Quantum Entanglement: Fundamentals, measures and application

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Week-03

Lec10: Quantum Measurements

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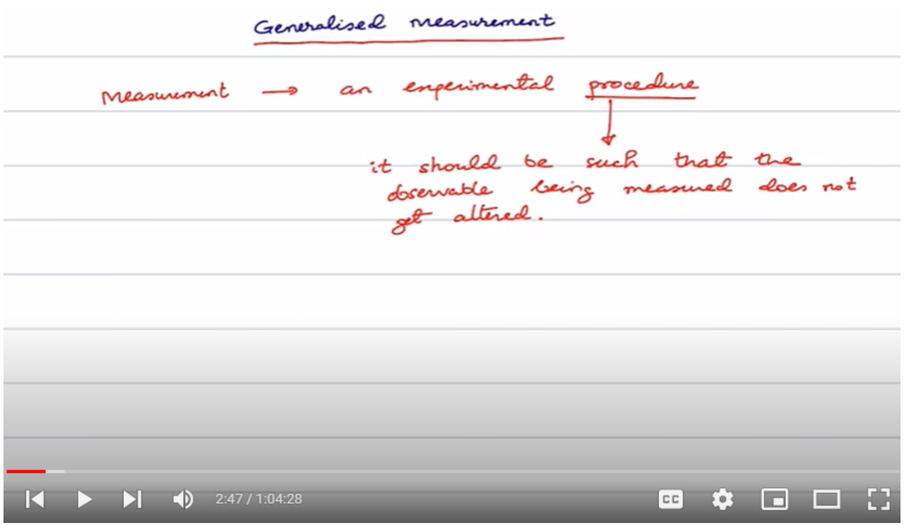

Hello, welcome to lecture 1 of module 3. This is lecture number 7 of this course. In this lecture we are going to discuss about quantum measurement process and this concept is extremely important in the context of quantum entanglement as you will see later and this measurement process in quantum mechanics is fundamentally different from the classical measurement process. Please note that the classical information theory is formulated independently of the measurement of the system. This is because of the reason that you are always going to get the same result if the system processes the same information which is totally different in the quantum information processing. So let us discuss about measurements now and you will find this concept a little bit technical but I try to give as much example as possible. So let us begin. Measurement is basically an experimental procedure meant to determine the value of a physical observable. That's what we mean by measurement. So it's an experimental procedure to determine the value of a physical observable and this procedure has to be such, in fact it has to be carefully designed so that it should be this procedure should be such, it should be such that the observable being measured being measured does not get altered. And in quantum mechanics measurement has a very important role and it's a it's a very critical concept.

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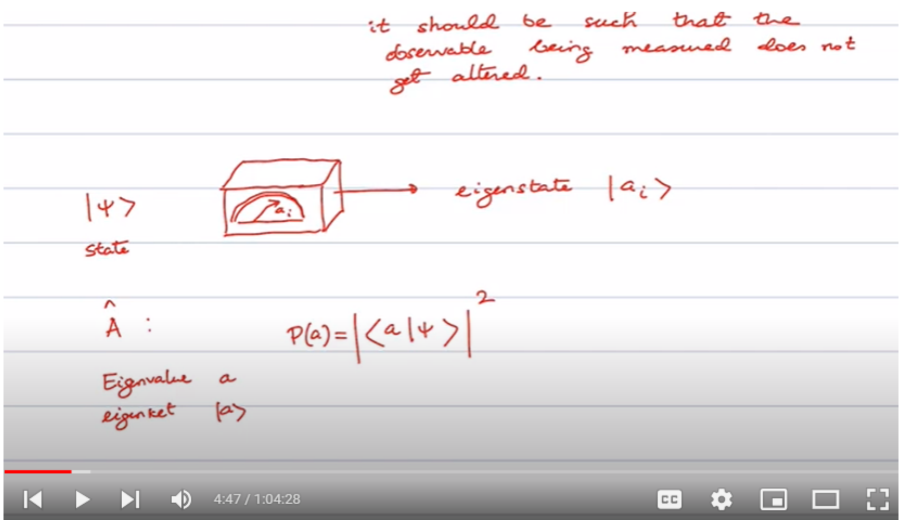

By making a measurement on the system what we mean by in quantum mechanics is that we project the system state vector into one of the basis vector that the measurement equipment defines. Let us say we have this is pictorially speaking suppose this is my system and this system is in the state say ket psi and by making a measurement what is done is that this state vector ket psi is projected into one of the one of the eigenvalues let us say ai so and the corresponding eigenstate say eigenstate ket ai. So because of the measurement this state vector ket psi is projected into one of the eigenstate ai and measurement of an observable suppose anyway all of us we already know that any physically observable quantity is always represented by an operator in quantum mechanics and measurement of an observable a of a system in the state psi yields an eigenvalue a of the operator and corresponding eigenstate is say ket ket a suppose because of the measurement eigenvalue i get is eigenvalue we get is a and the corresponding eigen ket is ket a and the probability of getting the eigenvalue probability of getting the eigenvalue a is given by mod square of this quantity right this is what we know it is also one of the postulates of quantum mechanics.

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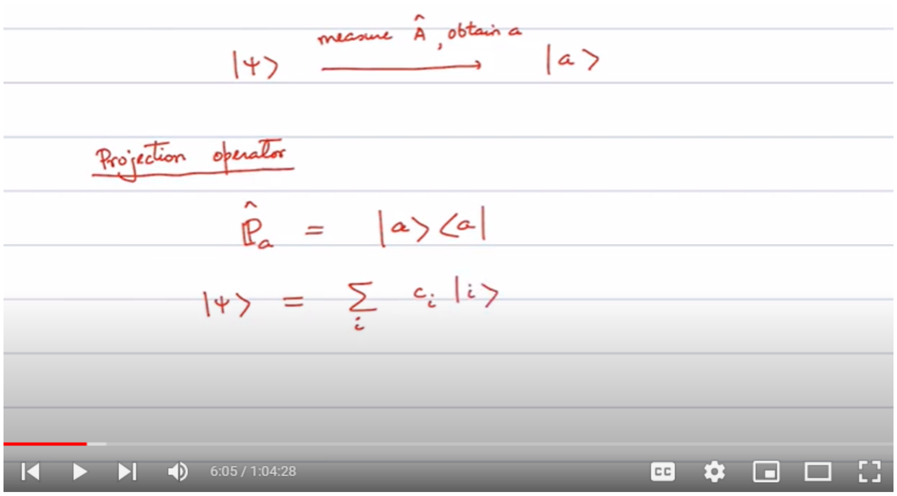

So basically measurement causes the state of the system ket psi to collapse into an eigenstate ket a so if you say measure the observable a then you are going to obtain the eigenvalue a or in other words you the state is getting collapse into the eigen ket a to describe this process by an operator acting on the state generally we introduce the so-called projection operator.

So projection operator i think most of you may know just let us have a quick recap what we mean by projection operator once again so the action it's basically action of the projection operator is to project the state along another state suppose the projection operator let me define is say this one ket a bra a so this is the projection operator and if suppose my state is ket psi and i can write it as a superposition of eigenkets say this is the superposition principle.

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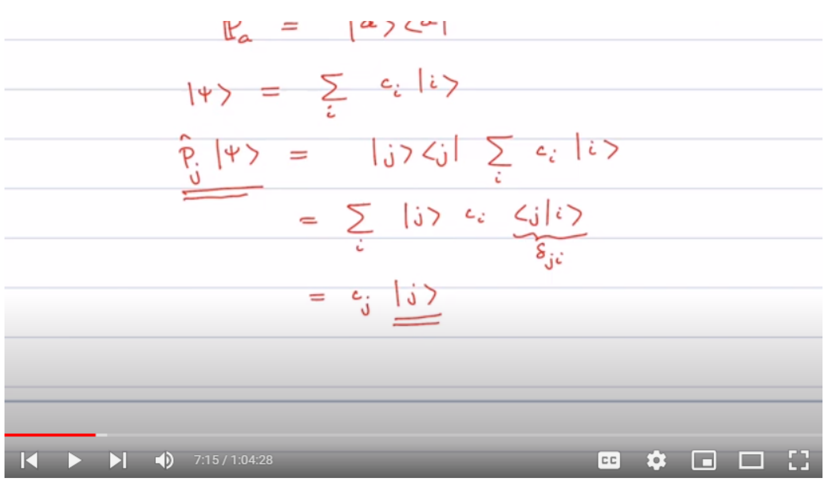

Then the projection operator say pj it if it is operated on the state vector ket psi what it is going to result is this projection operator pj is ket j bra j and it operates on ket psi which is the superposition of the eigenkets uh like this right and i can now write this quantity as follows i can write it as say ket j and this i am having ci is just a number and i have z i so this quantity is nothing but kronecker delta j i so because of this i am going to get simply cj kj so as you see the projection operator pj projects the state vector along the direction of the ket j okay.

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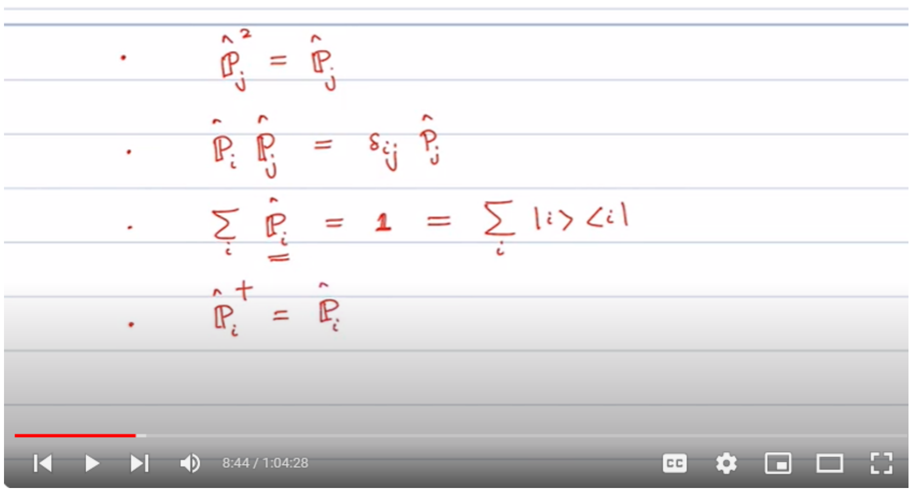

And there are many properties of projection operator some of the notable properties are very easy to prove for example if you can easily verify it pj squared is equal to pj right that it's very easy to see another property is the projection operators are mutually orthogonal if you have two projection operator different projection operators say p i and p j and it's we equal to delta i j p j that means if i is not equal to j then the p i p j product is going to give you zero and projection operators this is very important projection operators are complete basically these are projectors so you have this is basically the so-called completeness condition because this is equal to identity operator and in fact it is easy to see that this results is because p i is nothing but ket i bra i and as we know that this is nothing but the so-called completeness condition and also projection operators because has to be hermitian and it is also easy to see p i dagger is equal to p i okay this is Hermitian.

