Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture –5
Two-level Systems- II .

Hello! Welcome to the third lecture of this course. In the last lecture we discussed a generic 2-level system and without any external driving. In this lecture let us see how a 2-level system behaves under the influence of time dependent driving. So, let us begin.

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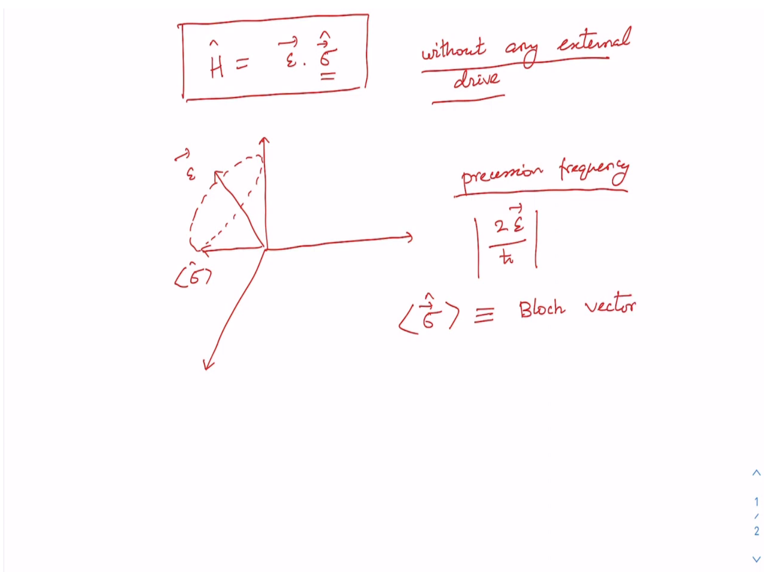

In the last class we established that the Hamiltonian of a generic 2-level system will now, say TLS, is given by or we can actually express it or put in this particular form say H is equal to epsilon dot sigma where sigma is the Pauli vector and this is a time independent Hamiltonian and this is without any external drive. And from our analysis we learned that this Pauli vector, this sigma, precesses around this epsilon vector.

And we plotted it, actually we can plot the expectation value of sigma and it turns out that it precesses around this vector epsilon with precession frequency twice epsilon by h cross, mod of twice epsilon by h cross. So, in fact the dynamics of a 2-level system when it is not externally driven can be discussed or described by evolution of these Pauli matrices. So, this quantity, expectation value of sigma, is known as the so-called block vector. So, once we know this block vector effectively we know everything about a 2-level system.

Now let us see what happens if a 2-level system is put under driving. So, 2-level system under time dependent driving, so, this is what we are going to discuss today in this class and you will see very nice physics will come out because of that.

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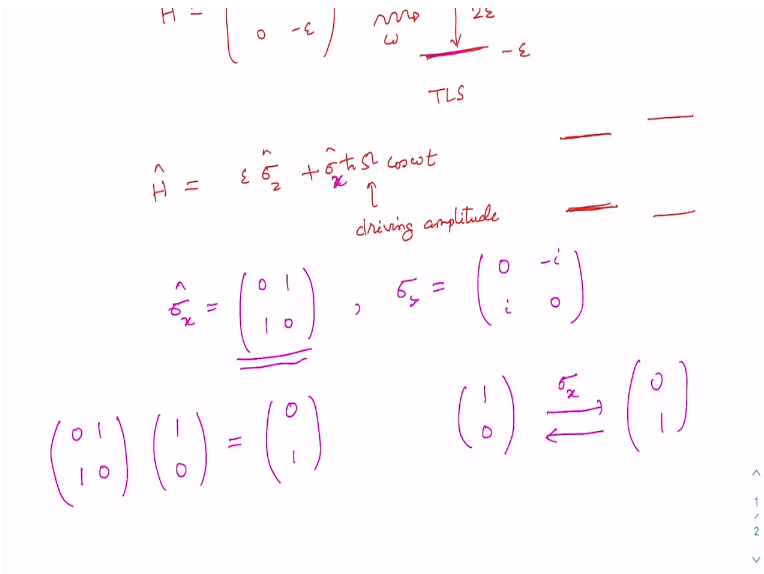

So, what we will do we will begin with a Hamiltonian which is already diagonalized. So, let us say our Hamiltonian is of this from say epsilon sigma z. Now at the moment we are not applying any driving by the way and this sigma z is you know is the z component of the Pauli matrix, it is 1, 0, 0, -1. So, in matrix form this Hamiltonian basically have epsilon, 0, 0, minus epsilon. So, this or we are having a configuration like this where the ground state has say energy is represented by minus epsilon and plus epsilon. So, energy gap is twice epsilon.

