Quantum Entanglement: Fundamentals, measures and application

Prof. Amarendra Kumar Sarma

Department of Physics

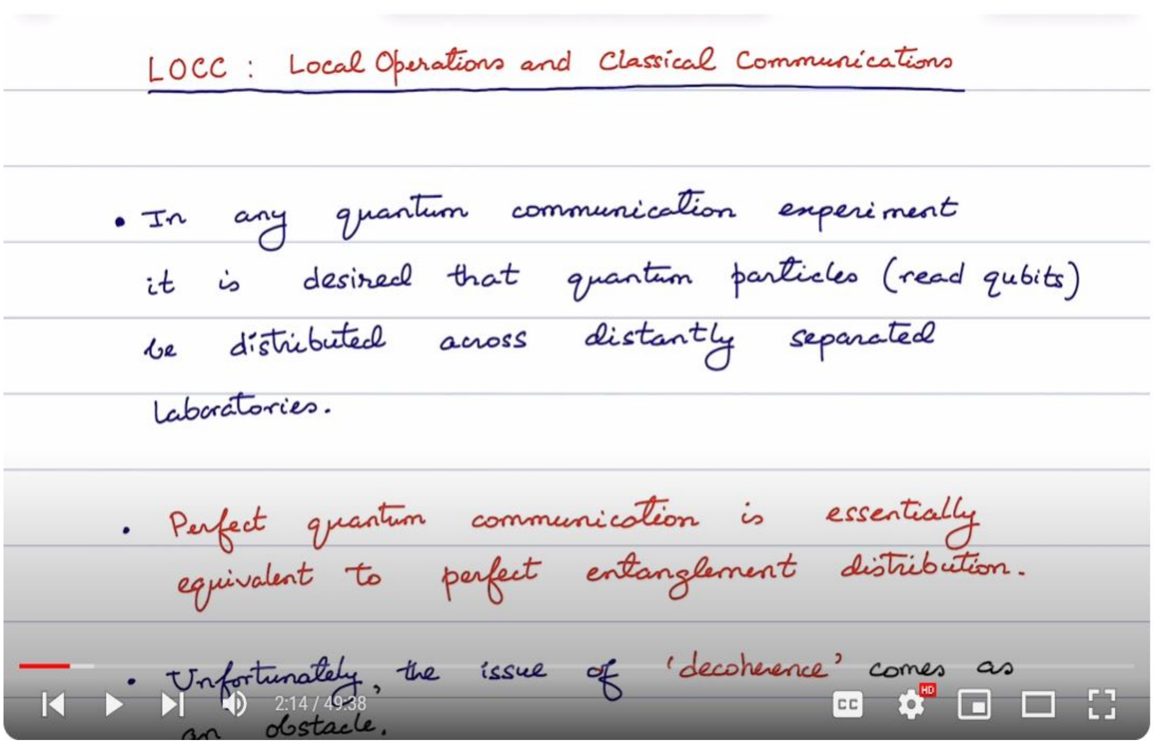
Indian Institute of Technology-Guwahati

Week-03

Lec 11: Properties of Quantum Entanglement

Hello, welcome to lecture 2 of module 3. This is lecture number 8 of the course. In this lecture, we are going to discuss about some important property of quantum entanglement. In the process, we will also discuss quantum gates, which is a very important topic in quantum information science. At the outset, I will discuss the very key concept of quantum classical communication and local operation. And this is an important topic and you will find it very useful when we discuss applications of quantum entanglement in module 4.

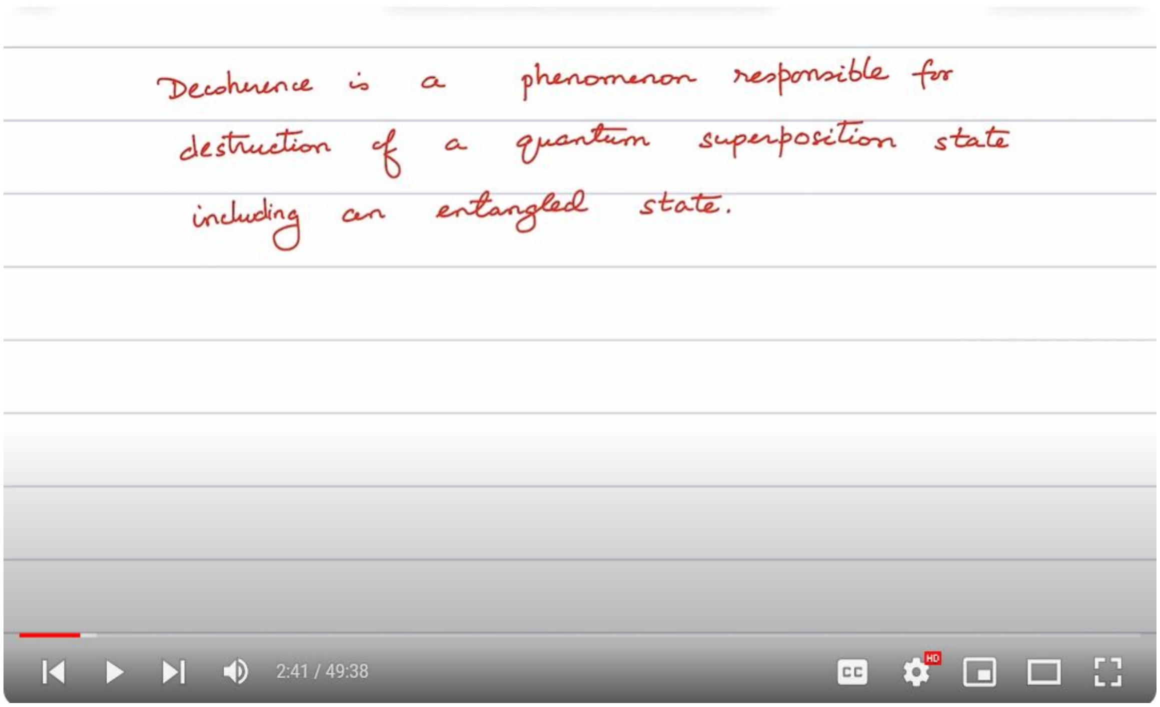
(Refer Slide Time: 02:14)



Let us first discuss LOCC or Local Operations and Classical Communication. In any quantum communication experiment, it is desired that we should be able to distribute quantum particles or qubits across distantly separated laboratories.

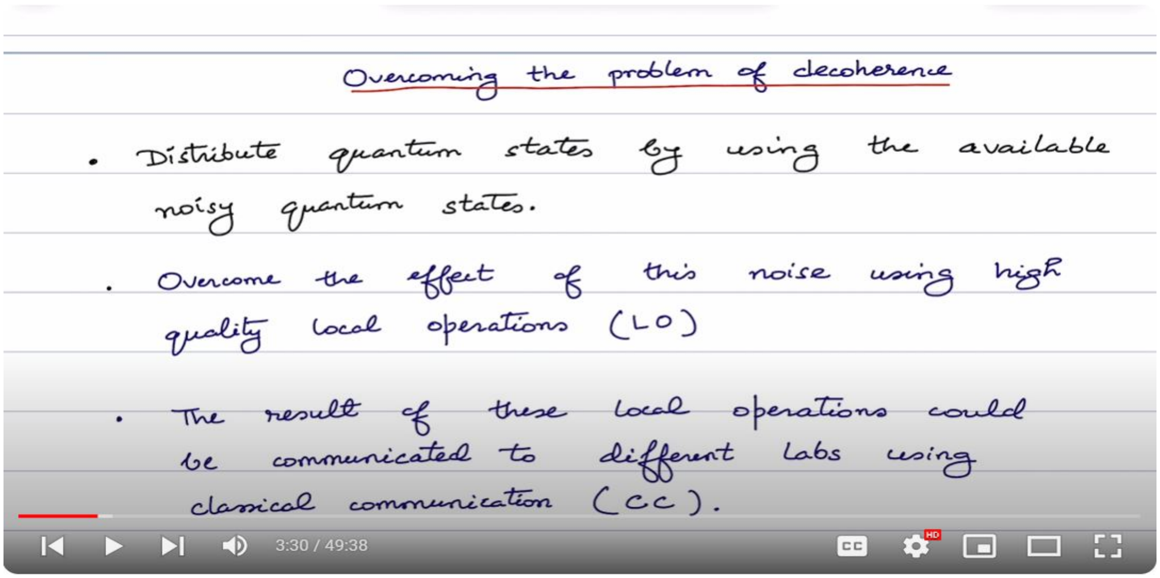
It is clear and obvious that once we prepare a set of entangled states, we would like to distribute and send them to various laboratories situated at many distant places. If we can transport a qubit without any decoherence, then any entanglement shared by that qubit will also be distributed perfectly. As you know, perfect quantum communication is essentially equivalent to perfect entanglement distribution. Unfortunately, the issue of decoherence comes as an obstacle.

(Refer Slide Time: 02:41)



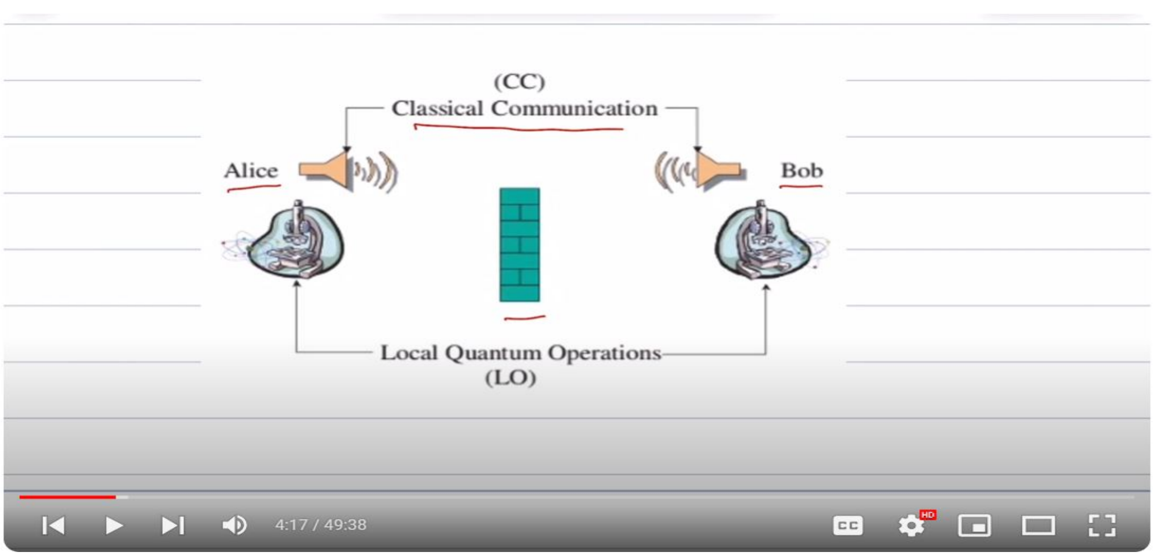
I am sure you know what decoherence is. Decoherence is a phenomenon responsible for destruction of a quantum superposition state, including an entangled state. Decoherence generally occurs when the quantum state interacts with noise or environment. In a sense, decoherence or the effect of noise, invariably forbids us to send quantum states over long distances.

(Refer Slide Time: 03:30)



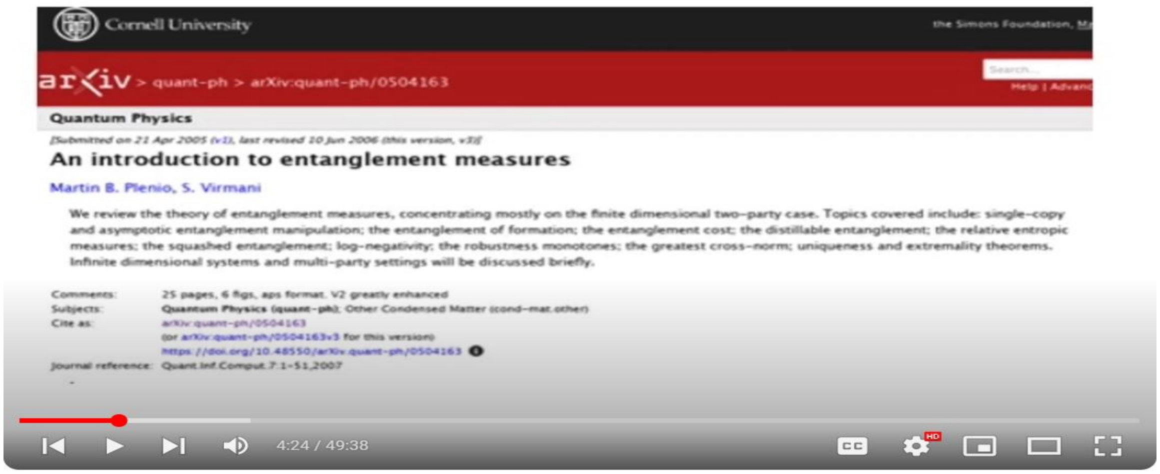
Now one way to overcome this problem is to distribute quantum states by using the available noisy quantum channels that is available to us. And then overcome the effect of this noise using high quality local quantum processes in the distinctly separated labs. This is termed as local operations. In this way, we can avoid decoherence induced by communication over long distances. The result of these local operations, local quantum operations could be communicated to different labs using the so-called classical communication. And this we can do by using the standard available telecom technologies.