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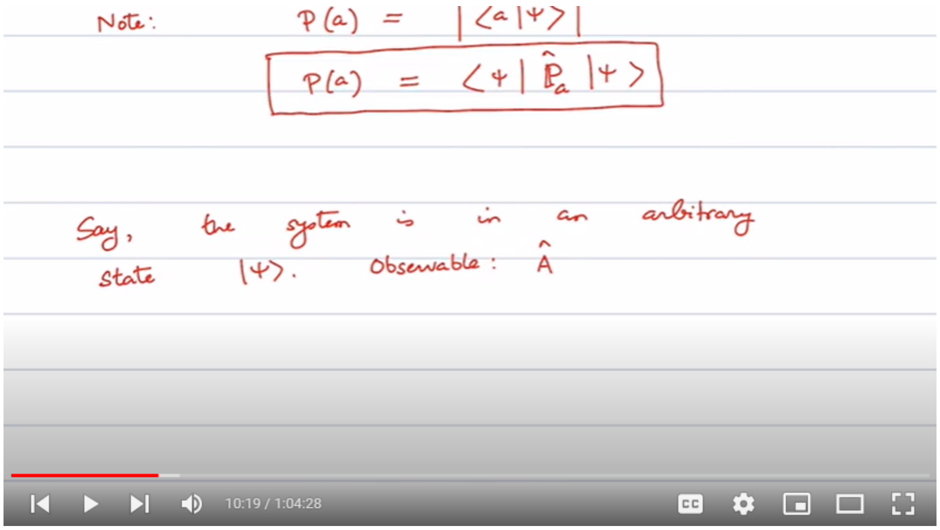

Note this particular it's very easy to see also that the probability of getting the eigenvalue a if the state is in the system is in the arbitrary state ket psi the probability of getting the eigenvalue a would be given by mod square of this quantity and which actually i can write also as this psi p a right psi okay so the projection operator p a projects ket psi into eigenstate ket a and the probability is can be written in this in terms of projection operator this can be written very simply by this expression. Now let us understand the measurement process a bit more clearly to do that let us say a system is in an arbitrary state ket psi say the system is in an arbitrary state ket psi okay and the observable is represented by operator we are interested in measuring an observable and this observable is represented by the operator a.

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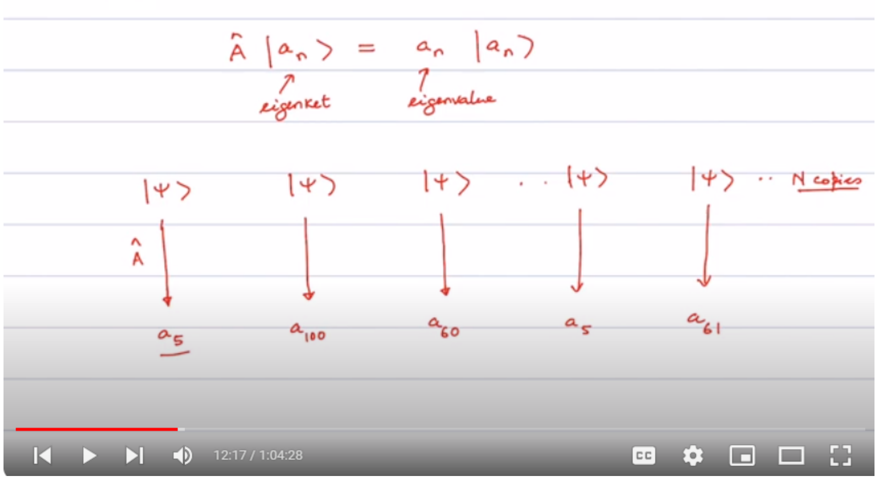

And this operator a satisfy the eigenvalue equation a ket a n is equal to a n ket a n and a n these are as you can see these are eigenkets and this is your eigenvalue okay and measurement of the observable a on the system yields any of the eigenvalues right okay let me make you understand it by this.

Suppose we have a a number of copies of the system and the system is in the state ket psi suppose we have n number of copies like this ket psi ket psi like this n number of identical copies n copies we have okay n number of copies we have the same system exact system and we are making a measurement on the system and basically the observable a and if we measure make a measurement then because of the measurement if i make an individual measurement on ket psi then i may get any one of the eigenvalues eigenvalues let us say i get the eigenvalue say a5 if i make another measurement on this ket psi suppose i get this time a 100 all identical copies i have i make measurement on each one each of them at the same time then i am going to get sometime i will get a say 60 and suppose i make sometimes i may again get say a5 as i have got it here and like this okay suppose i here get again suppose a 60 or say 61 and so on any of the eigenvalues okay.

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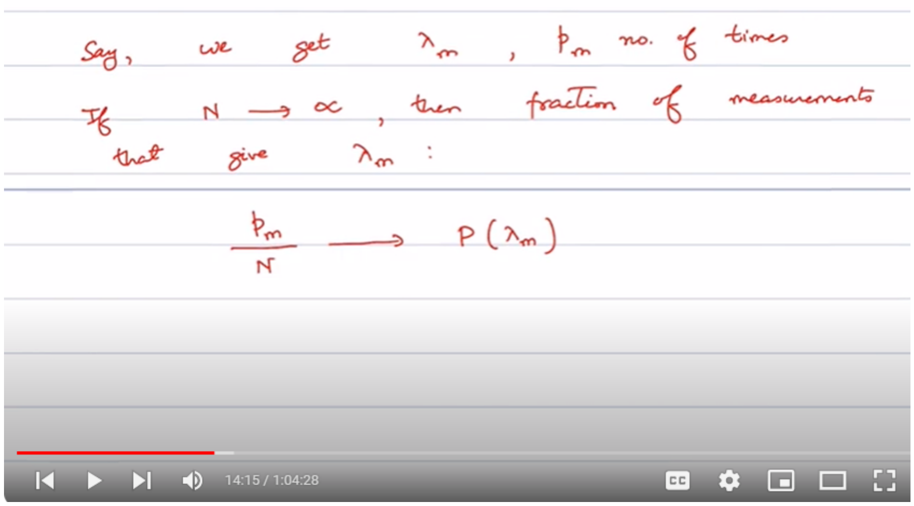

Then if suppose some eigenvalues because of this numerous number of measurements some eigenvalues you are getting again suppose some number of times let us say say you get suppose say we get say we get the eigenvalue lambda m pm number of times pm number of times okay that's the frequency at which we get the eigenvalue pm lambda m pm number of times we get now if say n is very large suppose n tends to infinity the number of copies of the system is very large then the fraction of measurement then fraction of measurements fraction of measurements so this is fraction of measurements that give that give lambda m is simply we get it pm number of times and n number of copies we have right so this is what we are going to have and this basically is nothing but the probability of getting the eigenvalue lambda m.

I hope you are getting the idea in other words what i mean to say is that quantum mechanics tells us the rate at which we will obtain a particular outcome when we have an infinite number of copies of the same exact systems okay.

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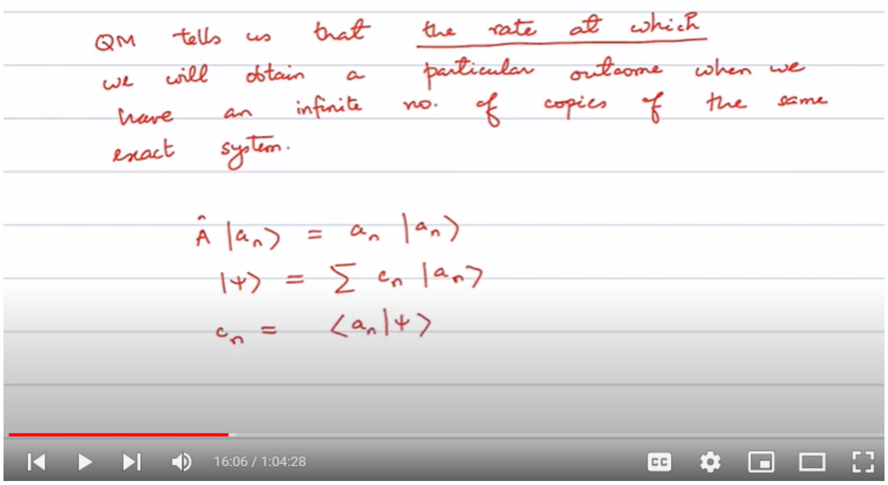

So basically quantum mechanics tells us quantum mechanics tells us that the rate this is important the rate at which the rate at which we will obtain we will obtain a particular a particular outcome when we have an infinite number number of copies of the same exact system this is important to understand because the essence of measurement is hidden here in this particular statement that i have written here let me make a little bit of more elaboration here.

Suppose if we have just one copy right rather than n number of copies if we just have one copy we can only know the probability of getting a particular outcome so if we have say this eigenvalue equation say a n a n this means that if the system is in an arbitrary state ket psi and which i can write it as a superposition of the eigenkets like this where cn is a complex coefficient and by now you know that cn i can write it as a n psi right.

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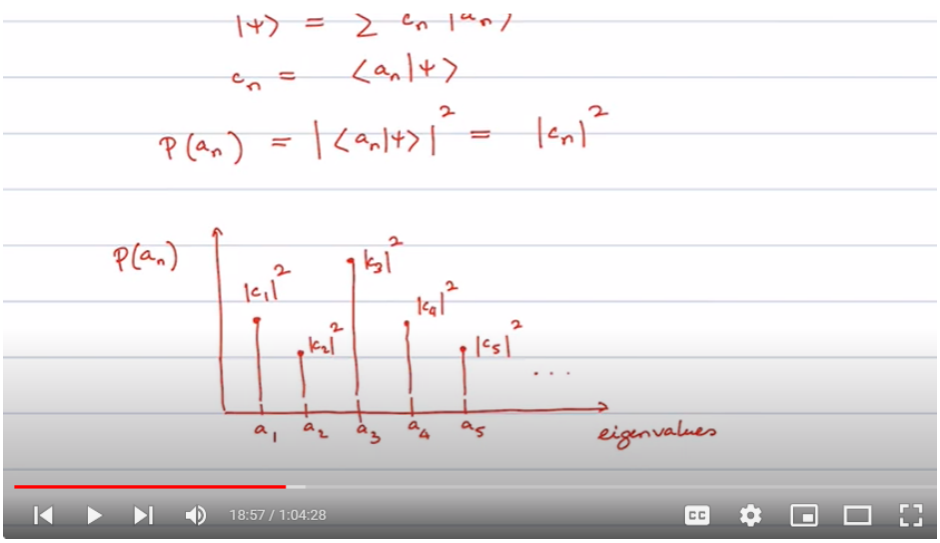

So the probability of getting the eigenvalue a n is simply would be mod of a n psi whole square and this is nothing but cn mod square. So now the process of measurement can be actually this described in a picture pictorially i can describe it as follows suppose let me let us say on let me draw the x-axis for x-axis and y-axis on the x-axis let me put the eigenvalues okay suppose i have these eigenvalues in the x-axis and okay let us say eigenvalues are say a1 a2 a3 a4 and so on like this a5 and so on we are having all the eigenvalues on the x-axis and along the y-axis let me put the corresponding probabilities suppose p of a n that's the probability of getting a particular eigenvalue so and the system is in an arbitrary state ket psi and these eigenvalues you know depends on the operator a only okay it depends on the operator a only and it does not depend on the state ket psi now each eigenvalue has a particular height given by the square of the absolute square of the coefficient suppose the probability of getting a1 is going to be given by c1 mod square and in this prob in this diagram it would have some height or corresponding to a1 the probability is mod c1 square corresponding to a2 the probability would be say c2 mod square corresponding to a3 let us say it is c3 mod square and so on corresponding to a4 it may be the height may be this much c4 mod square for a5 it may be like this right it would be c5 mod square it would be c5 mod square and so on that's how you will get so basically what you are getting is the is an distribution it's a probability distribution that you are basically obtaining here and higher the height higher is the probability of obtaining that particular eigenvalue so if we have a very large number of copies of the exact system that means ideally say n tends to infinity.