Now let us drive this 2-level system externally that means suppose we apply some kind of a time dependent or a pulse. Suppose we are throwing at this 2-level system then what happens in that case? So, in that case we can have it like this; sigma z plus now you are applying a time dependent driving suppose we are applying a cos sinusoidal wave like this cos omega t and so it has this driving has frequency say omega. This is the driving frequency and you need to have it as the dimension of energy.

So, let us say it has energy h cross omega where omega is the so-called driving amplitude and it has to be a 2 by 2 Hamiltonian again. So, let us say I have sigma z. But, if I have this kind of driving because sigma z as you see here it has these diagonal elements only. So, 1 and -1, that means it can only affect this energy levels and by applying this driving we are not going to get any interesting physics as such because what it basically means that these energy levels would be shifted upward and downward like this, no kind of no transition is going to takes place.

So, if we really want to have nice physics out of it. So, what we will do? We will apply a drive like this say it would be sigma x or sigma y because we know that if I have sigma x and this Pauli matrix is basically you have it is 0, 1, 1, 0 or even we can have sigma y also because it has this off diagonal elements are non-zero there. So, this you know it is basically going to induce transition between the energy levels.

So, we can for example if it is in the ground state, if I represent the ground state by say 1, 0 or if a state is like this and if I apply this Pauli matrix sigma x then we immediately will get that it would become simply 0, 1 that means you are going from 1, 0 to 0, 1 under the application of this Pauli matrix sigma x. That means the transition is going to takes place from say ground state to the excited state or excited state to the ground state and so on..

So, therefore this is going to be an interesting Hamiltonian to study here. So, we will consider our Hamiltonian to be of this type. Let us proceed further.

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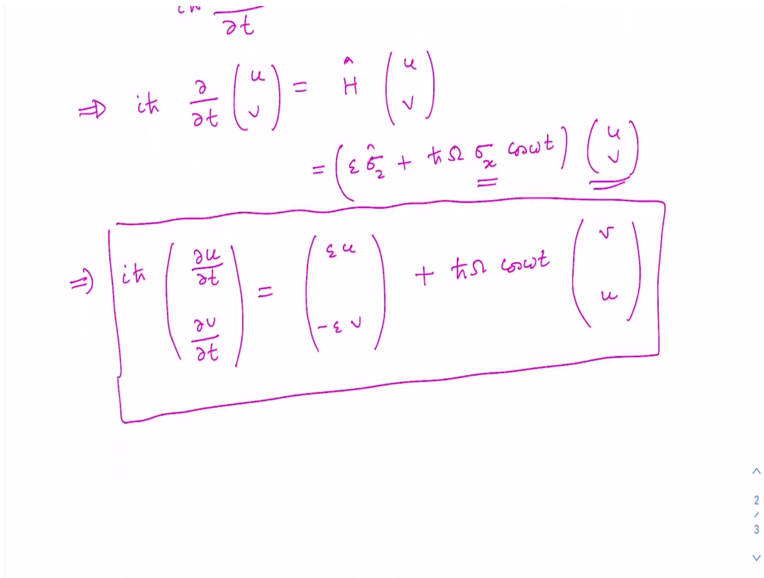

So, I have my Hamiltonian now like this. So, I have epsilon sigma z plus h cross omega sigma x cos omega t. So, let me tell once again that this h omega is the driving, h cross omega is basically the driving energy and I can name omega as driving amplitude and omega is the driving frequency okay. So, this is what I have. So, this is the Hamiltonian now we are going to analyze okay. First of all, what we will do, we will try to solve the so-called Schrodinger equation.