(Refer Slide Time: 04:17)



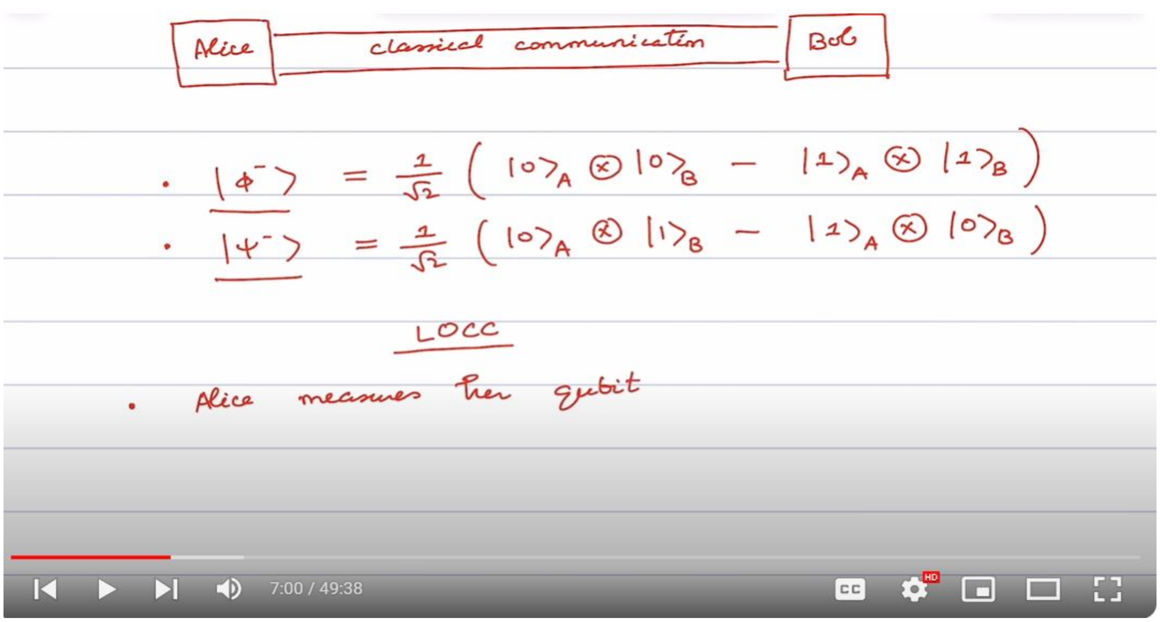
Now in this figure, here you can see a standard quantum communication settings between two labs run by Alice and Bob. Alice and Bob may perform any generalized measurements that is localized to their respective laboratories and the result of the measurement is communicated classically. The brick wall indicates here the fact that no quantum particle or qubits are getting exchanged coherently between Alice and Bob. This set of operations is generally referred to as LOCC, that is local operations, local quantum operations and classical communications.

(Refer Slide Time: 04:24)



A good amount of material of this lecture is based on this article including the image that I have just shown.

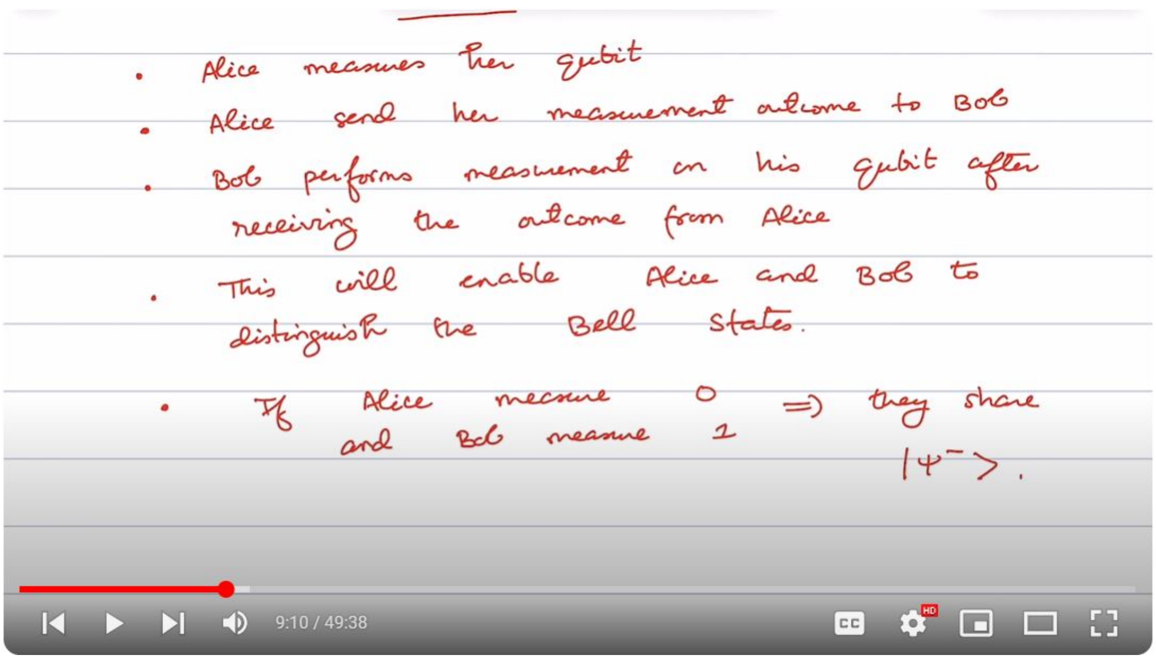
(Refer Slide Time: 07:00)



To give you a simple perspective on LOCC, let me provide you this example. Let us say Alice and Bob are located at two distant laboratories and they are sharing two Bell states, say Phi minus and Psi minus. And you know that Phi minus is this Bell state is given as 1 by root 2, 0, k0 belongs to Alice and the second k belongs to Bob, 0B, second qubit. And we have 1 and 1 here. And another Bell state is Psi minus which is given as 1 by root 2, first qubit belongs to Alice and the second qubit belongs to Bob.

It is a typical EPR kind of EPR, this particular state Psi minus, so 0B. So this is what we have, both Alice and Bob are sharing each other. Now Alice and Bob can choose one of the two shared states but the information about which state it is exactly is lacking. And Alice and Bob have been provided with a communication channel, a classical communication channel they have been provided with. So this is a classical communication channel and this may be an internet or a telephone.

(Refer Slide Time: 09:1phi)

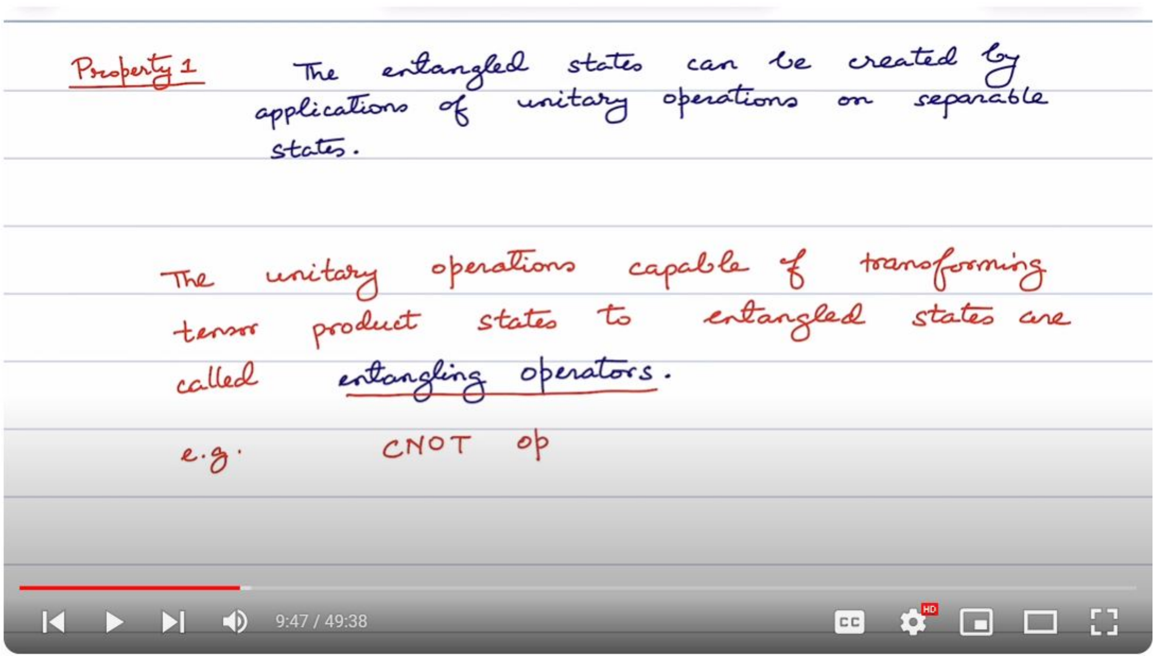


Okay, now by using LOCC actually Alice and Bob can distinguish whether they are sharing the state Phi minus or they are sharing the state Psi minus. To do so, Alice has to measure her qubits. In LOCC as you know, LO means local quantum operations and Alice does her own quantum operations, local quantum operation.

So here the quantum operation she does is, Alice measures her qubit and after getting the outcome, getting the result, she sends the information. Alice sends her measurement information or outcome, her measurement outcome to Bob. Measurement outcome to Bob by classical channel. And after Bob receives this, Bob then makes his own measurements. Bob performs measurement on his qubit, measurement on his qubit after receiving the outcome from Alice measurement.

Okay, I hope you get it. Now if that is so, then certainly they would know what state is shared by them. This will enable Alice and Bob because then Bob can also communicate this information that what the measurement outcome to Alice. This will enable Alice and Bob to distinguish the Bell states. I hope you get it. That means for example what's going to happen is this. Suppose if Alice measures zero and Bob measures one, this implies they share the Bell state Psi minus. It's as simple as that.

(Refer Slide Time: 09:47)



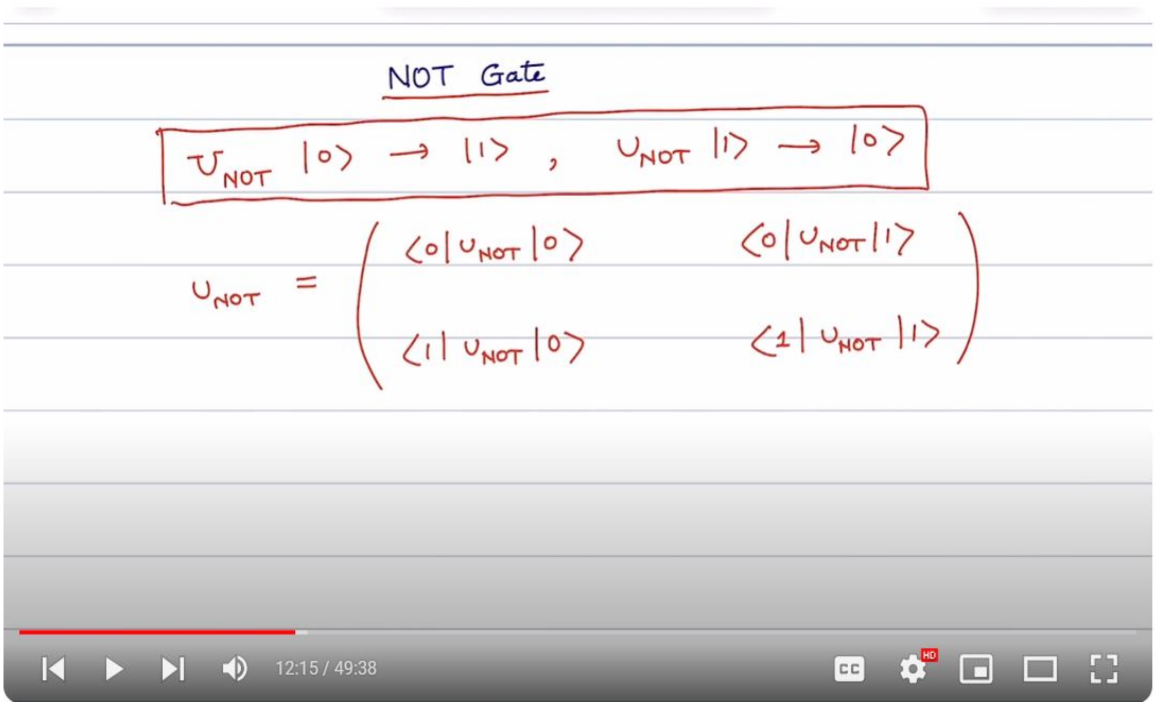
Now we already know what quantum entanglement is, so let us now discuss some of its properties. The first entanglement properties that I would like to discuss is related to entanglement creation. The entangled states can be created by applications of unitary operations on separable states.