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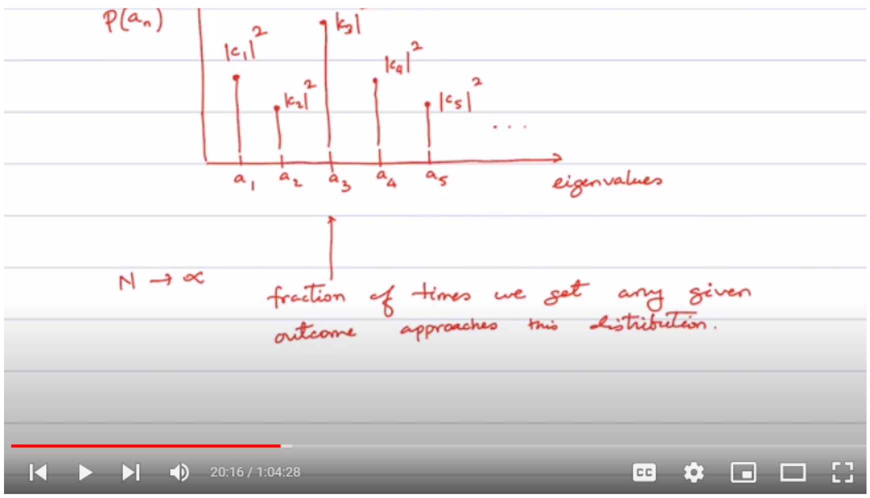

If n tends to infinity a very large number of exact copies of the system we have the fraction of time we get any given outcome approaches this particular distribution okay the this particular probability distribution means the what i mean to say is that the fraction of times we get any given outcome any given outcome approaches this distribution. I hope you get the idea here however for any individual measurement we cannot know beforehand what will be the outcome okay unlike classical physics in quantum mechanics we cannot predict the precise outcome of measurement of a physical quantity instead what quantum mechanics tells us is the precise probability distribution of all possible outcomes of that measurement.

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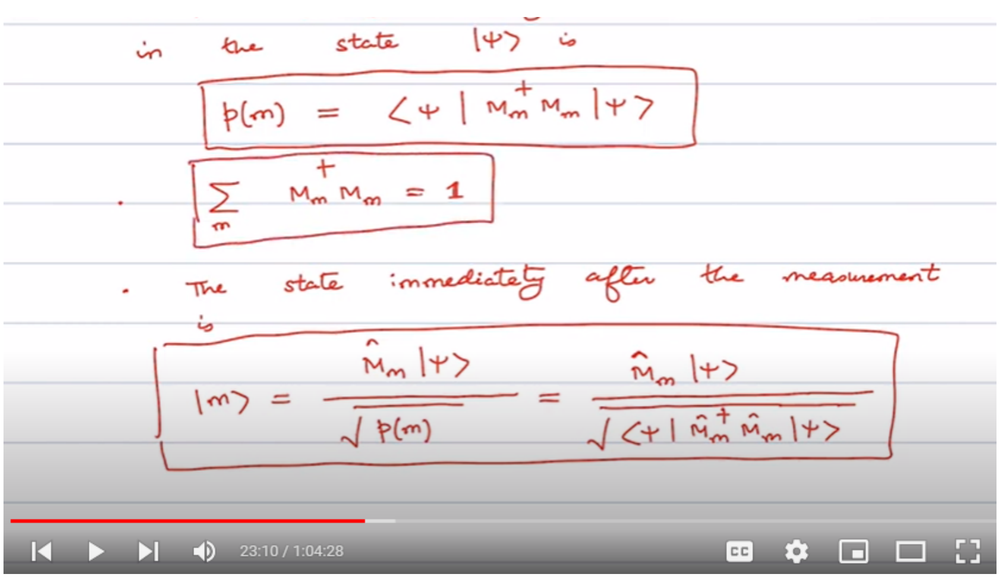

Now to be more formal we construct a measurement operator say mm such that the probability such that the probability of obtaining an outcome probability of obtaining an outcome say m in the state ket psi is given by this expression p of m which is probability of obtaining the outcome m is given by the expectation value of m m dagger mm this is the expression and here mm as i said is the measurement operator and this measurement operator satisfy the completeness condition sum over all indices m small m m m dagger mm is equal to identity and the state which we can get immediately after measurement is given by the state immediately after the measurement is this will be more clear to you if i give you an example i will give you very soon now the state immediately after the measurement is ket m is equal to equal to mm ket psi this is the measurement operator mm and divided by square root of p of m okay or if i write the full expression then this would be mm ket psi and here i have expectation below this m m dagger mm i'm not going to write the operator sign again and again so i'm just writing here but later on i will avoid to put a uh you know this sign here cap sign okay now let me give an example so these are these two are very important result this is also in fact this is also these three are very important result in the context of measurement operators.

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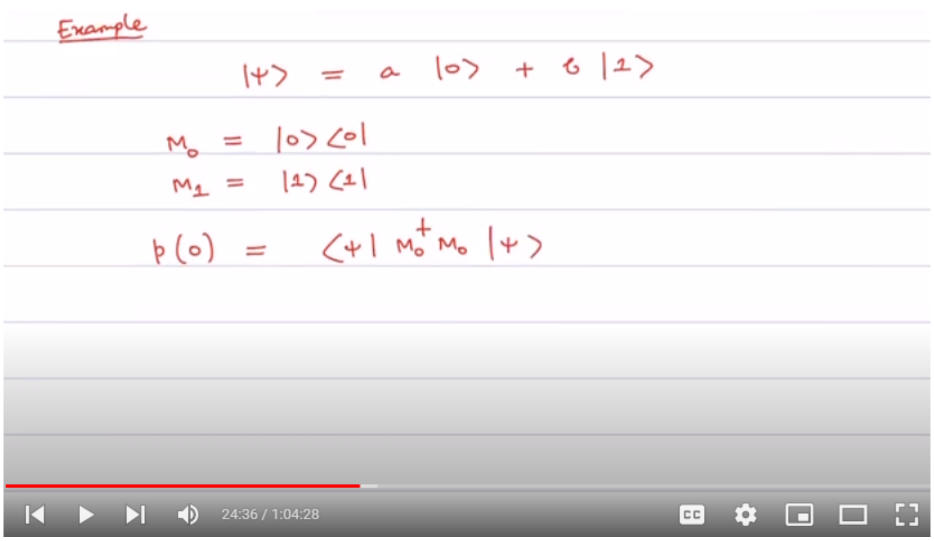

To give you an example let us say we have a superposition state like this ket psi is it's a qubit state it's a superposition of ket 0 and ket 1 a and b are the complex coefficient and we measure this particular state to see if it is in the state ket psi ket 0 or ket 1 okay so the measurement operators here we define them like this we have we have two measurement operators and these are the projection operators one is ket 0 bra 0 like this and another measurement operator is m1 that would be ket 1 bra 1 these are the two projection operators and getting the outcome 0 that means getting the the system in the state ket 0 so we said it is p of 0 there's a probability of getting the state to be in the in the ket state 0 is given by this expression now it will be m0 dagger m0 ket psi.

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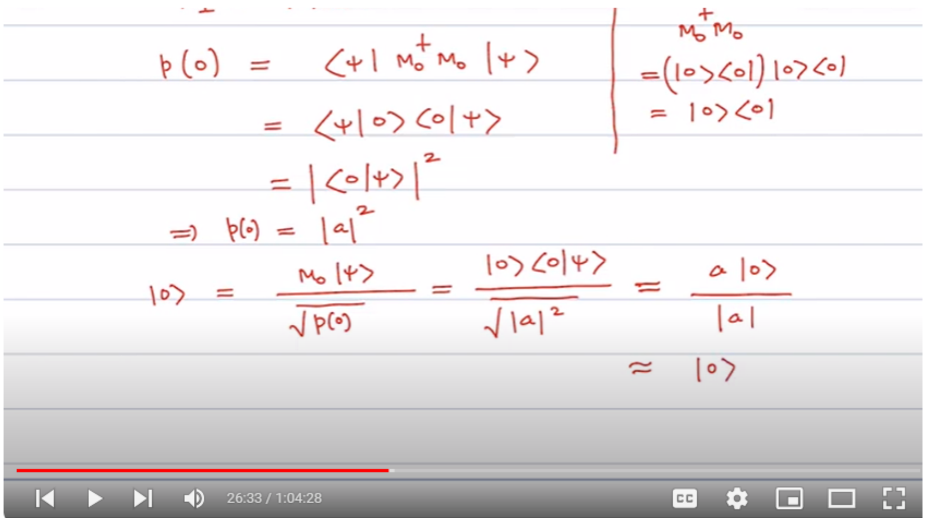

Okay let us open it up so if you open it up first of all as you can see m0 dagger m0 is simply it is m0 dagger is m0 is ket 0 bra 0 right and m0 dagger is the opposite of that so that would be simply this one this is your m0 dagger and which is same as m0 and now we have simply ket 0 bra 0 and therefore here let me put ket 0 bra 0 psi and this guy is nothing but modulus of modulus square of this quantity and this is again nothing but mod of a square this is basically a known result to us i'm just showing you the application of this measurement operator and as i said the state immediately after the measurement is now ket 0 so as per our definition we have m0 ket psi divided by p of 0 square root of p of 0 which is we have m0 is ket 0 bra 0 applied on ket psi and divided by square root of mod a square right p p0 this we have already worked out and this implies that we are going to have it would be simply a0 right divided by mod mod a okay so this is actually nothing but this is nothing but the state ket 0.