So, Schrodinger equation i h cross say del psi del t time dependence Schrodinger equation we have to solve because we have now a time dependent Hamiltonian. So, this is we have H psi which psi is a column vector. So, let us say it has this component like this u, v and then I can write it as i h cross delta t u v is equal to H u, v. These are operators. So, please pardon me if I do not put the operator but you should understand that I am discussing quantum mechanics.

So, this Hamiltonian this is an operator here. Now let us apply this Hamiltonian. I have epsilon sigma z plus h cross omega sigma x cos omega t u, v. So, what I am going to get? I will simply get i h cross, I will have del u del t here del v del t and you can see that epsilon sigma z if it is applied there you will simply get it as epsilon u minus epsilon v, the first term on the right hand side and the second terms because sigma x basically takes this reverse this elements there.

So, you will get it as h cross omega cos omega t and here you will get v and it would be u okay.

So, this is what we will get. So, basically, we get 2 set of equations, 2 couple equations we get.

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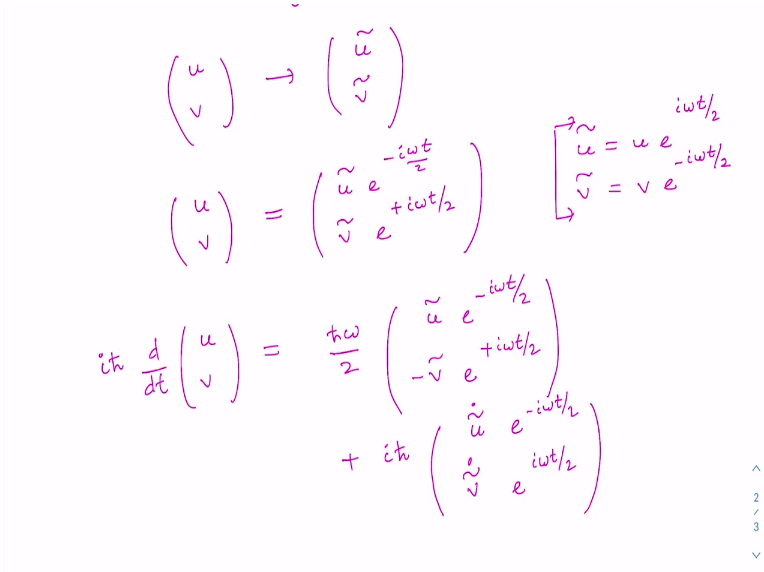

And now this equation is basically not analytically solvable though it looks a very simple equation, these 2 coupled equation; one you will get from the first row and another one you will get from the second row. These are coupled equations but because of the presence of this term this equation is not analytically solvable. So, we will apply a trick here and this trick basically says that, we will know more about it later on, it says let us go into different variables or go into a rotating frame of reference rotating frame of reference because I just want to get rid of this time dependent part ultimately and we'll see how by clever trick we can do that. So, let us go from this variable u v to some other variable u tilde and n v tilde where I defined my new variable u tilde v tilde like this and they are connected to all variables u v like this.

So, u is equal to u tilde e to the power -1 omega t by 2 and v is v tilde e to the power +i omega t by 2. So, I am just changing the variables like this, u I am taking it like this or u tilde is basically u e to the power or I can say u tilde is equal to u e to the power + i omega t by 2 or v tilde is equal to v e to the power -1 omega t by 2. This is the connection between this new variable and the whole variable. Now, if I do that and then if I you know, put it in this couple differential equations then let us see what we are going to get.

So, let me go by term by term first of all I have i h cross d of dt u, v. So, I will get if I just take the time derivative what I will get is h cross omega by 2 and here I will get u tilde e to the power -1 omega t by 2, you can try to do it yourself, v tilde +i omega t by 2 and you will also get i h cross, I am now talking about the left hand side of this Schrodinger equation and here you will get i h cross u tilde time derivative of u tilde and e i omega t by 2 and this would be v tilde e to the power i omega t by 2 okay.