The unitary operations capable of transforming tensor product states to entangled states are called entangling operators. There are many entangling operators, for example the so-called CNOT operator is an entangling operator that we are going to discuss soon.

Now let me digress a little bit. Let me first discuss and remind you about unitary transformation.

A unitary transformation is mathematically speaking carried out by unitary matrix capital U. It is unitary if capital U dagger U is equal to capital U U dagger is equal to I. I is the identity matrix. Let me in this context discuss about quantum gates. As you know in quantum information, informations are stored using qubits. A unitary transformation can be used or a set of unitary transformations can be used to change the state of the qubit or transform the state of the qubit. And these unitary transformations are termed as quantum gates.

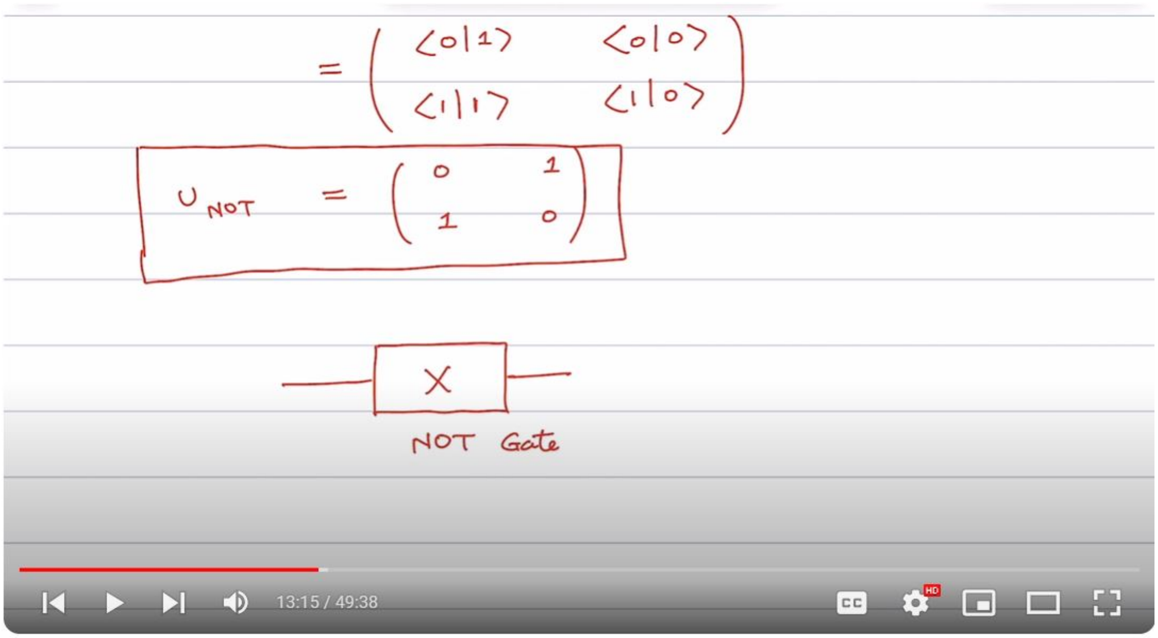
(Refer Slide Time: 12:1phi)



One very simple quantum gate is the so-called NOT gate. Quantum NOT gate is easy to understand because it is analogous to the so-called classical NOT gate. And as I said quantum gates are represented by unitary operators. And here a NOT gate is going to be represented by this unitary operator UNOT. And when it operates on ket 0, it converts it to ket 1. And when it operates on ket 1, it converts it to ket 0.

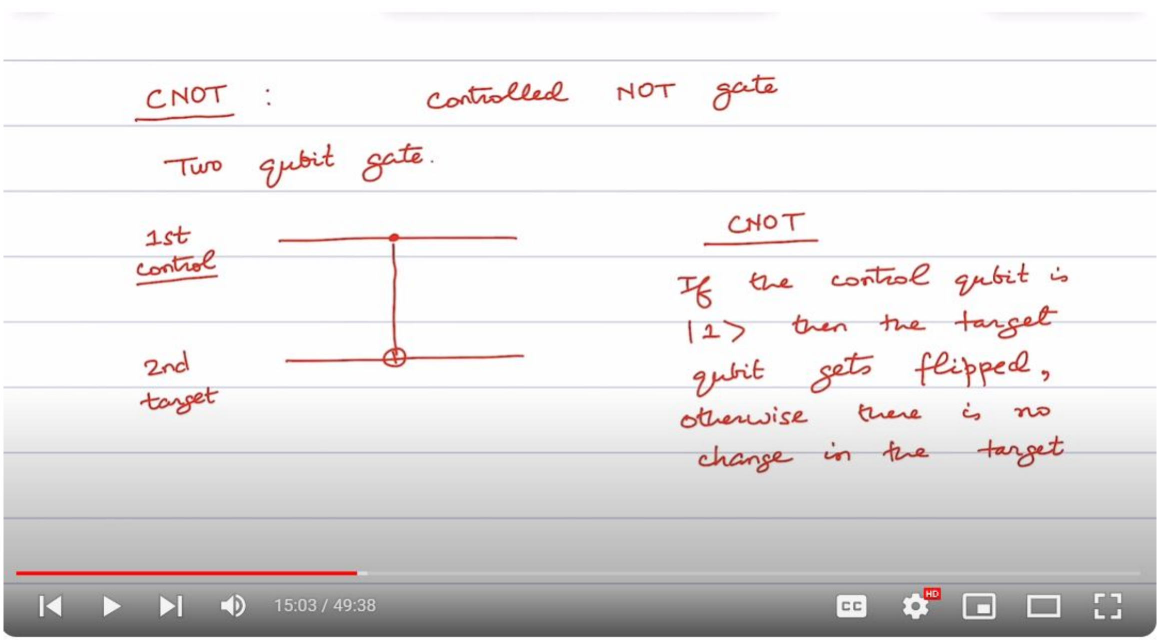
The matrix representation is easy to get here. In the computational basis, ket 0 and ket 1, I can write the matrix form of UNOT gate as follows. The first element in the first row would be 0 UNOT 0. Second element in the first row would be, in the first column would be 0 UNOT 1. And the second row, the first element in the second row would be 1 UNOT 0. And here the second element in the second row would be 1 UNOT 1. Right?

(Refer Slide Time: 13:1phi)



So obviously now because of these operations, you can immediately write here. It would be 0 1 and this is going to be 0 0. This one is going to be 1 1 and this one is going to be 1 0. And obviously from here I get the matrix representation of UNOT as 0 1 1 0. And this is the procedure we can adopt for writing the form of the matrix for any quantum gates. This quantum gate is symbolically represented by this symbol. You just put an X here. This is the representation of a NOT gate.

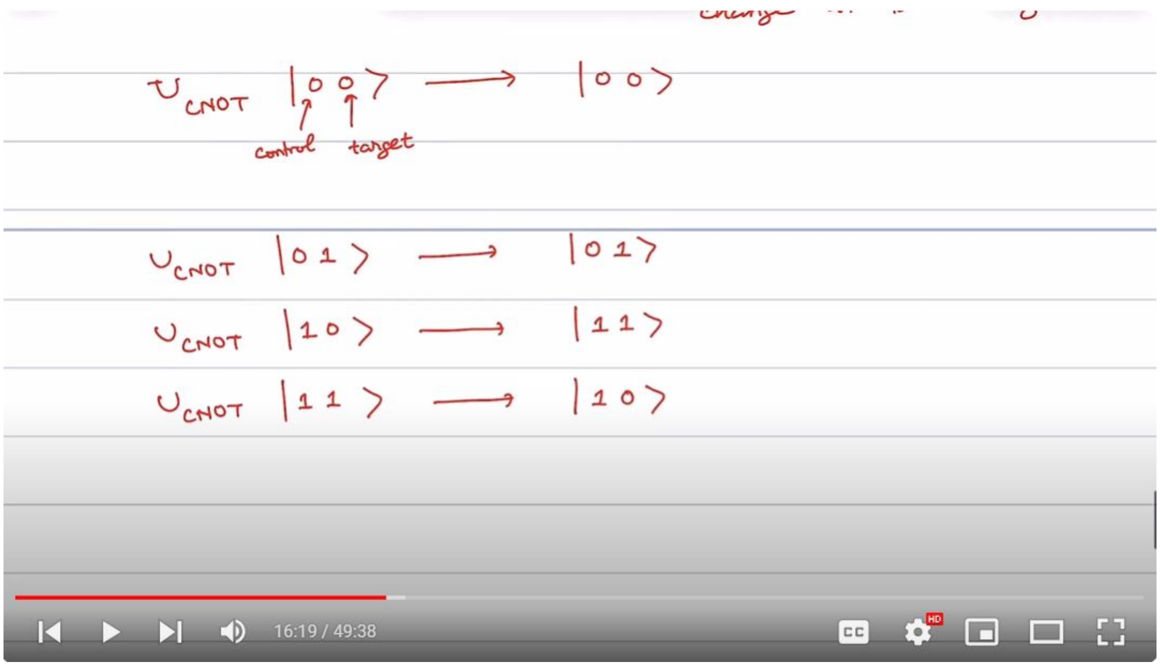
(Refer Slide Time: 1phi:03)



Now there is another gate that is very interesting and that already as I mentioned that is the so called CNOT gate. And using this quantum gate one can obtain entangled quantum states. And this is called CNOT means it is called control NOT gate. It is a 2 qubit gate. It is a 2 qubit gate. That means that 2 qubits are put as input. You have 2 channels here. And in the first channel you will put first qubit and in the second channel you are going to put the second qubit. The first qubit is called control and the second qubit is called target. Target bit and this is control bit. And a CNOT gate is represented by this symbol. Okay.

And the rule by which CNOT gate operates is as follows. If the control bit or the control qubit. If the control qubit is 1 then the target qubit. The target qubit gets flipped. That's the rule. Otherwise there is no change. Otherwise there is no change in change in the target.

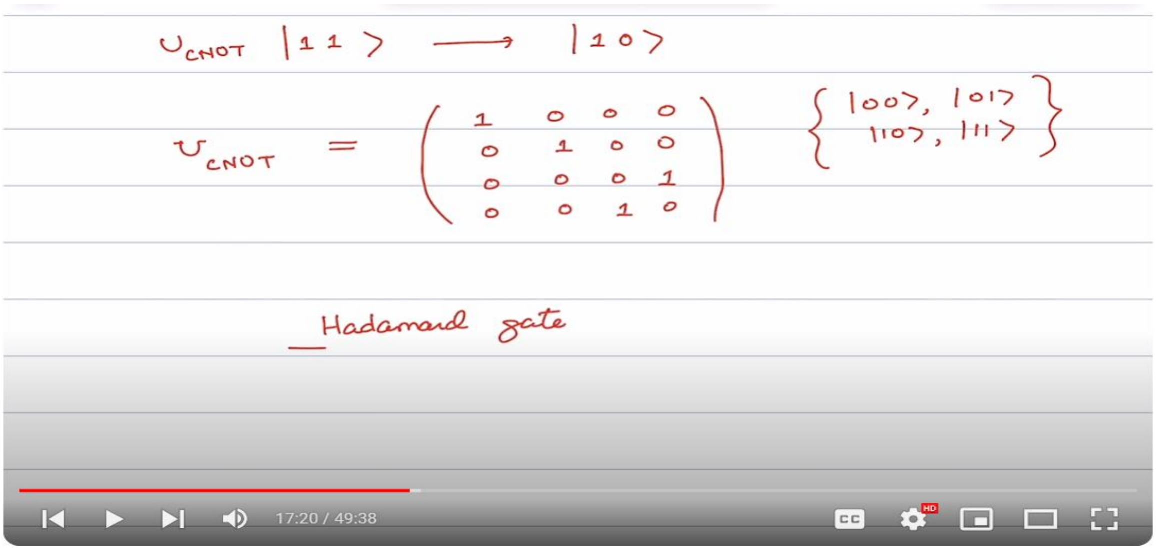
(Refer Slide Time: 16:19)



You can understand it very easily if I give you the operation rule by this unitary transformation. So it's going to be represented by a unitary operator say UCNOT. When you have your input as 0 0 and this is the first qubit and this is control and this is target. Because now the control qubit is 0 so target qubit does not flip and at the output you will simply get 0 0. And if the input qubit are say 0 1 here also output is going to be simply 0 1 because the control qubit is 0 the target remains as it is.