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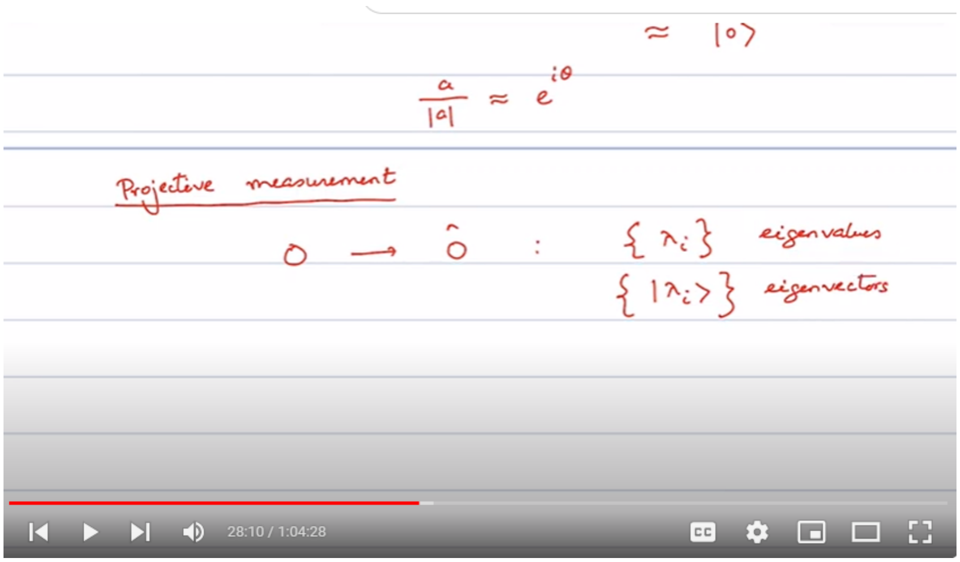

Similarly for the other case m1 we can have it now here you just have to note that the quantum state is defined up to a phase and here this quantity a by mod a it can be at the max e to the power some some phase factor would be there and and anyway this phase factor as you know it's not going to play any role here.

Now let us discuss about the projection measurement and which is a special case of the generalized measurement scheme so we are now going to discuss about projection or basically not projection let me the terminology appropriate terminology would be projective measurement projective measurement which is a special case of the generalized measurement okay so say to understand that let us we have an observable o and which can be of course observable o it's represented by an operator o cap in the quantum mechanics and this has say eigenvalues are say lambda i and the corresponding orthonormal eigenvectors are represented by ket lambda i like this right this is what the set of eigenvalues corresponding to the observable o and these are its eigenvectors.

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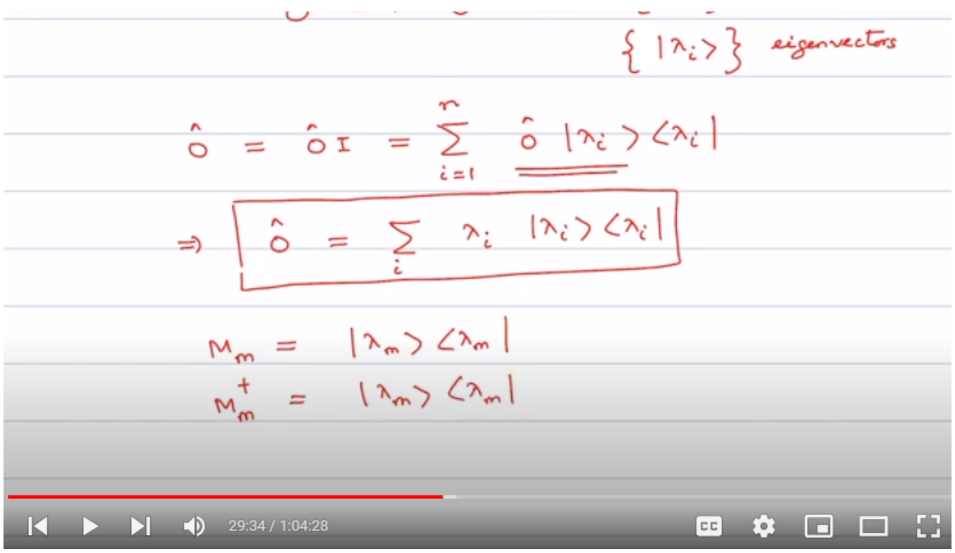

So in that case i can now represent this operator the so-called spectral decomposition i can do this we have done and it's actually it's very simple also you can have i is equal to 1 to say n o cap and this is your lambda i lambda i okay this is the completeness condition i am using o lambda i because the eigenvalue equation is going to be satisfied so therefore i can write o operator o cap is equal to i sum over i here it will be lambda i this is an eigenvalue it is lambda i ket lambda i bra lambda i okay the operator anyway i can represent it in this form it's called known as the spectral decomposition of the operator o now let us make the measurement operator i am talking about projective measurement so let me define a operator m measurement operator in this form so lambda m lambda m and the corresponding uh its hermitian conjugate would be again as you can see it would be ket lambda m bra lambda m.

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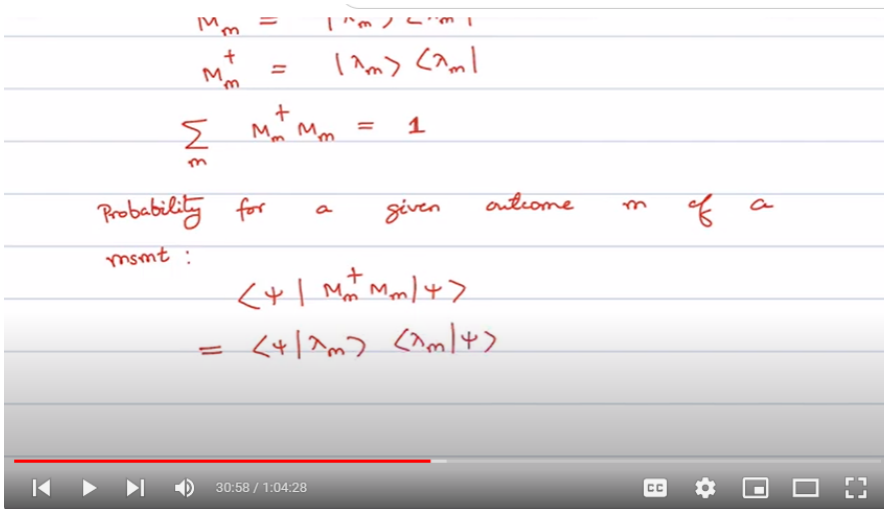

It's very easy to see that sum over all this thing is nothing but if you just do it m m m dagger m m you will find that this is identity so therefore this particular property of measurement operator is anyway satisfied by this projective operator also.

Now the probability for a given outcome it's a kind of repetition but we are doing it in the context of projective measurement probability maybe later on it will be more clear to you if i give more examples and i will do that so for probability for a given outcome for a given outcome m of a of a measurement let me write m as m t for measurement is then expectation value as i we have this result general result this is what you will have and this in the case of projective or this thing what we are having this is psi and this is lambda m right and this is again lambda m ket psi.

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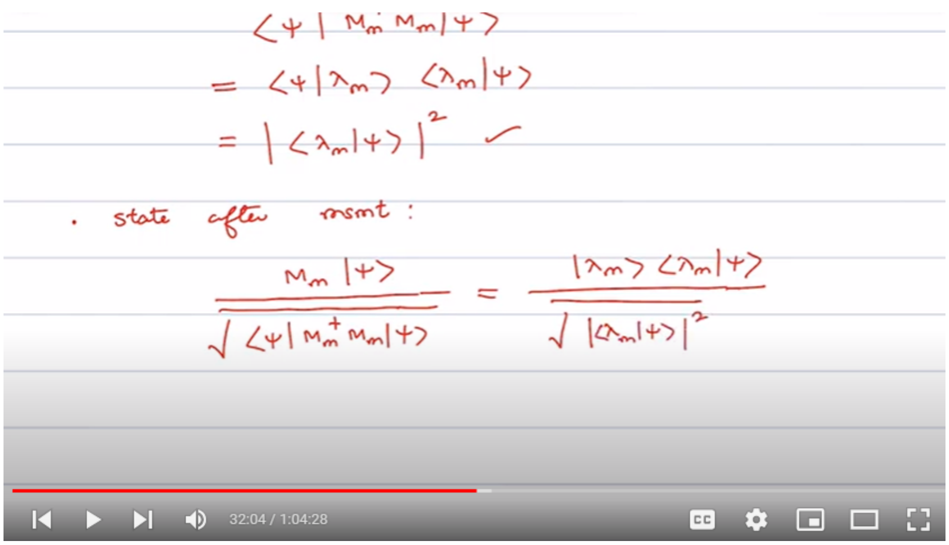

This guy is nothing but mod of mod square of this quantity lambda m psi mod square okay which is exactly the absolute square of the expansion coefficient of ket psi for that eigenstate.

Now what about the state after measurement so this is one result we have and state after measurement msmt that means measurement is going to be m m ket psi divided by square root of p of m which is here psi m m dagger m m ket psi right so if i open it up then you will get it would be lambda m lambda m ket psi divided by square root of this result already we have so let us put it that would be modulus square of this quantity inside the bracket lambda m psi mod square.

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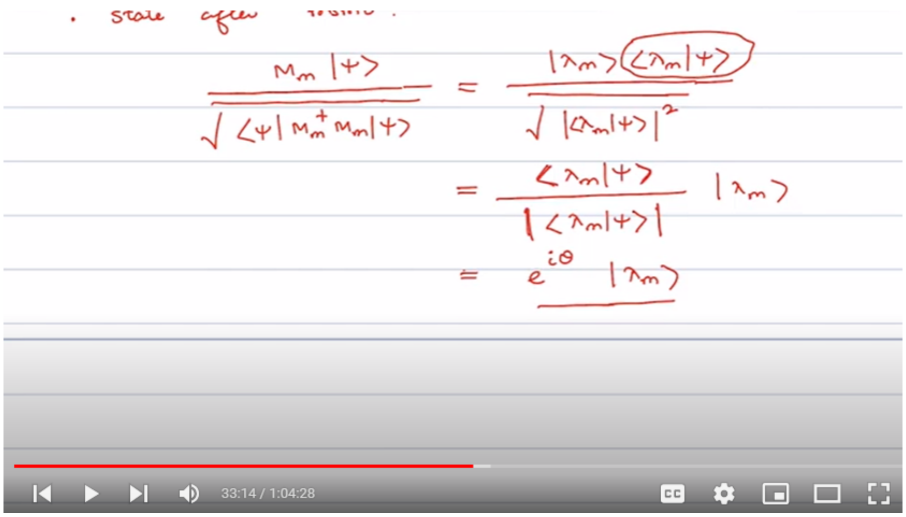

So from here i get it as this is just a number right this is just a number so let me write lambda m psi it's a complex number divided by lambda m psi okay and this is lambda m ket lambda m and this as you can see it is nothing but e to the power i theta because this is a complex number i can always write it as e to the power i theta into the modulus of this quantity this modulus get cancelled out and you'll be left out with e to the power i theta lambda m okay here theta is an arbitrary phase and just like lambda ket lambda m is the is an eigenstate similarly e to the power i theta ket lambda m is also an eigenstate of the observable o or the operator o cap.

