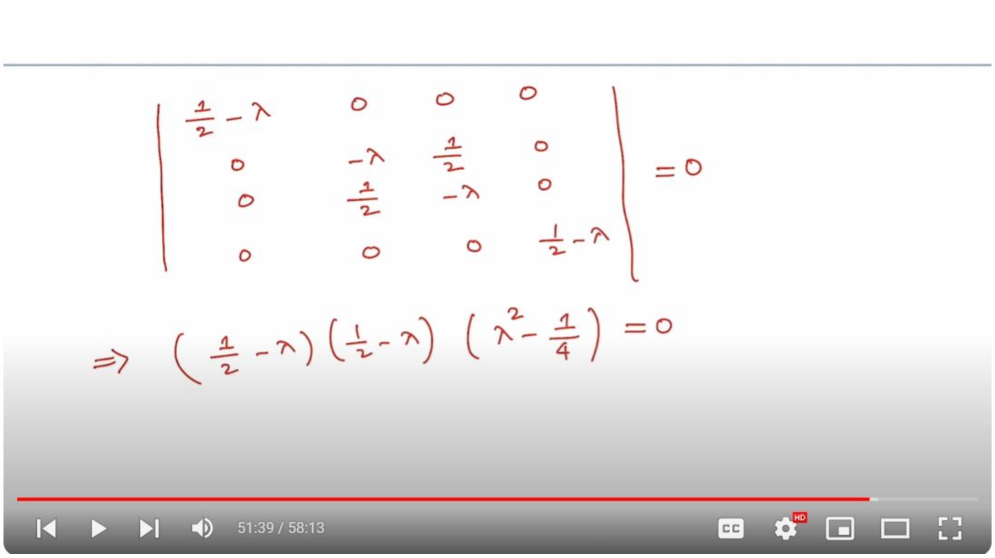

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{2}-\lambda\right)\left(\frac{1}{2}-\lambda\right)\left(\lambda^2-\frac{1}{4}\right)=0$$

To answer that you just have to find out the eigenvalues of rho Tb and to do that you have to form the characteristic equation, let us do that quickly. The characteristic equation would become determinant of half minus lambda 000, 0 minus lambda, half 00, half 200, half minus lambda, this is equal to 0 and thereby I will be able to form this equation that would become half minus lambda. Half minus lambda and lambda squared minus 1 by 4 is equal to 0.

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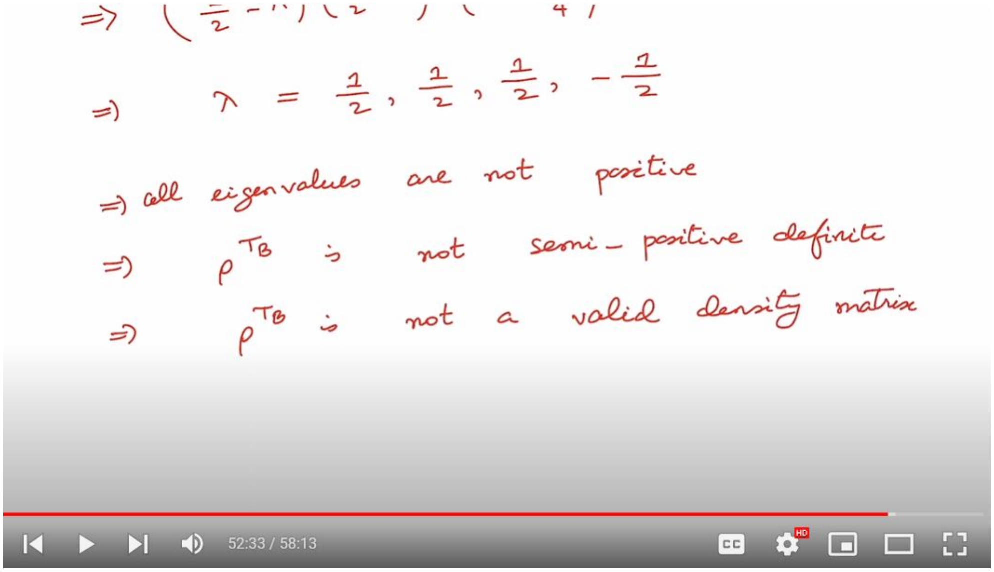

$$\Rightarrow \lambda = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$$

⇒ all eigenvalues are not positive

⇒ ρ^{TB} is not semi-positive definite

⇒ ρ^{TB} is not a valid density matrix

This will give me the eigenvalues as one half, one half, one half and minus half. So as you can see this clearly means that eigenvalues are not all the eigenvalues are positive, eigenvalues, all eigenvalues are not positive.

Right, this is what it means and it implies that rho Tb is not semi positive definite. What does it mean? This means that rho Tb is not a valid density matrix. So as per the PPT criteria this implies that system A and B are, the state is not separable and A and B are entangled. So I hope you get the idea here.