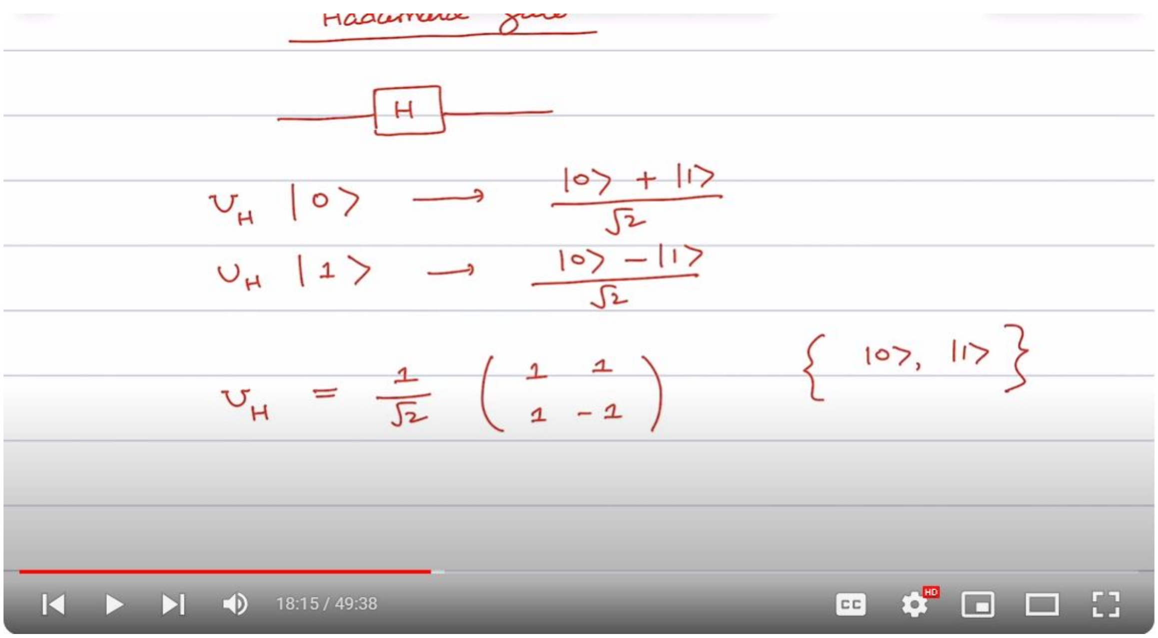
And if now the control qubit is 1 and target is 0 then control target qubit now gets flipped and 0 would become 1. On the other hand if your control qubit is 1 and the target is 1 the target qubit gets flipped at the output and the output you are going to get is 1 0.

(Refer Slide Time: 17:20)



So this is the operation of a CNOT gate and the matrix representation of the CNOT gate can be easily worked out. And if you worked out the matrix representation will look like this. You will have 1 0 0 0 0 1 0 0 0 0 0 1 0 0 1 0. While I write this matrix form the basis states that I am using here I am going to use a two qubit basis the basis state would be 0 0 0 1 1 0 1 1. Because it is a two qubit system. Right.

(Refer Slide Time: 18:1phi)

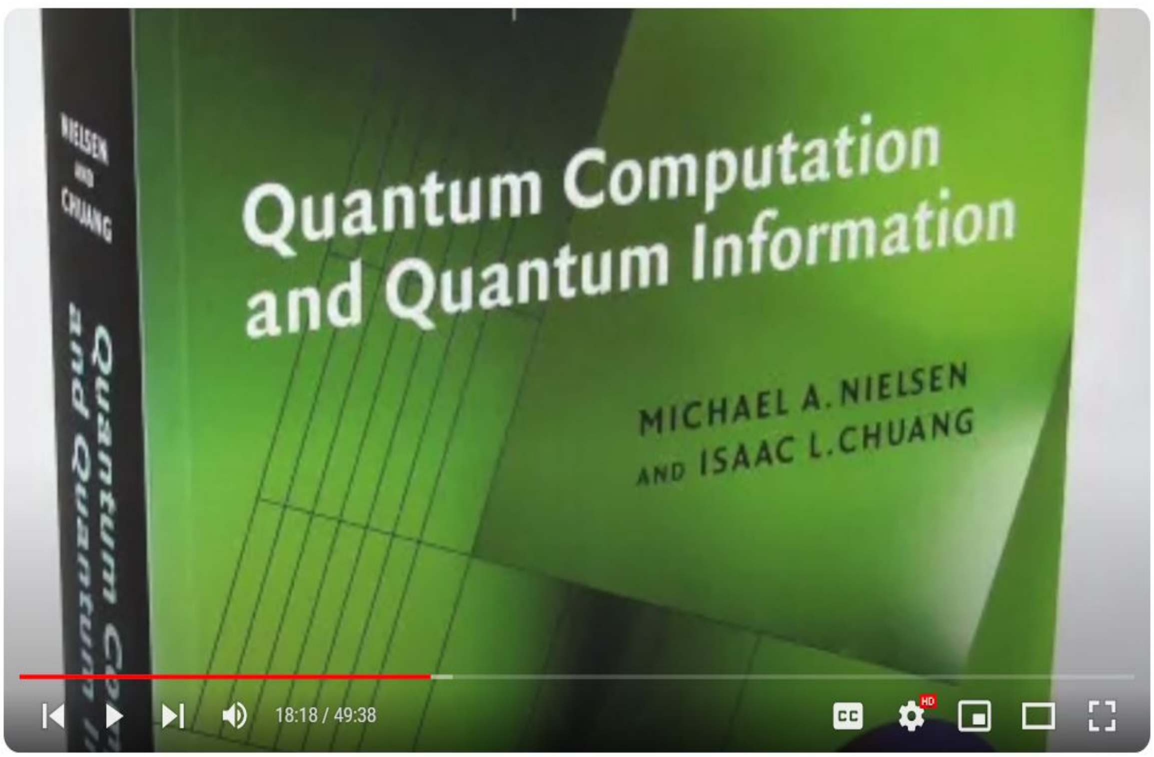


Now let me quickly discuss about another useful and relevant gate that is called the Hadamard gate.Hadamard gate. And a Hadamard gate is represented by this symbol it is a single qubit gate so it is represented by this symbol. Okay.

And its operations can be defined this way. When the Hadamard gate operates on ket 0 it converts it to a superposition state ket 0 plus ket 1 by root 2. And if it operates on the qubit or state ket 1 it converts it to superposition state ket 0 minus ket 1 by root 2.

Now here also you can write down the matrix form for this Hadamard gate and it would be 1 by root 2 1 1 1 minus 1. Because it is a single qubit gate the basis states are here ket 0 and ket 1. Okay.

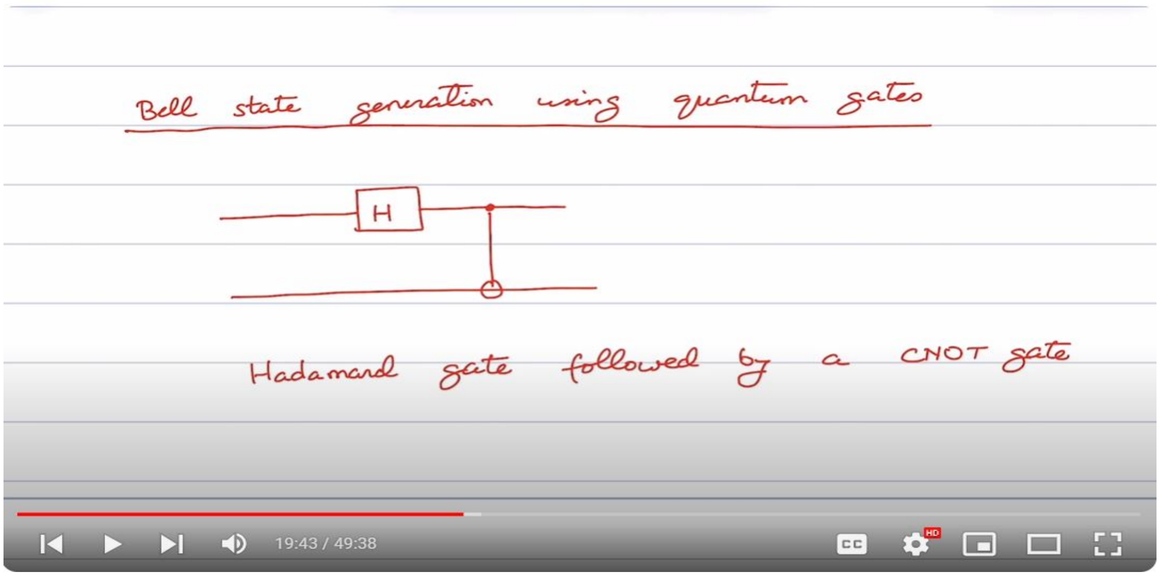
(Refer Slide Time: 18:18)



You can read more about quantum gates in the classic book by Nielsen and so on.

Now let me give you an example how these gates can be quantum gates can be used to obtain Bell states.

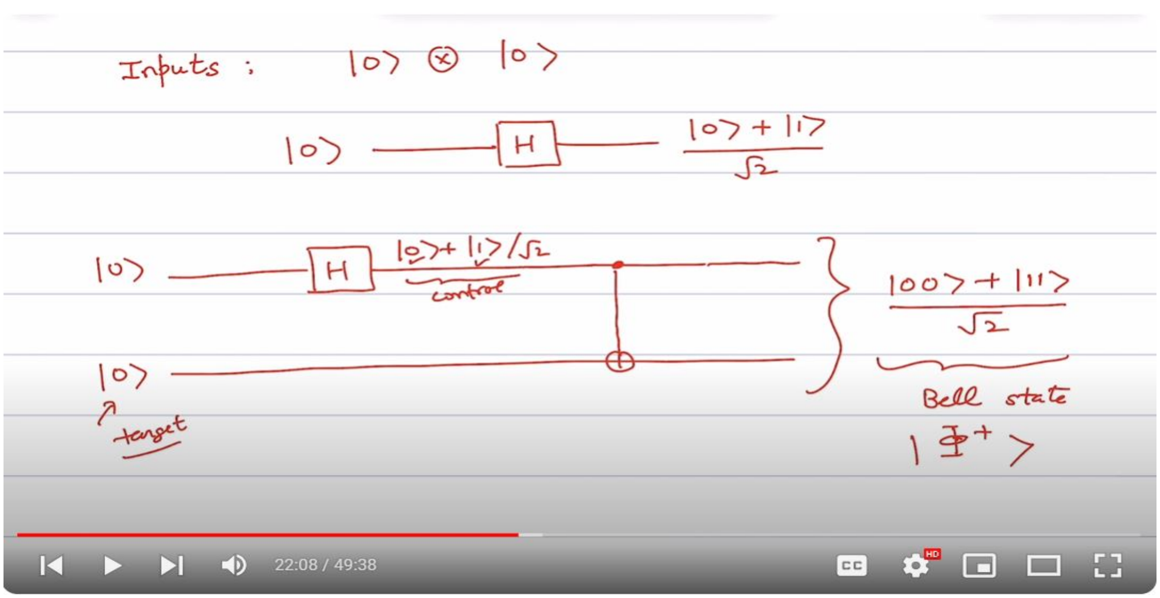
(Refer Slide Time: 19:43)



So now I will briefly discuss Bell state generation. Generation using quantum gates. Quantum gates.

To do that let me consider this quantum circuit. Here I have a 2 qubit gate but 2 gates I am going to use. First I have a Hadamard gate and input of this Hadamard gate is going to be the output of this Hadamard gate is going to be the input of a CNOT gate. And I have this a CNOT gate is there followed by this Hadamard gate. So I hope you are getting it. So this is the Hadamard gate. We have in this structure we have a this is a quantum circuit. And we have a Hadamard gate followed by followed by a CNOT gate.

(Refer Slide Time: 22:08)



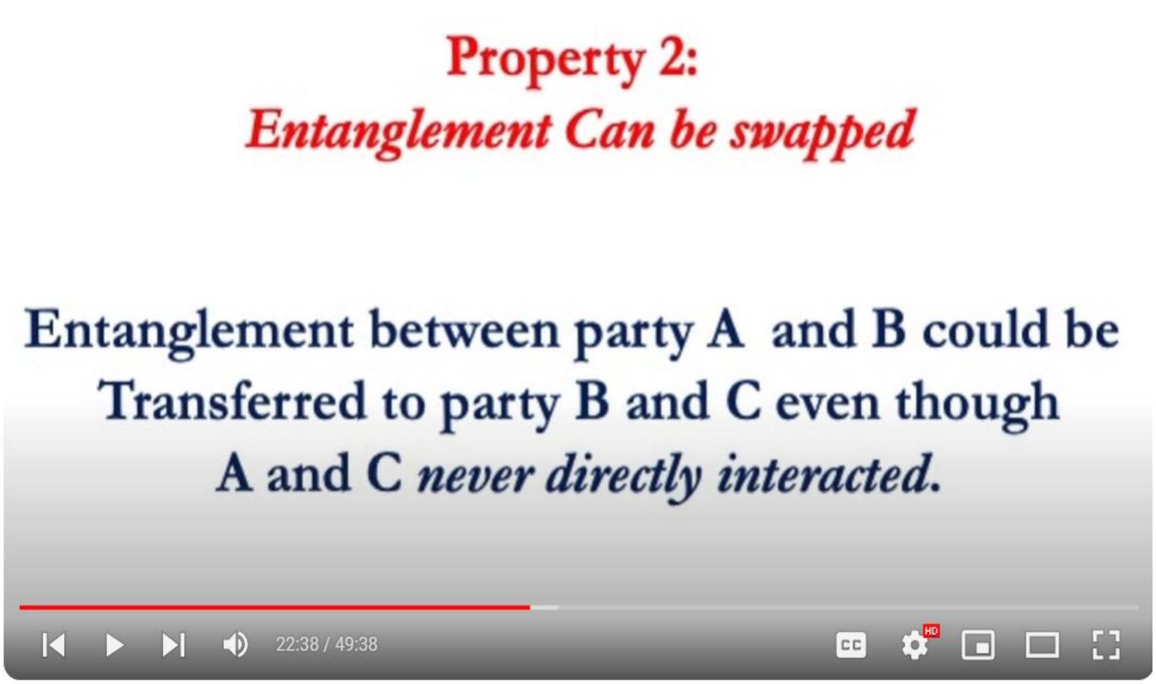
As an example let me consider the input as 0 0. That means the first qubit is put 0 is put as the input here to this Hadamard gate and here I am putting the another qubit 0. Then as it passes through the Hadamard gate.