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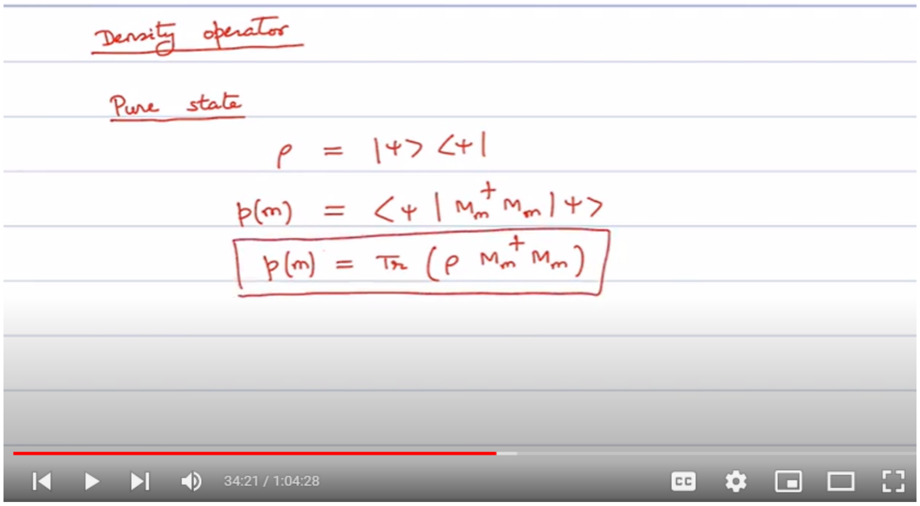

Okay so this generalized measurement scheme can easily be actually generalized to density operator as well so let me now extend this concept to the case of density operator formalism also extended to density operator case first let me do that for pure state that is very easy and simple actually it's very straightforward and as you know a pure state is represented by density operator rho is equal to this ket psi bra psi and therefore the probability of an outcome p of m is going to be this is our result from generalized measurement scheme so i can now write it in the using because this is nothing but the average right kind of expectation value and we know from our density matrix formalism this i can write is as trace rho m m dagger mm so this is the result i have in the in terms of density operator now.

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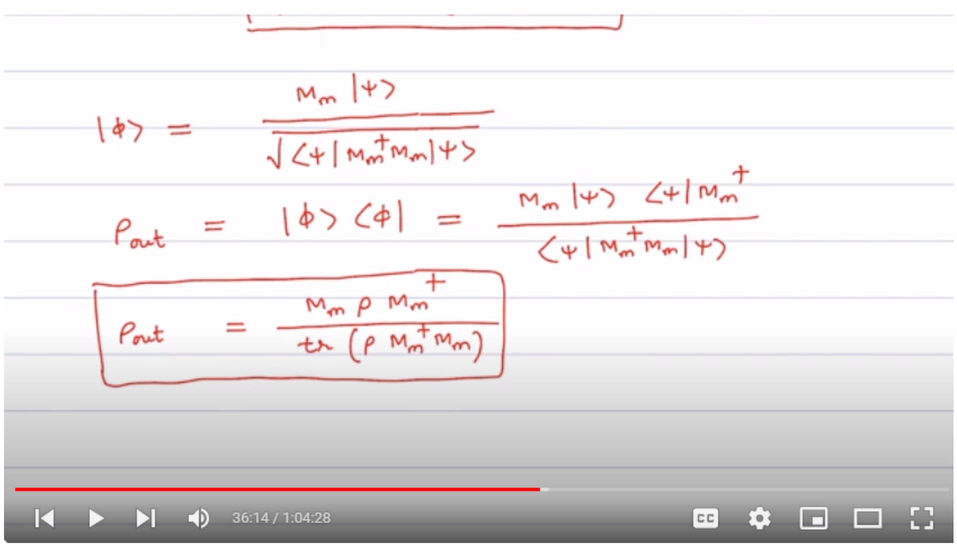

Now what about the state when the state changes after a measurement with result m the density operator the density operator corresponding to okay first let me write the state after measurement as you we know that we are going to get the state to be say this one let me name it as say ket phi after measurement i get the state to be like this i have this psi m m dagger mm ket psi the corresponding density operator that means at the output after the measurement i would get you will just for pure state is very simple this is what you will have and if i open it up what i am going to get is this m m ket psi this bra psi m m dagger divided by in fact you will get two terms and square root will go away and you will be left out with m m dagger m m ket psi very straightforward i think all of you can see this and this would be the output that means the after measurement the state would be the density operator would be m m rho m m dagger divided by trace of rho m m dagger mm okay so this is also an important result and this we have got in the context of a pure state but the extension to the mixed state is also very straightforward and in the case of mixed states we are going to get actually the similar result.

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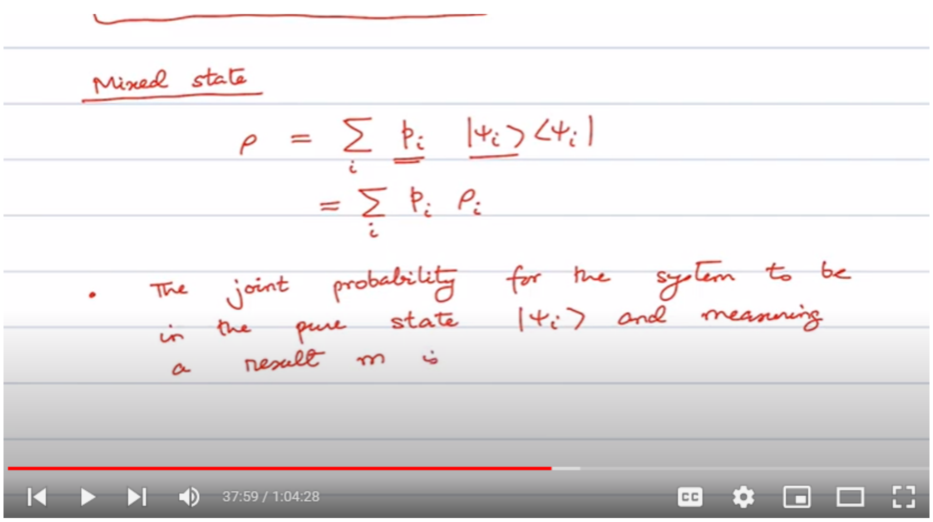

So let me just quickly discuss that as well for mixed state our density operator for mixed state is given by this expression p i is the probability ket psi i ket bra psi i so this is the density operator for the mixed state and this actually i can write as sum over i's this is probability this is probability p i and this i can write the density operator for the pure state gets psi i right ket psi i i can write it the density operator rho i is referring to the density operator corresponding to the pure state ket psi i so this is the expression i can write now the join probability let me write because you see in the case of mixed state two probabilities are involved here one is the so-called classical probability and then the quantum case which we have discussed in great detail when we have discussed density operator formalism now the join probability join probability for the system for the system to be in the pure state to be in the pure state ket psi i.

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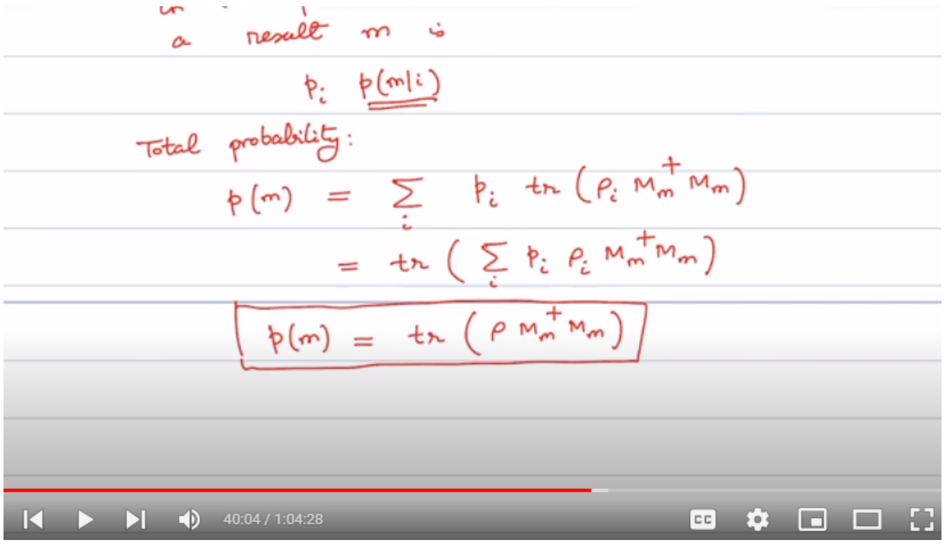

And measuring the result measuring a result say m is it would be first this is the probability p i that is the probability to pick up the pure state ket psi i and then getting the result m outcome m is p of m at say i right when you have picked up ket psi i and then getting the outcome so this is basically the join probability and now you have to sum over all i's to get the total probability for getting the outcome m as a result of measurement so total probability total probability total probability of getting the outcome m right because you have so many pure states are involved here every pure state is designated by one particular ket psi i so you have to sum over all i's there and if you sum over all i's then here you have p i and this quantity for pure state case you have already know that will be trace of rho i m m dagger m m so this is what you will get.

And this is very straightforward and very simple because of trace operation i can write trace here and put the summation side inside the trace operation trace operator then we have sum over i i have here p i rho i please note the symbol carefully okay this rho i'm writing kind of an in italicized way for you know density operator thing and i have here m m dagger m m and this is nothing but as you can see this is trace of rho rho m m dagger m m okay so as you see once again we get the familiar expression that we have obtained for pure state also same expression similar expression.

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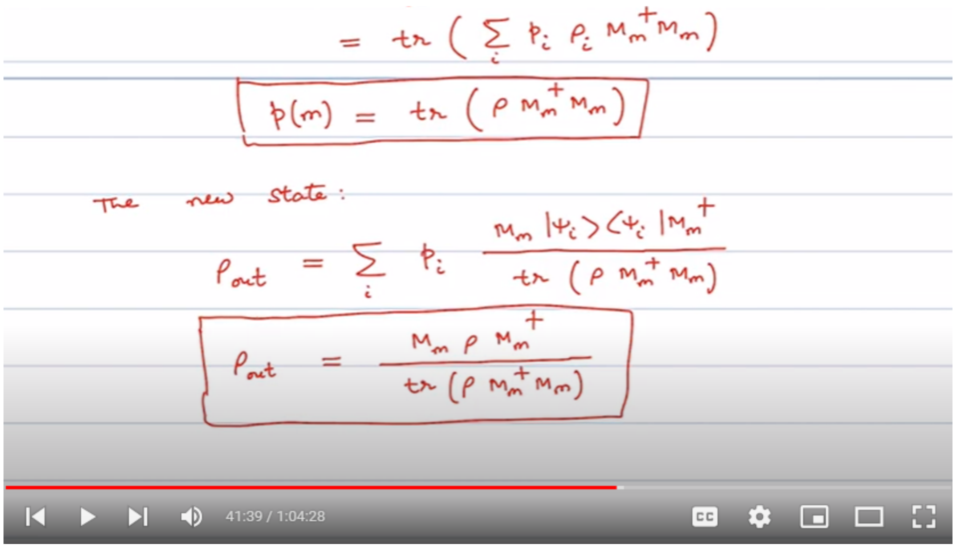

The new state after measurement of the mixed state becomes this also you are going to get exactly the similar one i don't want to elaborate on it more but you can check it yourself this would be again you have to sum over all i's here you have this p i and then you have m m ket psi i right and psi i m m dagger this we have done for pure state case and here you have trace of rho m m dagger m m and this is going to result in this particular expression m m rho m m dagger divided by trace of rho m m dagger m m so here only expression looks similar only point you have to keep in mind is that this density operator rho that i am writing here refers to the mixed state and if it is a pure state then you have to write the density operator pertaining to the pure state okay. So as you can see the general measurement scheme covers all operations that can be performed on a quantum system.