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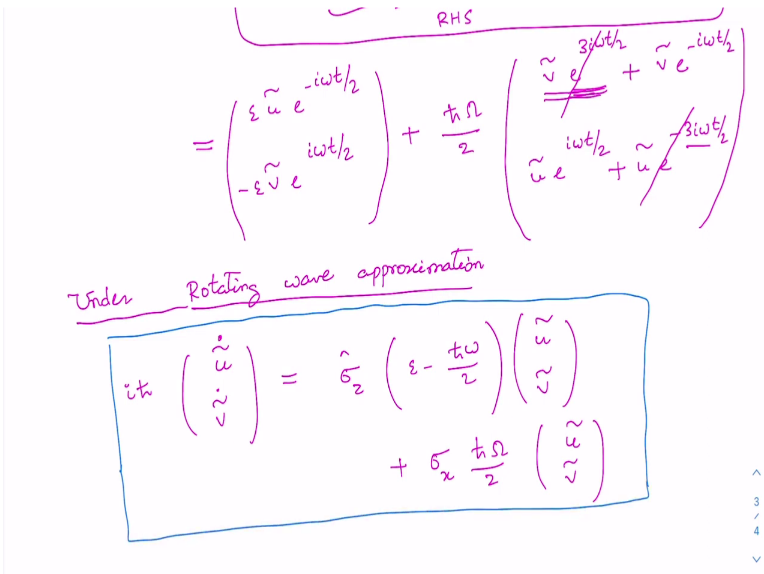

Now going further, so this is the right-hand side, what about the left hand side? The left-hand side okay, now I have this. So, let me this is what. So, this has to be equal to the left hand side I had, let me just keep it like this epsilon u minus epsilon v okay u tilde e to the power -1 omega t by 2 and it would be v tilde e to the power +i omega t by 2. This is the first term on the left hand side and you have here h cross omega, you have this cos omega t term, so, this I can write it as this cos omega t term I can write it as e to the power i omega t + e to the power -1 omega t. So, this is what I will have and here I have as you can see v and u there. So, it would be v tilde e to the power i omega t by 2 and you have u tilde e to the power -1 omega t by 2. Now what we are going to see is that if you just look these both sides and this is your left-hand side and this is your right hand side.

You see left hand side and right hand side if you compare only the terms you know that is having this coefficient say e to the power -i omega t by 2 on both sides or say e to the power + i omega t by 2 as in the column vector in the lower term there that would become only resonant okay, they would become prominent because it would become resonant but there if you see from here okay I think let me explain it little bit more properly.

If you look at this term okay let me just write here this right hand side let me write again epsilon u tilde e to the power -1 omega t by 2 minus epsilon v tilde e to the power + i omega t by 2 and this term I can write let me keep h cross omega by 2 here and then I have in the upper here I have v tilde e to the power 3 i omega t by 2 and I will have plus say I will have v tilde e to the power -i omega t by 2 okay and in here in the lowers this one I will have u tilde e to the power +i omega t by 2 and the other term would be plus u tilde e to the power minus 3 i omega t by 2.

Now you can compare. Let me write the right, if you just look at the left-hand side also okay. Now we can compare left hand side and right-hand side okay. It is a bit difficult to write everything here but I hope you will get the point if you compare both sides then you see only the term which is having coefficient e to the power -1 omega t by 2 on both sides in the first row okay, on both left hand side and right hand side would become resonant.

And on the other hand, the terms that is having this, this is going to be out of phase and this would be non-resonant and in fact this particular term would not be able to contribute much. So, therefore we can cancel it okay and similarly here we can in the second row we can cancel this particular term which is having 3 omega t by 2 and then, this is basically known as the so-called rotating wave approximation.

So, under this rotating wave approximation where we have actually cancelled the highly oscillating term like this, 3 omega by 2 and 3 omega by 2 minus 3 omega by 2 and plus 3 omega by 2. This is known as rotating wave approximation and under rotating wave approximation I can now write my time dependent Schrodinger equation like this in the basis u tilde, in this variable u tilde and v tilde.

So, this is what I will have and so, let me just write the final equation. You will be able to, you should be able to show that you will going to get this equation, this is what you are going to get u tilde v tilde plus sigma x h cross omega by 2 u tilde v tilde. So, this is the key equations that we obtained, these are coupled differential equations we get under the so called rotating wave approximation.