Let me just write it properly. So this is what my input. This is my input to this structure circuit. This is my inputs. Now when the first qubit passes through the Hadamard gate. It basically gets converted to 0 plus 1 by root 2. So let me write here again. When I have put 0 as my input at the output of the Hadamard gate. I am getting 0 plus 1 by root 2. And that is the input for my CNOT gate. That is my control gate. And at the output what I am going to get.

You see if I have control and here the input is 0. So I am going to write in the second qubit of the control gate. That is the target. Target is 0. So at the output obviously what you are going to get if your input is 0. The target does not get flipped. So you are going to get 0 0.

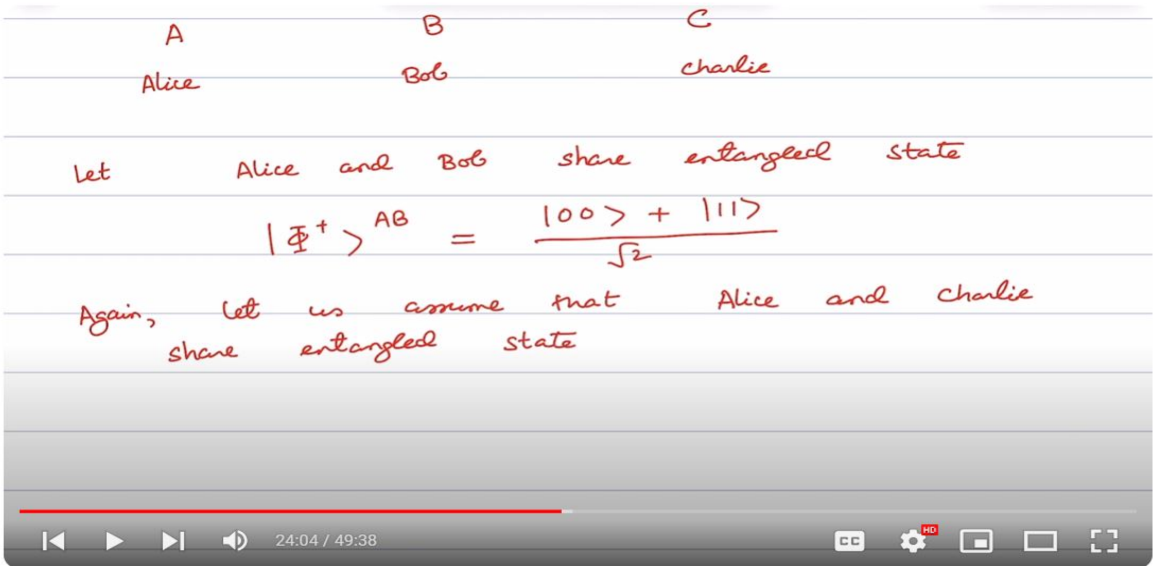
But if the control is 1. Target is going to get flipped. Now here this is my control. So at the output I am going to get if my input is 1. The control is 1. The target is going to get flipped and it will become 1. Right? So I will get 0 0 plus 1 1 by root 2. And you know this is one of the, it's a Bell state. One of the Bell state. Okay, I think this is phi plus. You can verify it. You can check it. This is phi plus. Maybe we will do couple of problems in the problem solving session.

(Refer Slide Time: 22:38)



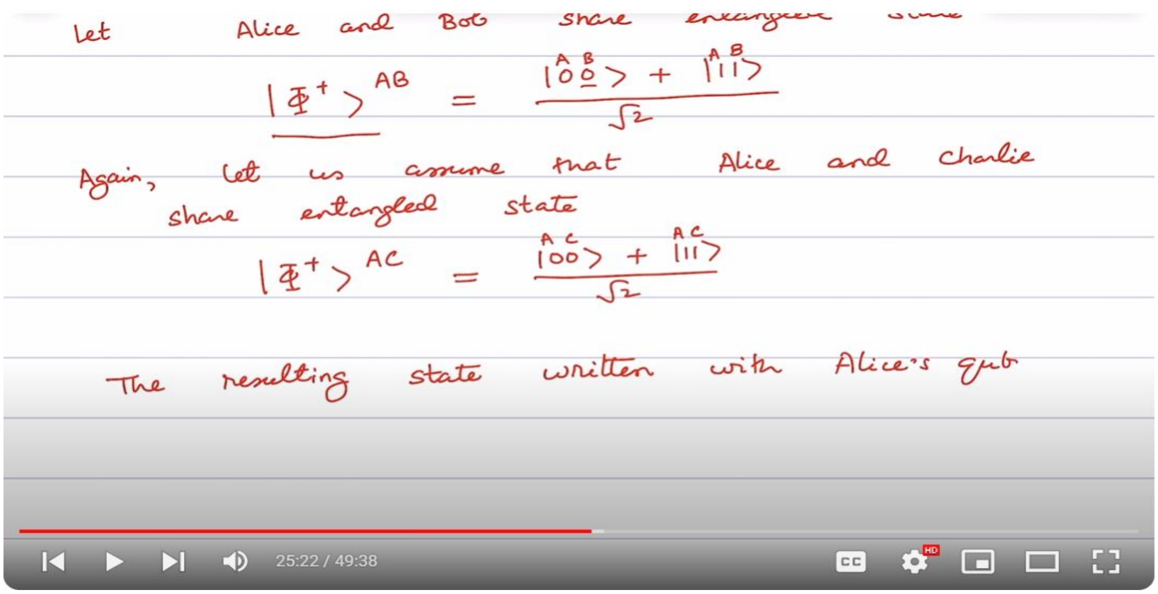
Now let us go back discussing properties of quantum entanglement. One of the most important and useful property of quantum entanglement is that Entanglement can be swapped. It basically means that entanglement between parties A and B can be transferred to parties B and C even though A and C never directly interacted.

(Refer Slide Time: 24:04)



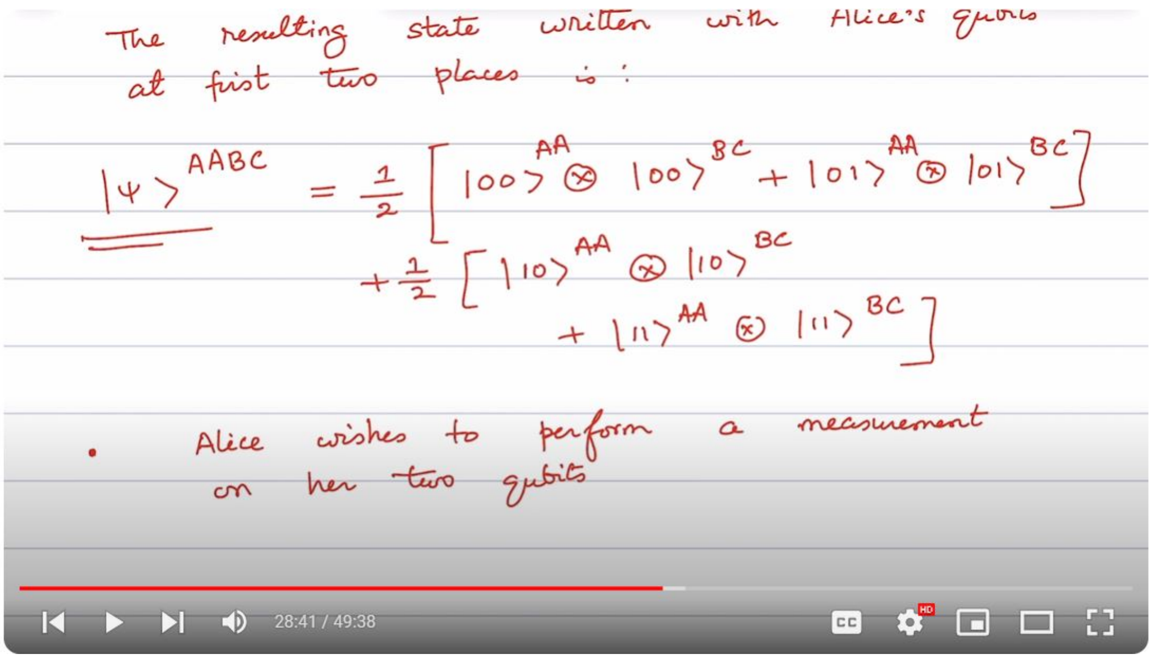
To understand this property let us consider this example. Say A, B and C are three parties represented by Alice, Bob and Charlie. Okay? Let Alice and Bob, let us say Alice and Bob share entangled state, entangled states, phi plus. And it is, it's a Bell state. So this Bell state is 0 0 plus 1 1 by root 2. This is shared by Alice and Bob.

(Refer Slide Time: 2phi:22)



Again, again let us assume that Alice and Charlie share entangled state, again say phi plus. Now this is shared by Alice and Charlie. That is 0 0 plus 1 1 by root 2. The first qubit is belongs to A here in this case and second qubit belongs to Bob. And here the first qubit belongs to Alice, the second qubit belongs to Bob, here the first qubit belongs to Alice, and this belongs to Charlie, this belongs to Alice and this belongs to Charlie. And what we are ultimately going to show you by this protocol that this entanglement state can be shared between Bob and Charlie. So that's what we are going to do using this protocol.

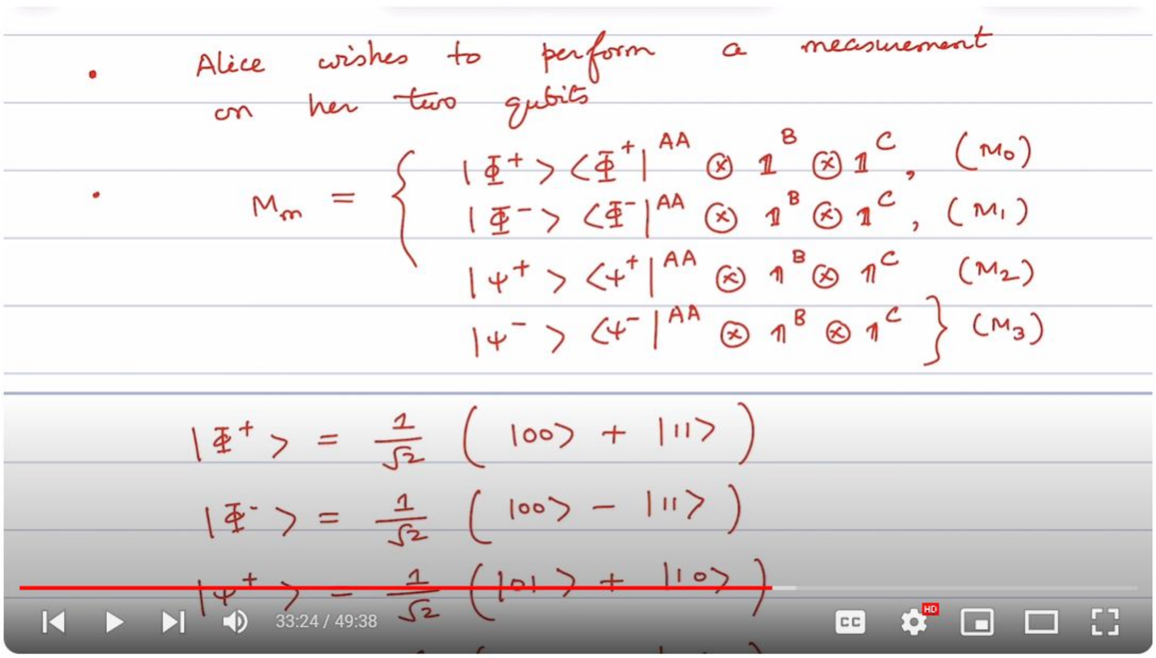
(Refer Slide Time: 28:41)



Now you can write a resulting state with respect to Alice qubits at first two places. The state, the resulting state, because of this share, resulting state return with Alice qubits at first two places, at first two places is, so let me write this state a a b c, first two places is belonging to Alice now, so we are going to have this. So I think you can make it out, if I write these two things together, you can easily understand what are the combinations I can get.