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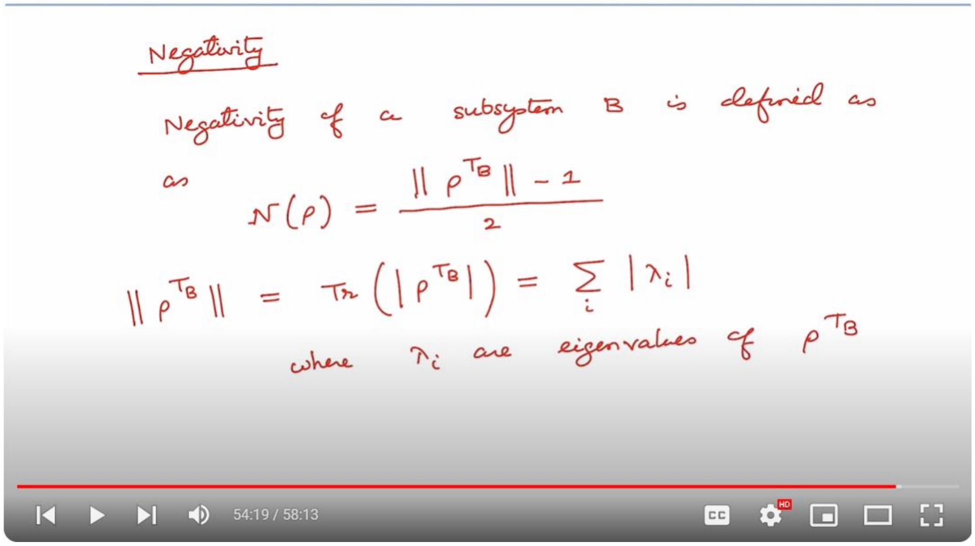

Negativity

Negativity of a subsystem B is defined as

as

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_B}\| - 1}{2}$$

$$\|\rho^{T_B}\| = \mathrm{Tr}\left(|\rho^{T_B}|\right) = \sum_i |\lambda_i|$$

where λ_i are eigenvalues of ρ^{T_B}

This PPT criterion leads us to a simple entanglement measure called negativity. It is represented by the symbol italicized N, capital N, it's a function of the density operator rho is equal to trace norm over the partially transpose matrix of rho. After taking the partial transpose over the subsystem B minus 1 divided by 2 where this trace norm of rho Tb is equal to trace over the modulus of rho Tb. Which in fact means that it is taken the sum over the modulus of the eigenvalues of rho Tb where lambda i are eigenvalues of rho Tb.

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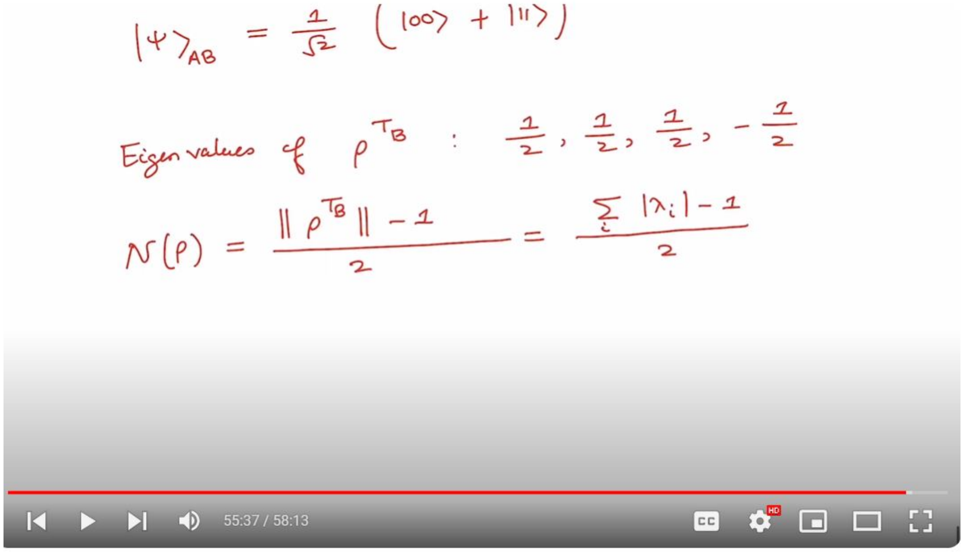

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Eigen values of ρ^{T_B} : $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_B}\| - 1}{2} = \frac{\sum_i |\lambda_i| - 1}{2}$$

Now we can understand this negativity concept very simply by this example that I have discussed just a few minutes back where we considered a composite system represented by the state vector 1 by root 2 ket 00 plus k11. And we have calculated the partial transpose over the subsystem B and we found the eigenvalues of rho Tb and the eigenvalues of rho Tb where we calculated that eigenvalues of rho Tb were calculated to be 1 half, 1 half, 1 half and minus 1. Therefore the negativity for the subsystem B can be calculated very quickly and this is trace norm over rho Tb minus 1 by 2 which is sum over the modulus of the eigenvalues divided by 2.