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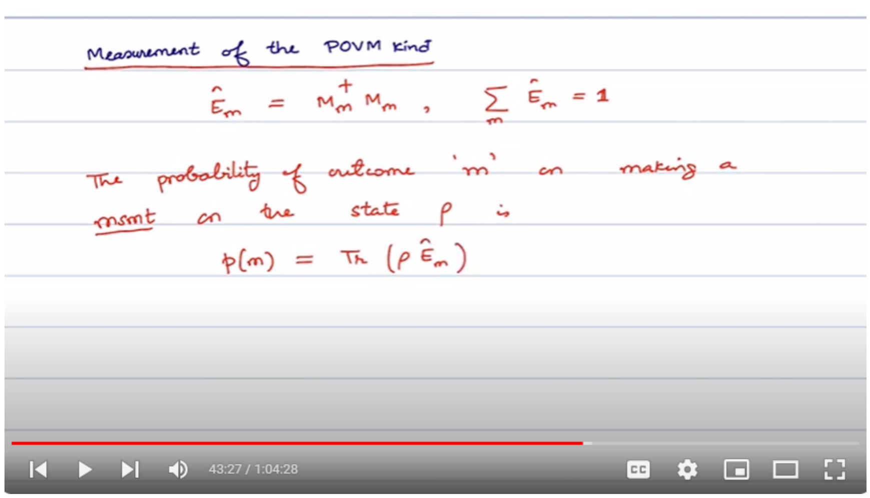

Many times we are not interested in the post measurement state of the system but we are interested in the statistics or the relative probabilities of outcome that we can collect by making a measurement on an ensemble this we can generally do by the so-called povm or positive operator value measurement formalism let us discuss it measurement of the povm kind let us consider the set of operators say em is equal to m m dagger mm and of course all measurement operators whatever it is they have to satisfy this particular condition so this has to be identity and the probability of outcome the probability probability of outcome m right on making a measurement on making a measurement let me write m as mt so this is i mean by this i mean measurement on the state measurement on the state rho is p of m is equal to trace rho em this already we know right and obviously for pure state.

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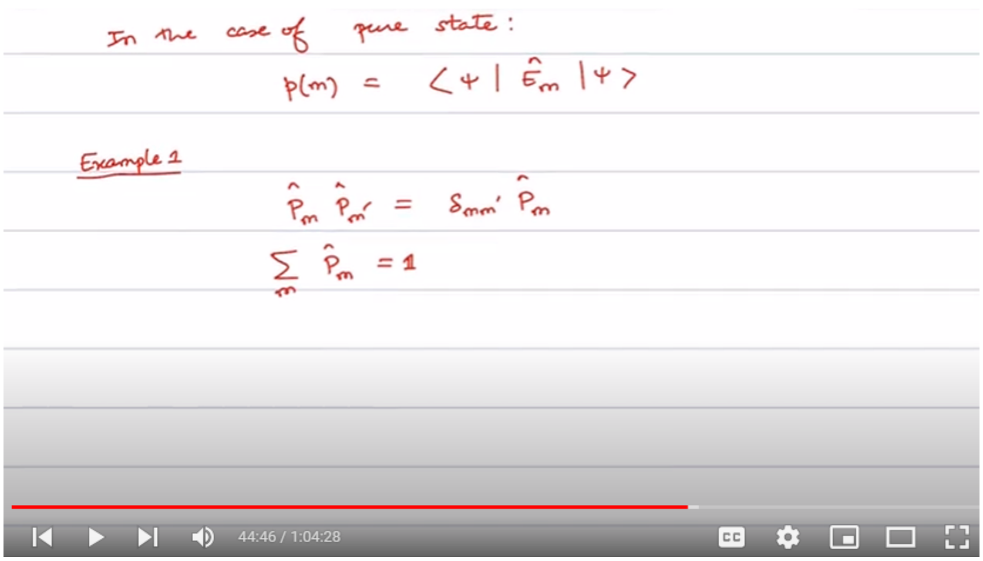

In the case of pure state in the case of pure state this particular expression will simply boil down to expectation value of this operator em in the state ket psi okay let me illustrate this povm by an example couple of example first let me begin with a trivial example say consider a projective measurement described by measurement operator pm these are projectors such that pm pm dash because you know the projection operators are orthogonal orthonormal delta m m dash pm this we already know and also sum over these operators is equal to 1 or identity.

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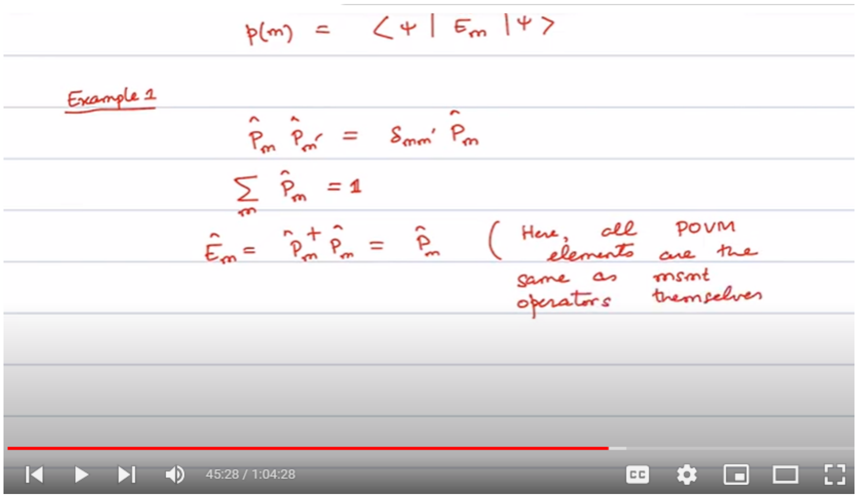

In this case all povm elements are same as measurement operators themselves because here em is equal to pm dagger pm and which is nothing but pm itself right so here in this particular example all povm elements all povm elements are the same as measurement same as measurement operators themselves operators themselves.

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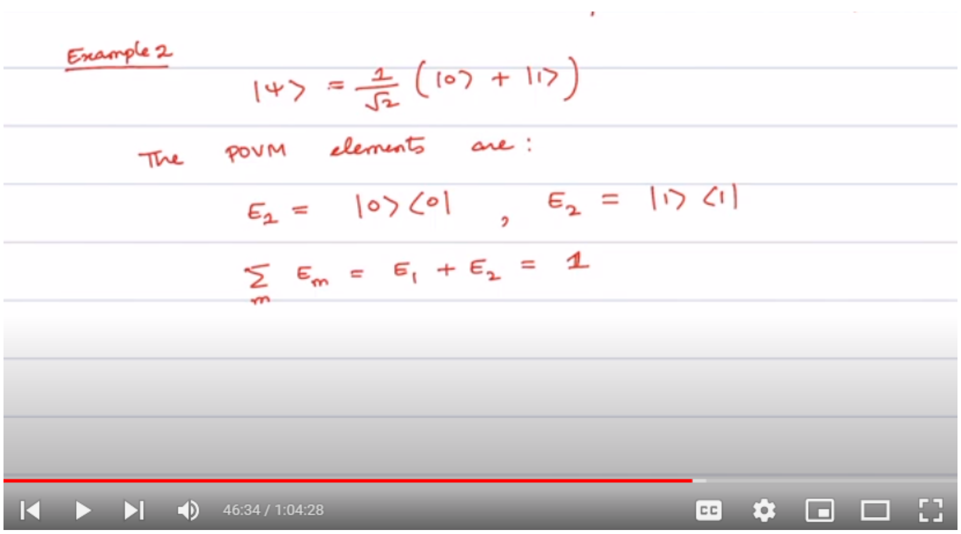

Maybe let me give you an another example this may not be very appropriate to you so let us say another example consider this one let us say ket psi is equal to a state is given it's a qubit system it is in a superposition of ket 0 plus ket 1 this plus ket 1 this qubit state what are the povm elements here it's also a trivial and you know what are the povm elements here povm elements are two elements are there one is projector ket 0 bra 0 and another one is e 2 is ket 1 bra 1 and it's very clear that sum over all these projectors which is basically e 1 and e 2 e 1 plus e 2 should be equal to identity event right.

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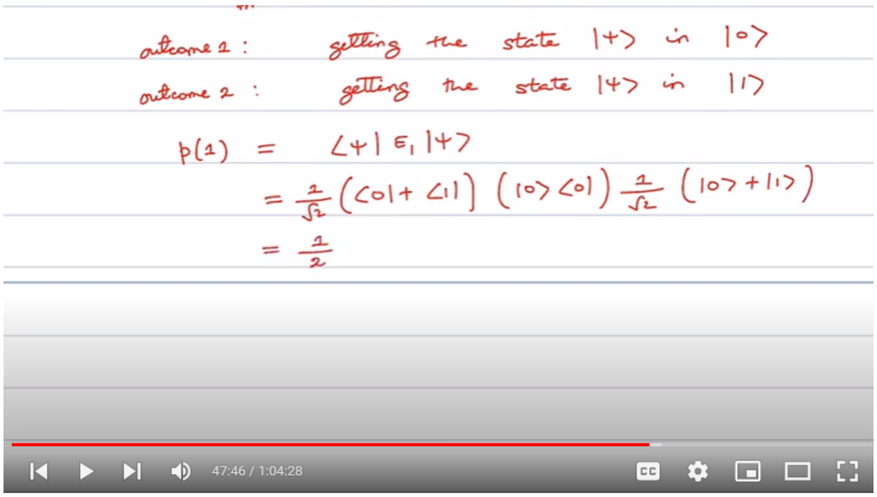

And there are two outcomes of the measurement so outcome outcome one two outcomes first outcome is say getting the state ket psi this qubit state in the state ket 0 and the second outcome that say outcome number two is getting the state ket psi in ket 1 okay and what about the corresponding probabilities probability of getting the outcome one is given by this expression and you can easily work it out it's very trivial so let me still work it out it is 1 by root 2 bra psi is bra 0 plus bra 1 here and e 1 is this projector ket 0 bra 0 and ket psi is 1 by root 2 ket 0 plus ket 1 and as you can very easily see that this is going to be equal to simply half right.