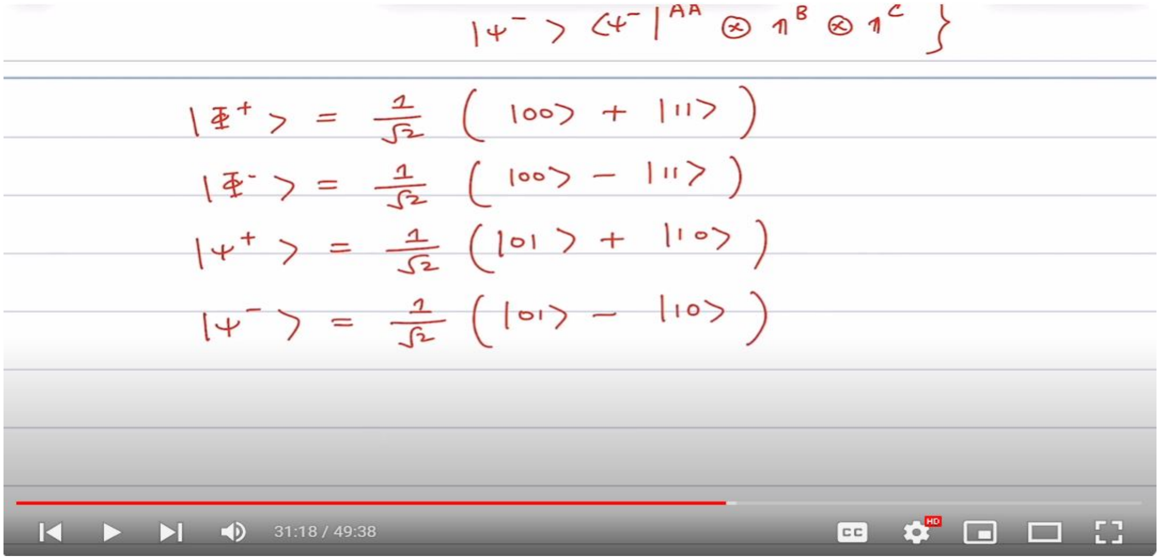
First two states belong to Alice, 0 0, and the second one, so just look at it, look at this one and this one, 0 0 a a, and then this 0 0 b c. So if you see this together, you will see what I mean by this. And then you are going to have, similarly you can write, the first two places belonging to Alice, 0 1 a a. Now here you look at this one and this one, 0 1 belongs to Alice, and 0 1 belongs to Bob and Charlie, right? As you can see, just look at it again, these two I mean to say, this is what you are going to have, and then you have another two possibilities, plus half, you are going to have, now you look at this one and this one, okay? So 1 0, 1 0, this belongs to Alice, and then you have, again you see here, this is your Bob and this is your Charlie, right? So 1, that would be 1 0, 1 0, right? 1 0, 1 0, this is belonging to Bob and Charlie, and then final one, now you again look at this one and this one, the first two places, 1 1 belong to Alice, and then 1 1 belonging to Bob and Charlie. I hope it's easy to see, so this is what you are going to have.

(Refer Slide Time: 30:18)



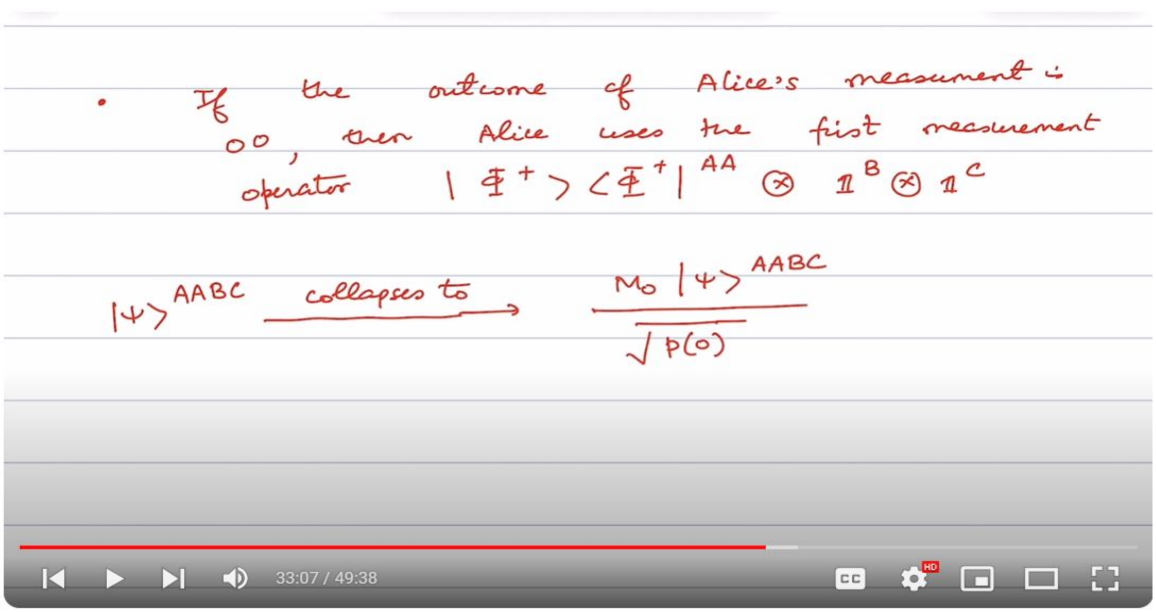
Now, Alice, now what Alice is going to do? This is a protocol that we are discussing, Alice wishes to perform a measurement on her two qubits, perform a measurement on her two qubits, when the state is this, right? So she designs measurement operators for this purpose, based on the Bell state, and these measurement operators she designs are as follows, so she uses Bell state phi plus phi plus, this is one measurement operator, and there is no measurement done by Bob and Charlie, so they are represented by this identity, this is one measurement operator, she is basically going to exploit all the four Bell states, so another Bell state she will use is phi minus phi minus, here you will have Bob and Charlie are not going to do any experiment, and then you have psi plus psi plus, I have already discussed about Bell states in great detail, please look at what I mean by these Bell states, psi plus psi plus, so basically she is designing four measurement elements, because there are four Bell states, psi minus psi minus, A A, 1 B, 1 C, so these are the measurement operators she is having,

(Refer Slide Time: 31:18)



So just let me give you all the Bell states for your reference, phi plus is 1 by root 2, 0 0 plus 1 1, phi minus is 1 by root 2, 0 0 minus 1 1, and psi plus is 1 by root 2, it is psi plus is 0 1 plus 1 0, and psi minus is 1 by root 2, 0 1 minus 1 0, these are the so called Bell states,

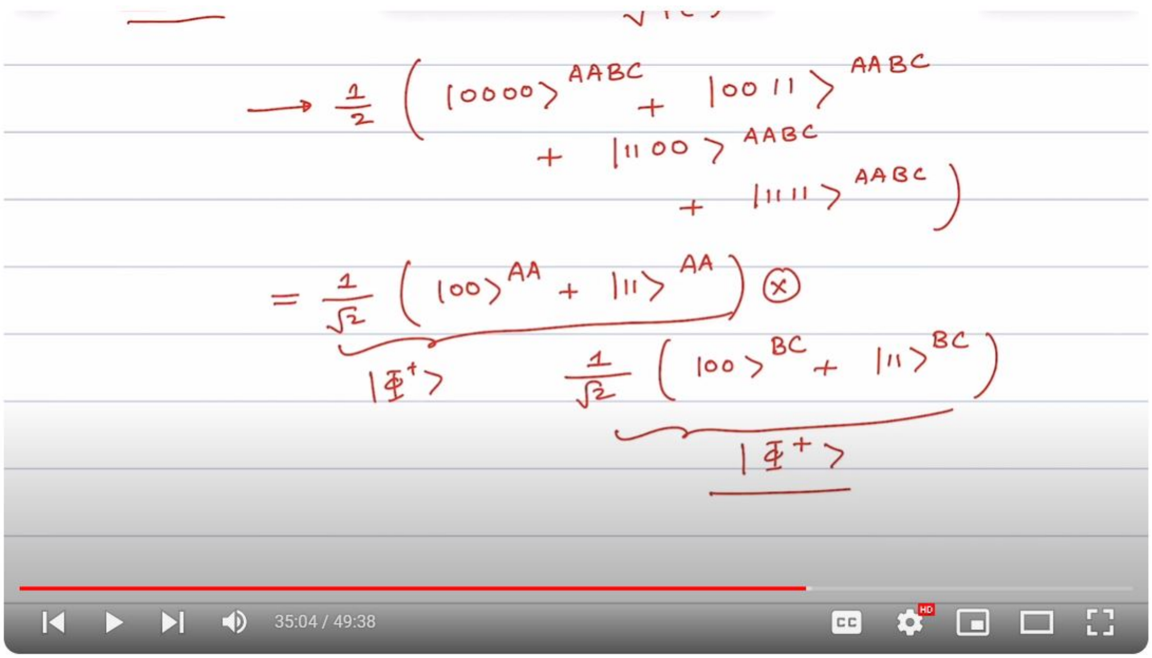
(Refer Slide Time: 33:07)



Now there are four possible outcomes, now say if the outcome of Alice measurement, if the outcome of Alice measurement, because Alice is the only one who is making, doing this measurement, if outcome of Alice measurement is 0 0, then the Alice uses the first measurement operator, Alice uses the first measurement operator, that's what it means, measurement operator, which is, let me write, the first measurement operator was phi plus phi plus, and Bob and Charlie does not do any experiments, so this is what you have, if that is so, the state psi A A B C, it collapses to, because of the measurement, this state collapses to, the state M 0 psi A A B C, divided by square root of P of 0, P of 0 is the outcome of first measurement, 0 0, to probability of getting the outcome 0,

we have already discussed these things in the earlier class, now M 0 by M 0, actually I mean, this is my M 0, this is my M 1, this is my M 2, these are the measurement elements, this is M 3, right?

(Refer Slide Time: 35:04)

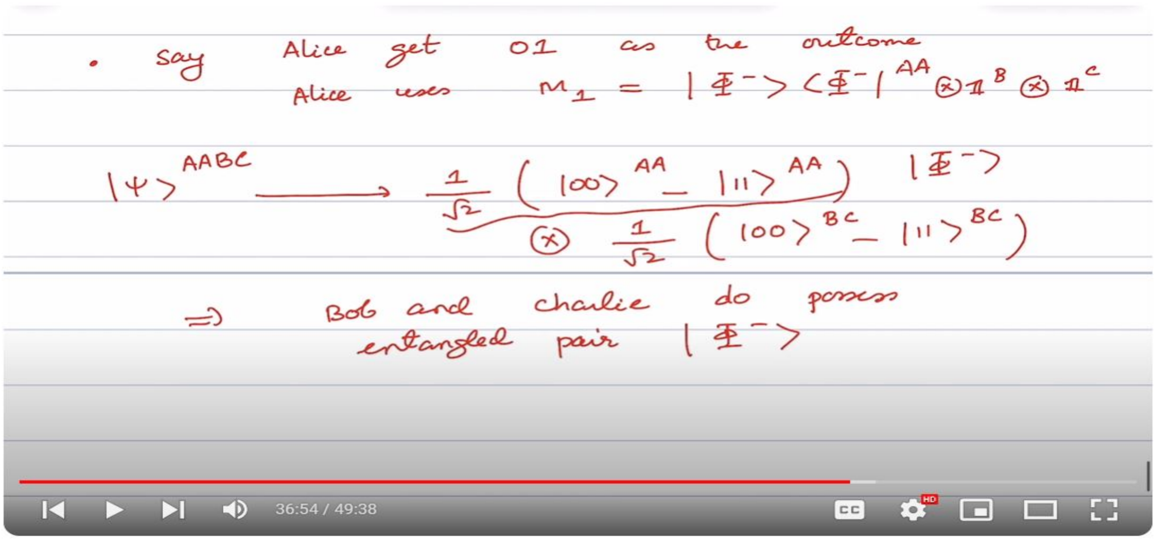


So you can work it out, and if you work it out, and we'll work it out, maybe in the problem solving session, in this thing in great detail, it's very simple to work out, if you work out, you'll find that the state, psi A A B C collapses to, it would be half, 0 0, 0 0, A A B C, plus 0 0 1 1, A A B C, plus 1 1 0 0, A A B C, plus 1 1 1 1, A A B C, and this can be actually expressed as, 1 by root 2, 0 0 A A, plus 1 1 A A, direct product with 1 by root 2, 0 0 B C, plus 1 1 B C.