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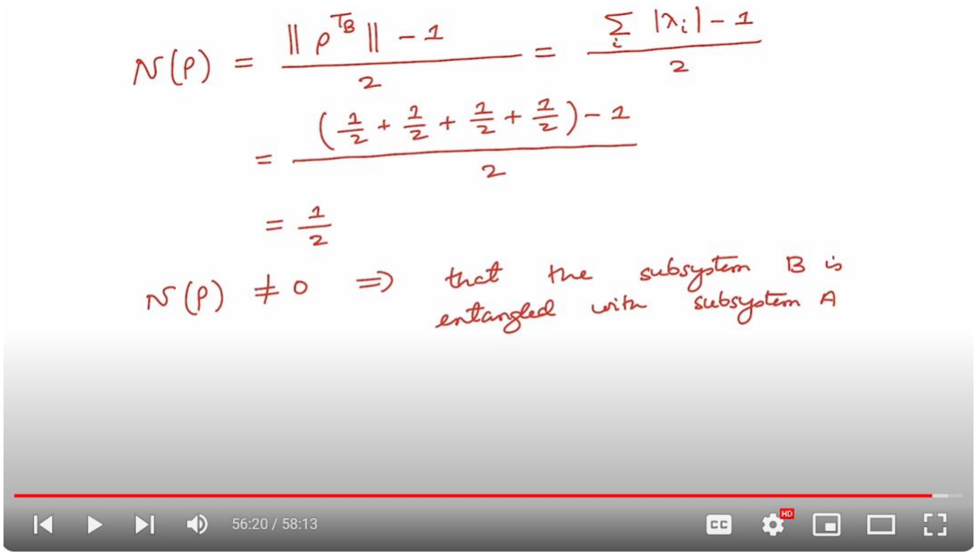

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_B}\| - 1}{2} = \frac{\sum_i |\lambda_i| - 1}{2}$$

$$= \frac{\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - 1}{2}$$

$$= \frac{1}{2}$$

$\mathcal{N}(\rho) \neq 0 \Rightarrow$ that the subsystem B is entangled with subsystem A

Because the eigenvalues we know we have to take the modulus so it would be 1 half plus 1 half plus 1 half plus 1 half minus 1 divided by 2 and this is going to give us 1 half. So as you see that the negativity is non-zero, n rho is not equal to 0 here. This implies that the subsystem B is entangled with subsystem A.

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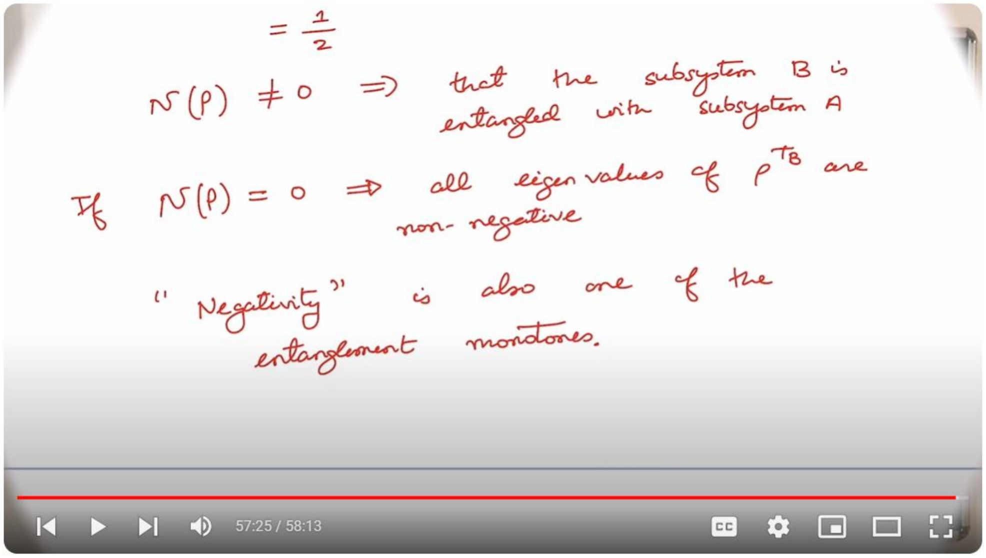

$= \frac{1}{2}$

$\mathcal{N}(\rho) \neq 0 \Rightarrow$ that the subsystem B is entangled with subsystem A

If $\mathcal{N}(\rho) = 0 \Rightarrow$ all eigen values of ρ^{T_B} are non-negative

"Negativity" is also one of the entanglement monotones.

On the other hand if n rho negativity is 0 this implies that and in fact it ensures that all eigenvalues or all eigenvalues of rho Tb are non-negative.

Thereby it means that the system A and B are not entangled. Sometimes this negativity is also called entanglement monotone. Let me stop for today. In this lecture we have learnt about quantum entanglement measures and criterion, volume and entropy followed by the PPT criterion and negativity.

We will continue our discussion on entanglement measures in the next class. Thank you so much. See you in the next class. .