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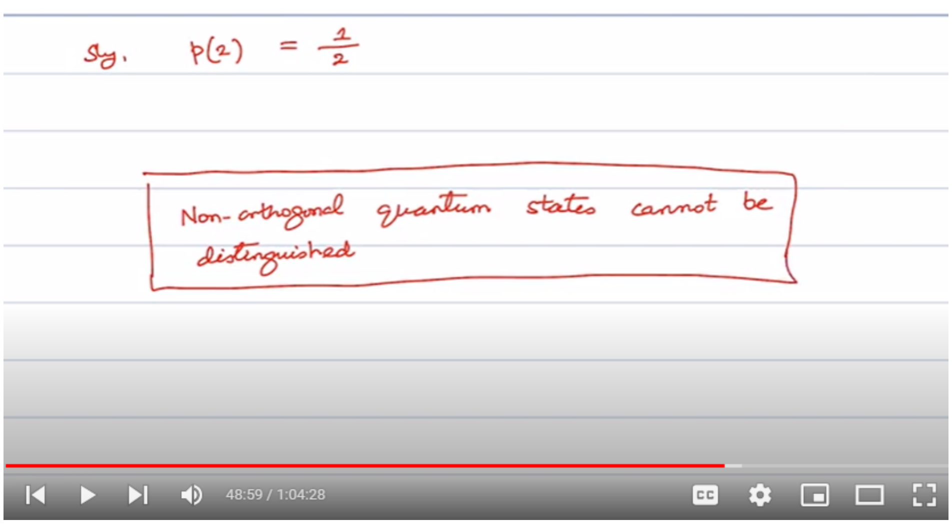

So similarly you can make it out that probability of getting the second outcome is also half and these results are already known to you because there is a 50 50 probability of getting the state either in ket 0 or ket 1 i just have illustrated you only by this example.

You can see the importance of povms when i talk about one important theorem or in quantum mechanics in particular related to measurement problem and this theorem is this that non-orthogonal quantum states non-orthogonal quantum states cannot be reliably distinguished non-orthogonal quantum states cannot be distinguished and this has very important replication in quantum information processing and many experiments so let us actually prove it.

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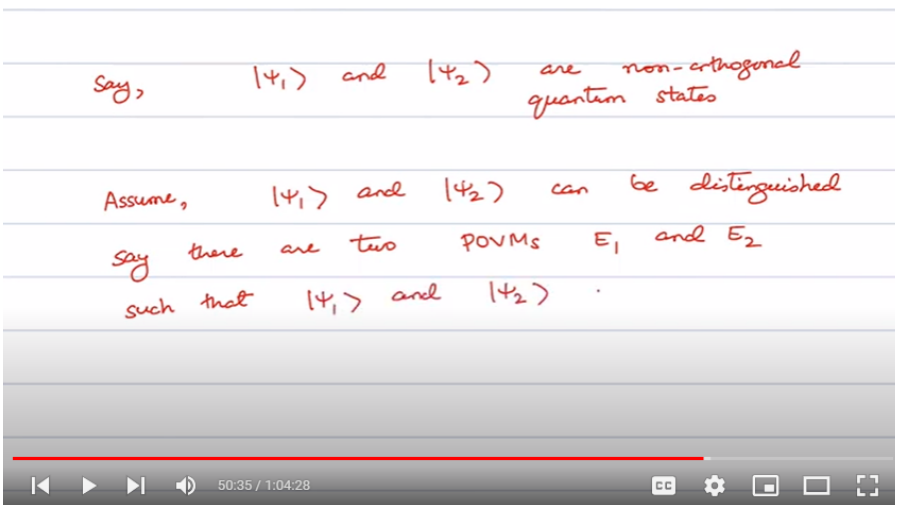

To prove that let me let me argue it otherwise let us say we have ket psi 1 and ket psi 2 and these are two non-orthogonal states say ket psi 1 and psi 2 are non-orthogonal non-orthogonal quantum states okay they are non-orthogonal quantum states and we will assume rather than assuming that because we have to prove that they cannot be distinguished so let us assume otherwise that assume ket psi 1 and ket psi 2 can be distinguished okay can be distinguished if this is not we are going to actually going to get some kind of a contradiction.

Now say there are two povms there are two povms two experiments are there to say there are two povms two measurements e1 and e2 such that such that they can be distinguished such that psi 1 and psi 2 can be distinguished can be distinguished reliably.

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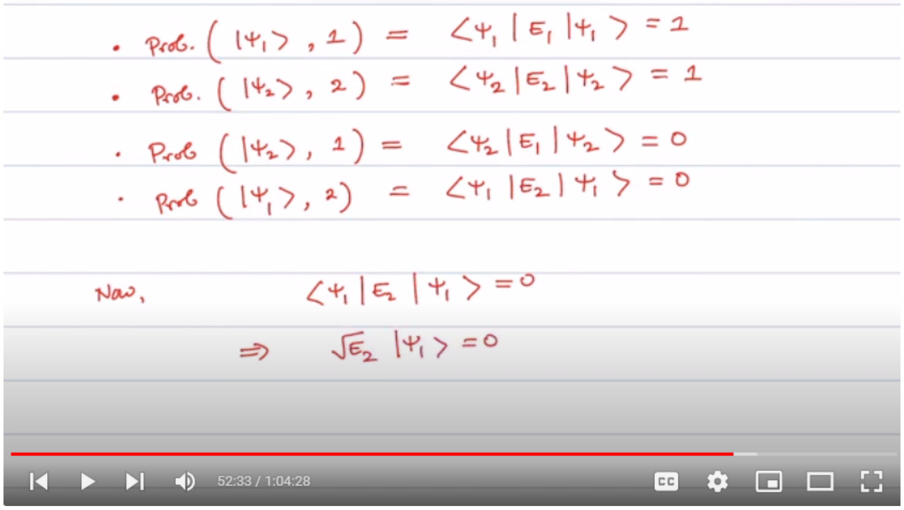

What does that mean this means that the probability of measuring ket psi 1 and getting the outcome 1 is 100 so you are going to get psi 1 e1 psi 1 is equal to 1 and probability of measuring ket psi 2 and getting the second outcome outcome 2 is also 100 that means psi 2 e2 psi 2 is equal to 1 or that also mean this is important results that also mean the probability of measuring psi 2 and getting the outcome is has to be is 0 so psi 2 e1 psi 2 is equal to 0 and probability of measuring the state measuring ket psi 1 and getting the outcome 2 is going to be 0 psi 1 e2 psi 1 is equal to 0 okay so this is what we mean that if psi 1 and psi 2 can be distinguished reliably by two povms e1 and e2. Now you see let us analyze it a bit further now we have this expression psi 1 e2 psi 1 is equal to 0 from here one can easily obtain square root of e2 psi 1 is equal to 0.

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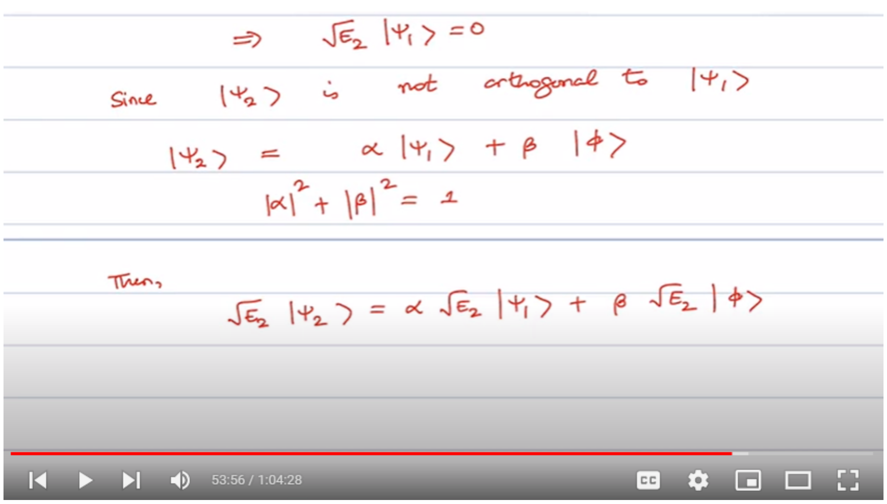

This is trivial to see this is what you are going to get now since ket psi 1 psi 2 is not orthogonal is not orthogonal to psi 1 ket psi 1 so therefore psi 2 can be decomposed actually into two components one component parallel to psi 1 and another component orthogonal to psi 1 so one component orthogonal parallel to psi 1 and other component is say phi which is orthogonal to psi 1 so this is what you should get of course with the condition that mod alpha square plus mod beta square has to be equal to 1.

So what is what is the you know result of this thing this means that then what you will have then you can see that square root of e2 psi 2 psi 2 i'm now writing it as alpha psi 1 plus beta phi so let me just write it alpha square root of e2 psi 1 plus beta square root of e2 ket phi okay.

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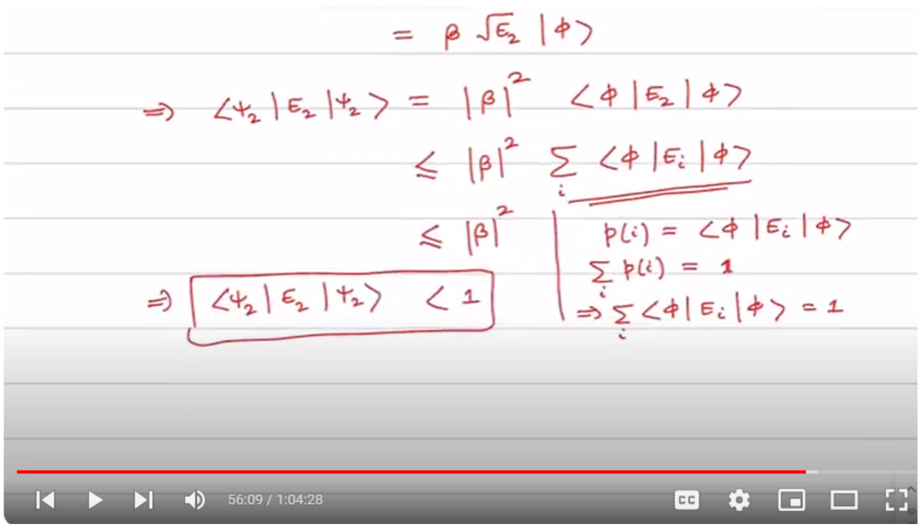

Now this is going to be equal to simply beta beta square root of e2 it would be beta square root of e2 phi because this guy is anyway equal to 0 by the end of this expression right so therefore we'll be left out with this one and this implies from here i can write psi 2 e2 psi 2 right this would be equal to mod beta square and you will have phi e2 phi i think this is also trivial to see and of course this has to be less than or equal to mod beta square if i take sum over all e i's all e's then i have this right e i phi.