Now what you see? You see that this particular state is nothing but the Bell state phi plus, and this state is, which belongs to now B C, Bob and Charlie, shared by Bob and

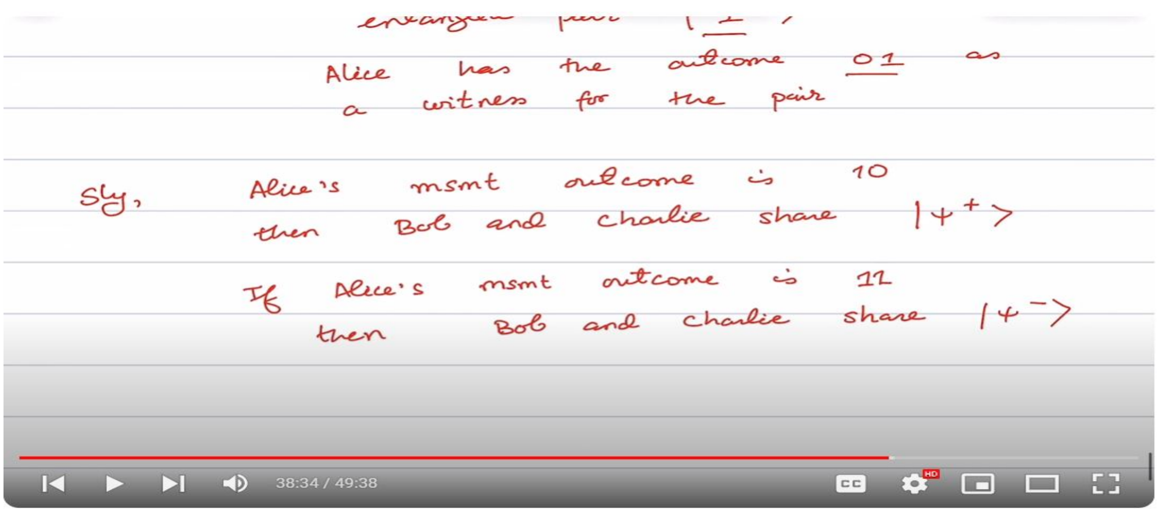
Charlie is phi plus, so it can be clearly seen that Bob and Charlie indeed possess the entangled pair phi plus, so this is what is entanglement sweeping is, now Alice has outcome 0 0 as a witness to this situation.

(Refer Slide Time: 36:54)



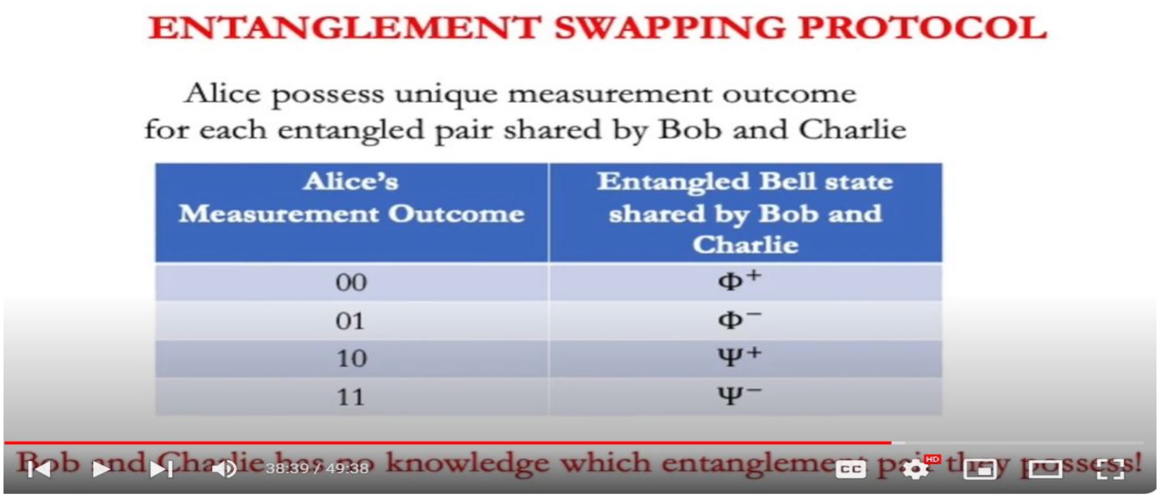
Now say, next say Alice, say Alice measures, or Alice get 0 1 as the outcome, 0 1 as the outcome, and in that case, Alice uses, in this case Alice uses measurement M1, which is, phi minus phi minus, A A, 1 B that is, Bob and Charlie does not do anything, does not do make any measurement, and it can be shown that in this case, psi A A B C, this particular state collapses, to this pair of states, that is 1 by root 2, 0 0 A A, minus 1 1 A A, and it would be 1 by root 2, 0 0 B C, minus 1 1 B C, so what it means, now it means, this implies that, Bob and Charlie, Bob and Charlie, do possess, do possess, entangled pair, entangled pair, this is, nothing but phi minus, right?

(Refer Slide Time: 38:34)



this state is phi minus, entangled pair, this is the entangled Bell state phi minus, and Alice has the outcome 0 1 as the witness, Alice, has the outcome, outcome 0 1, as the, as a witness, as a witness, for this, for the pair, the fact that Bob and Charlie is now possessing entangled pair, Alice can know that by making a measurement, if the outcome is 0 1, then she will know that, Bob and Charlie possess the entangled pair phi minus, similarly, you can easily guess, that if, Alice measurement outcome, outcome is 1 0, then, Bob and Charlie, share the entangled state, psi plus, and finally, if, Alice measurement, measurement outcome is 1 1, if Alice measurement outcome is, outcome is 1 1, then, then Bob and Charlie, Charlie share the entangled state, psi minus.

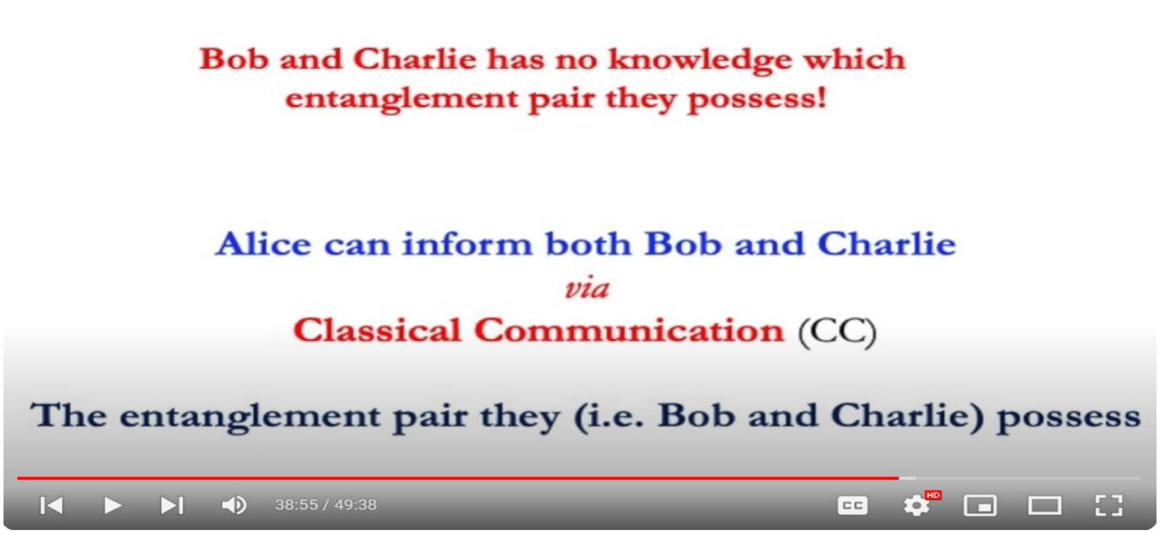
(Refer Slide Time: 38:39)



Alice's Measurement Outcome	Entangled Bell state shared by Bob and Charlie
00	Φ^+
01	Φ^-
10	Ψ^+
11	Ψ^-

So by the way, as you can see, Bob and Charlie has no knowledge, about which of the entangled pair they possess,

(Refer Slide Time: 38:55)



now for each entangled pair, as we have seen, Alice possess a unique measurement outcome, and she can tell, who is Bob and Charlie, through classical communication, that what entangled pair they are possessing,

(Refer Slide Time: 39:10)

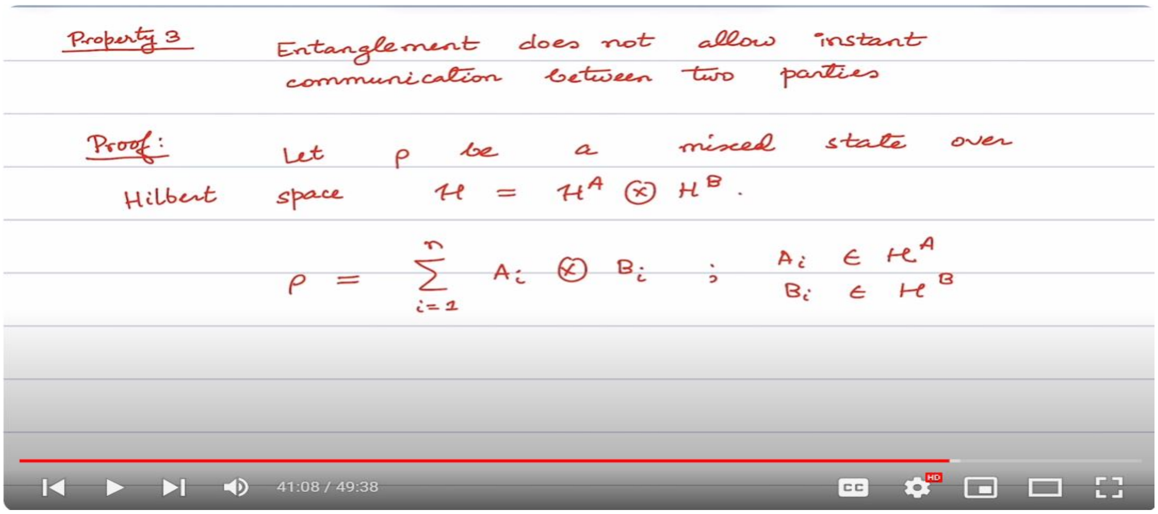
Bob and Charlie, who has never ever met before, can now possess entangled pair, with the help of a person in the middle, and that is Alice, okay.

(Refer Slide Time: 39:30)

Now let me discuss, another property of entanglement, this property of quantum entanglement, is related to no instant communication, entanglement does not allow, instant communication between two parties, this is an important property.

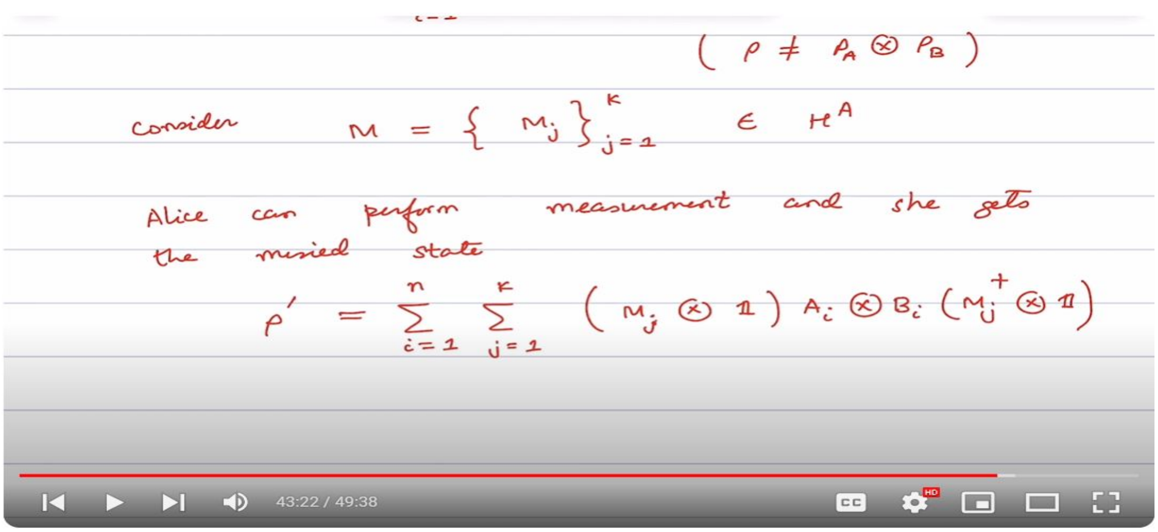
it clearly shows that, the so called spooky action at a distance, does not mean, that it is possible to have instant communication.

(Refer Slide Time: 41:08)



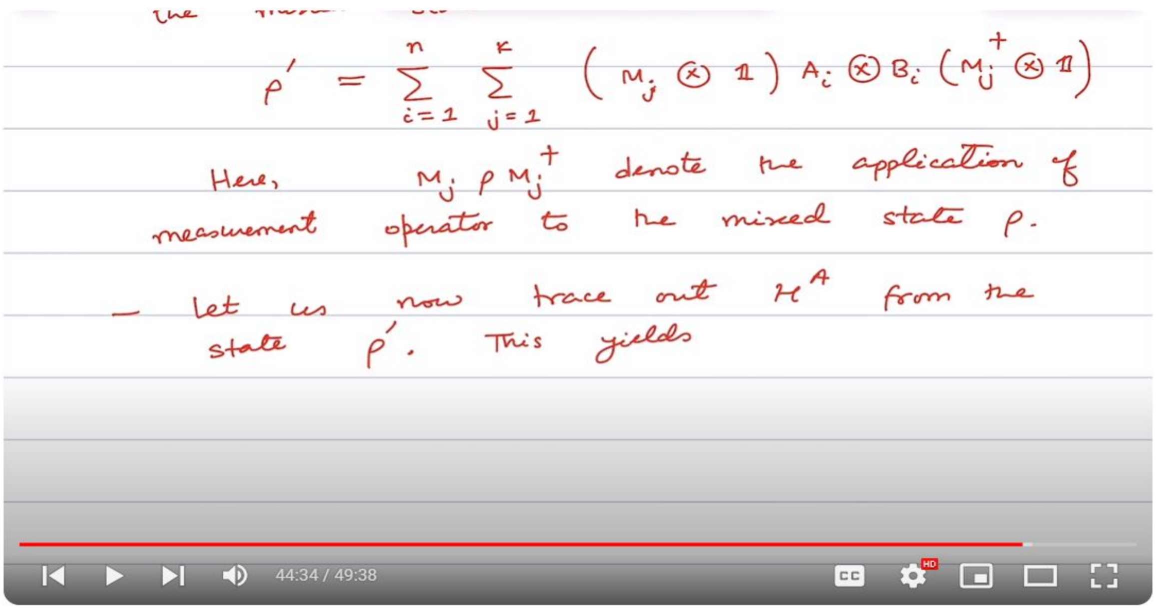
Let us prove it, the proof is a little bit technical, still I hope that you will be able to follow it, say, let rho be a, mixed state, be a mixed state, over, Hilbert space, over Hilbert space, H is equal to, HA, the tensor product between these two Hilbert space, A and B, A you can consider, to be the space belonging to Alice, HA, that's the Hilbert space, and HB is the Hilbert space belonging to, the gentleman, Bob, and every such mixed state can be written as a, sum of, superposition, basically sum of tensor product of linear operators, Ai and Bi, Ai belong to HA, and Bi is a linear operator belonging to, HB, so, here, Ai belongs to the Hilbert space, HA, and Bi is a linear operator belonging to, HB,

(Refer Slide Time: 43:22)



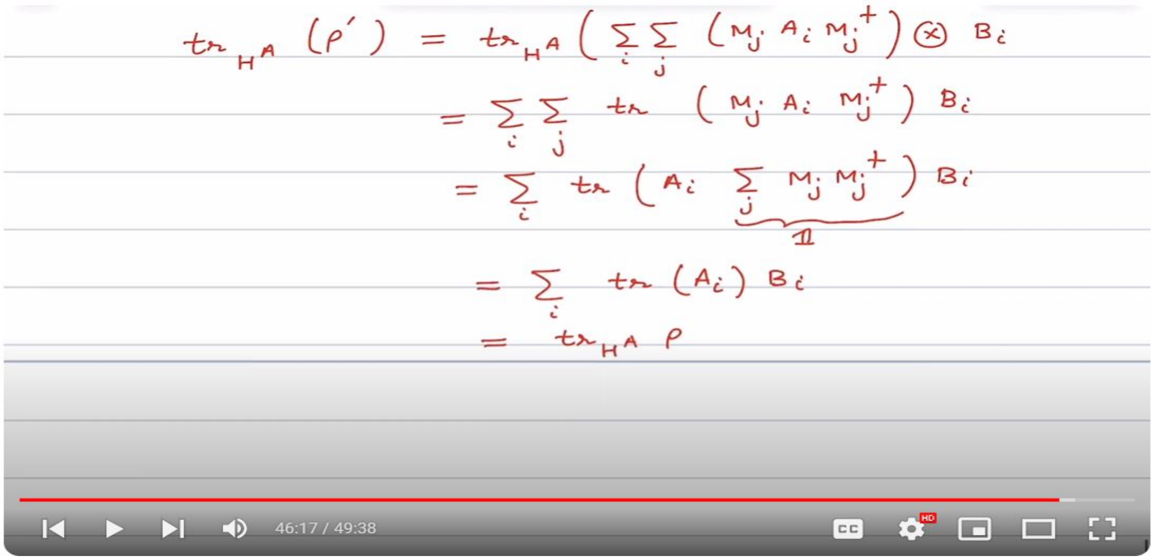
please note that, here I am not saying that, these are product state, rather, you should understand that, I am writing it in terms of linear operators, I am not writing rho is equal to, say rho A, rho B, this is not I am saying, okay, and, also, consider, some measurement operators, consider M is equal to, a set of measurement operators, M is equal to MJ, K number of measurement operators are there, say J is equal to 1 to K, and this belongs to, the Hilbert space, A, belonging to Alice, say, then Alice can perform a measurement, Alice can, perform, can perform, measurements, measurements, and, she gets, the mixed state, after making a measurement, she gets the mixed state, say, rho dash is equal to, I is equal to 1 to N, and there are, set of measurements, K's number of measurements are there, J is equal to 1 to K, so, Alice touches only, her qubits, MJ and, Bob's qubit, that is B, the linear operator B is, not getting touched, so this is what, we mean by this, right? And you have this, MJ dagger, identity, this means that only, measurement is done over, that of, Alice, place only, right? In Alice laboratory,

(Refer Slide Time: 44:34)



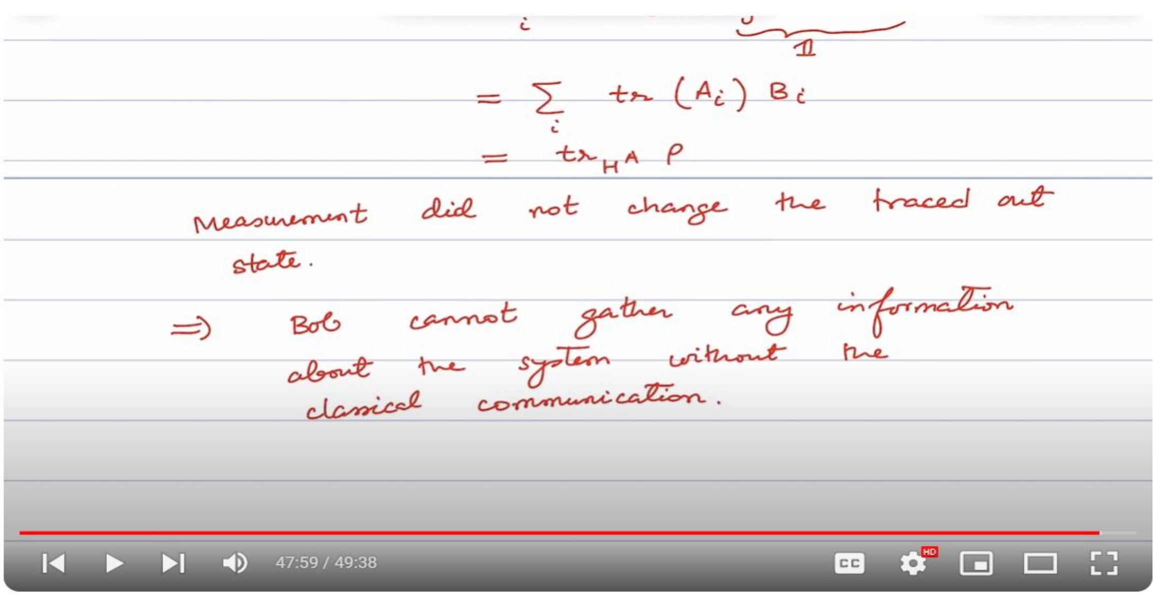
Here, here you can see, that, MJ rho, MJ dagger, denote the application of measurement, this denote the, application, application of, measurement operator, measurement operator, to the mixed state, to the mixed state, rho, ok, this actually we have also discussed earlier, in the, measurement topic, that we discussed in the last lecture, now let us trace out, let us now trace out, trace out, HA, from the state, from the state, rho dash, and this is going to give me, this else, I am just doing the trace operation, tracing out operation, it's just like working out the reduced density matrix, we have done so many problems related to that,

(Refer Slide Time: 46:17)



so when we are tracing out A, that means trace operation is going to be, done over, on Alice place only, so here I, let me write the state rho dash here, and it is, MJ AI, MJ dagger, tensor product with BI, so operation is done over Alice place only, so trace operation will be done accordingly, and because it's a linear operation, so I can take it inside, so IJ, trace, MJ AI, MJ dagger, BI, and this I can further write as, I, sum over I, trace, AI, sum over J, MJ, MJ dagger, BI, and we know the measurement operator property, this is nothing but the identity, and therefore, what I will get is, sum over I, trace, AI, BI, and this is nothing but, you can easily see that this is trace over, A, over the original mixed state rho, okay,

(Refer Slide Time: 47:59)

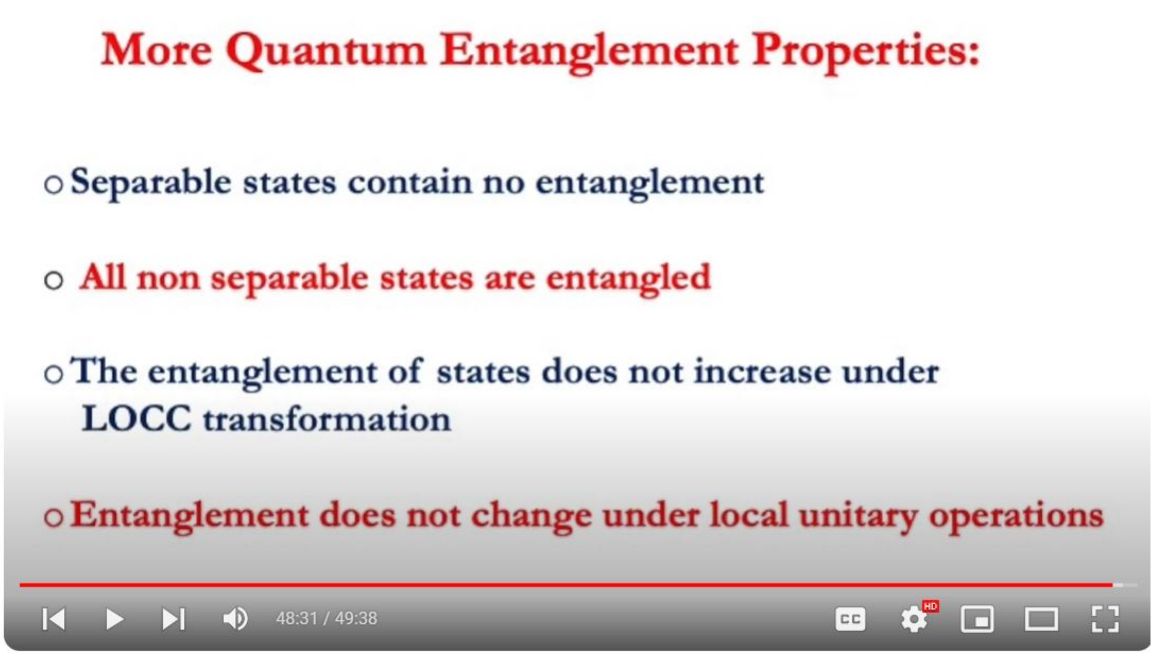


so what we have got from here, you have to think carefully here, this means that, the measurement, did not change the, traced out state, this implies, it's little bit technical, but if you think properly then you will get it, this means that measurement, did not change, the, traced out state, did not change the traced out state, okay, what does that mean? That means that when you are tracing out A, that means you are going to get the state of Bob basically, so Bob, because the tracing out operation did not change anything, so Bob cannot gather any information, any information about the system, without the classical communication,

so this basically implies that, Bob cannot, Bob cannot gather, any information, any information, about the local operation, that is done by Alice, any information about the, system, without, without the, classical communication, without the classical communication, and you know that classical communication, cannot happen faster than light, speed and therefore that means instant communication, is not possible, alright, that's how we can prove it, this is little bit technical.

I hope you get it, even if you did not get it, please don't worry, just remember that, entanglement does not allow instant communication between two parties, there are couple of more properties of quantum entanglement, let me simply mention them,

(Refer Slide Time: 48:31)



separable states contain no entanglement, I think this is easy to understand, all non-separable states are entangled, the entanglement of states does not increase under, L O C

C, that is local quantum operations and classical communication, transformations, and finally entanglement does not change under, local unitary operations.

Let me stop here for today, in this lecture we have learnt a lot about, some important properties of quantum entanglement, in the next lecture we will start discussing, quantification of quantum entanglement in the context of, discrete variable quantum mechanics, so see you in the next lecture, thank you so much. Thank you.