Okay now this has to be less than or equal to mod beta square why because because you know the probability of the outcome i is in the state shape ket phi because of the measurement e i is this and sum of all probabilities is has to be unity one okay so therefore it means that you should have sum over i phi e i phi is equal to one okay this is going to be equal to one so that's why it is less than mod beta square but this has to be less than one because mod alpha square plus mod beta square is equal to one so mod beta square has to be less than one okay but okay so what you get is this result you get but earlier what you have obtained this you got but earlier you got this expression that this you have taken it to be equal to one because you were able to distinguish psi 1 and psi 2 that is that is your assumption but what you have what resulted because of all these assumption you are getting in contradiction. So this is this is a contradiction this is a contradiction so what does it mean this means that we cannot we cannot reliably reliably distinguish distinguish orthogonal states orthogonal states actually non-orthogonal states right we cannot reliably distinguish non-orthogonal states ket psi 1 and ket psi 2.

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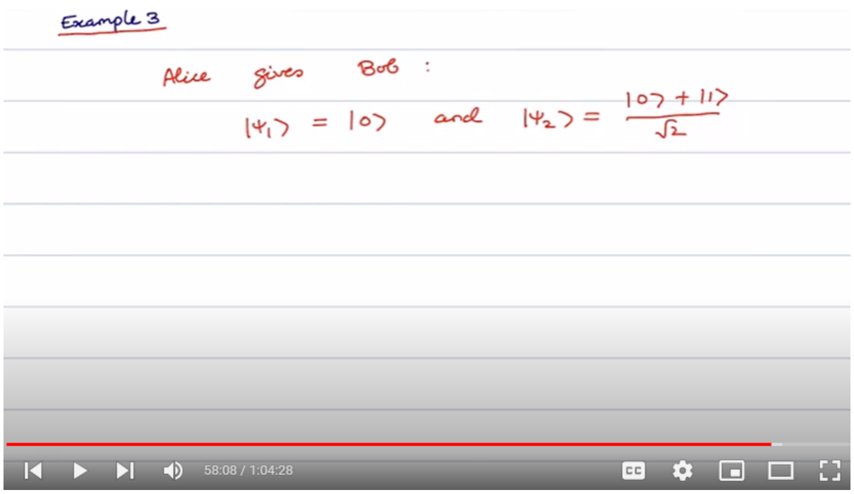

Now as a final example to illustrate POVM formalism let me give you this example let us say Alice gives Bob a qubit prepared in one of the two states that may say psi 1 is equal to ket 0 and ket psi 2 is equal to ket 0 plus ket 1 by root 2 okay now you can see that ket psi 1 and ket psi 2 are non-orthogonal so Bob find it impossible to determine whether he is given ket psi 1 or ket psi 2 with perfect reliability however it is possible to for Bob to perform a measurement which distinguishes the state some of the time but never makes an error of misidentification.

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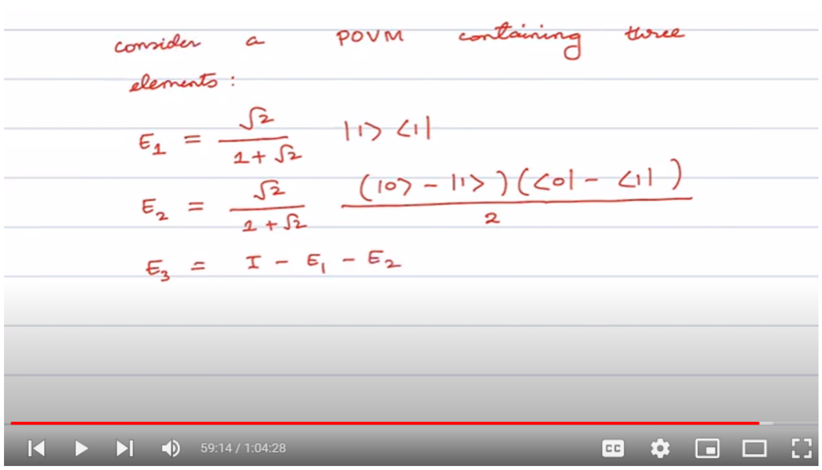

To understand that let us consider a POVM containing three elements consider a POVM containing three elements containing three elements very cleverly chosen elements given as this e1 is equal to root 2 divided by 1 plus root 2 ket 1 bra 1 this is one measurement one POVM element other one is e2 is equal to square root of 2 by 1 plus root 2 okay ket 0 minus ket 1 bra 0 minus bra 1 divided by 2 and e3 is equal to I minus e1 minus e2 .

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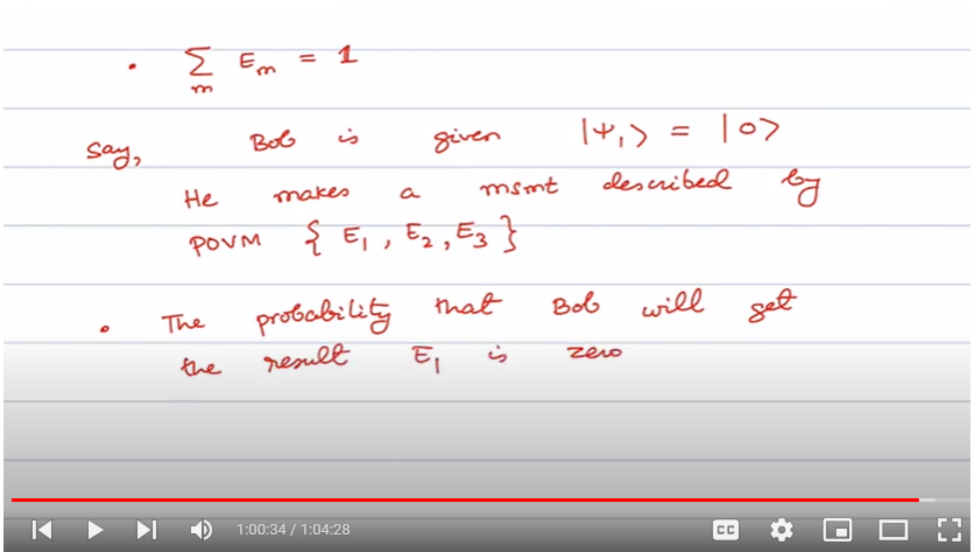

It's very trivial and straightforward to see that sum of all EMs because this guy has to be satisfied whenever you are considering some measurement operators this has to be equal to 1 so they form a legitimate POVM.

Now say Bob is given the state ket psi 1 okay ket psi 1 is equal to ket 0 he makes a Bob makes a measurement he makes a measurement described by this measurement described by these three POVM elements POVM which are e1 e2 and e3 okay.

Now the probability the probability probability very easy to see that probability that Bob will get the result e1 is 0.

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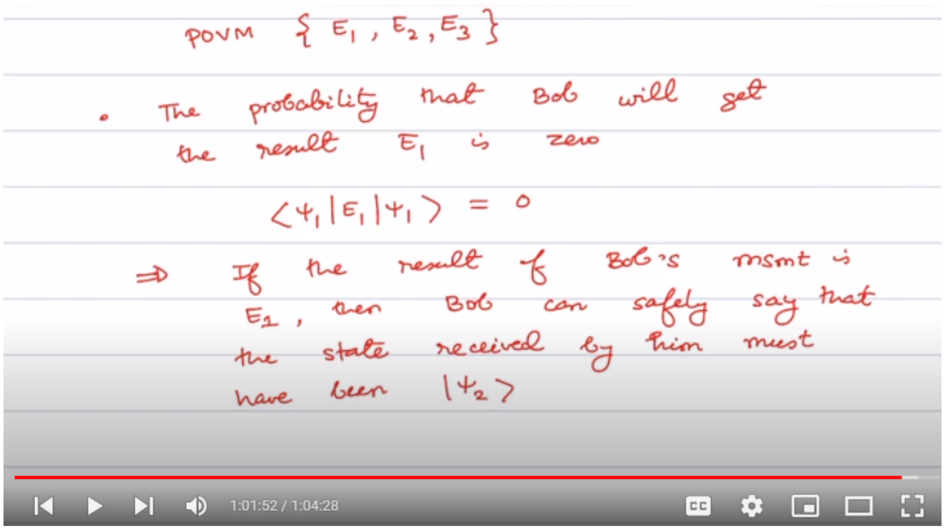

You can see that because if you find out this expectation value psi 1 e1 psi 1 if you work it out because e1 already I as I have defined here you see e1 if you take the operations ket psi 1 is ket 0 you can immediately see this is going to be 0 what does it mean this implies that if the result of his measurement is e1 if the if the result of Bob's measurement if Bob's measurement is e1 then then Bob can safely say Bob can safely safely say that the state provided by state received by received by him or provided by Alice to him must be must have been what must have been psi 2 not psi 1 right that's what it mean and in fact you can see if you can work out psi 2 e1 psi 2 you will find that this is going to be non-zero so therefore Bob has received the step psi 2 instead of psi 1.

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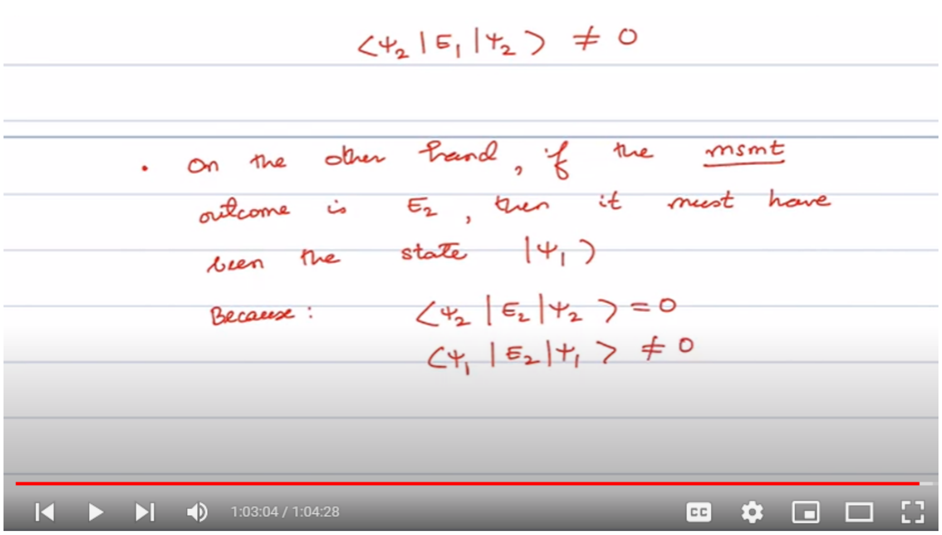

On the other hand on the other hand on the other hand if the measurement if the measurement outcome is e2 measurement outcome is e2 then then it must have been have been the state ket psi 1 that Bob has received right because because if you again mathematically see that if you calculate psi 2 e2 psi 2 you will find it to be 0 and you will find psi 1 e2 psi 1 is non-zero similar argument now sometime what may happen is this sometime Bob may get may get e3 and then Bob would not be able to then Bob cannot distinguish or cannot actually conclude cannot conclude what state is given to him what state is given to him I hope by this example you have seen the power of POVM measurements let me stop for today in this lecture we have discussed about quantum measurement process in the next lecture we are going to discuss about entanglement measures related to discrete variables so see you in the next lecture thank you so much.