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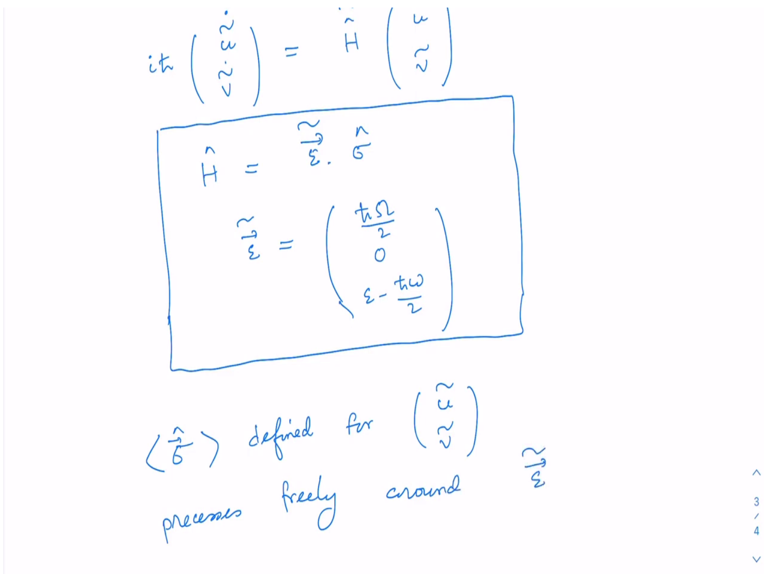

Now immediately you see that I can now write this in a compact form like this i h cross u tilde v tilde dot and this I have a new Hamiltonian now in this new variable u tilde and view tilde or in this rotating frame and where H I can express it as epsilon, I will explain it, epsilon tilde dot sigma where if you look at it this epsilon tilde vector okay, this has components like this you have x component, from here you see x component would be h cross omega by 2 it has no y component right, it has no y component it is zero and z component you can see from here it would be simply epsilon minus h cross omega by 2.

So, what you have seen that effectively I have now a time dependent problem. I am now turning it into a kind of a time independent problem by using the so-called rotating wave approximation okay and you immediately see that this block vector sigma okay, which is now defined for sigma now define say for in this new variable u tilde v tilde precesses freely around this new vector epsilon tilde okay.

So, very cleverly we have obtained or we are able to you know turn the whole time dependent problem into a kind of time independent problem using this rotating wave approximation and the new variable. Now let us analyze the problem further and let us see what happens at resonance.

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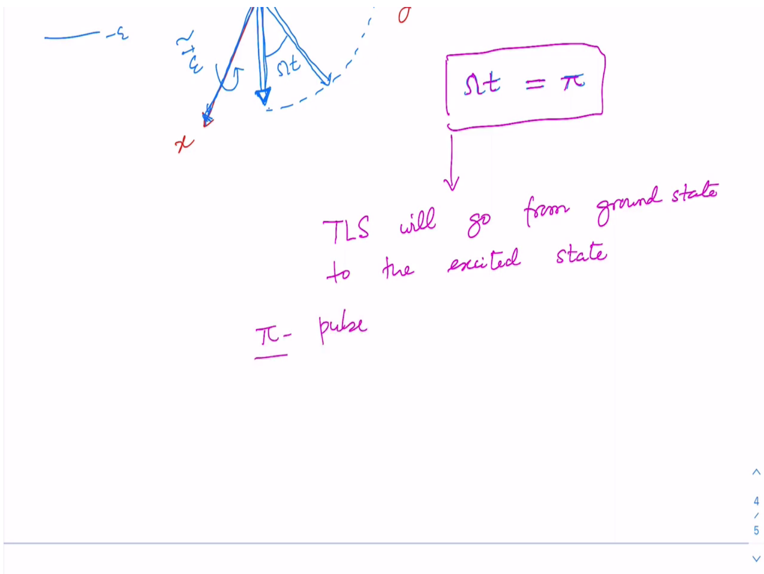

At resonance the energy gap between the energy level which is twice epsilon, that becomes equal to h cross omega or in other words when epsilon is equal to h cross omega by 2 we are having the condition of resonance and at the resonance this epsilon tilde vector would have only the x component. So, it would have x component as h cross capital omega by 2 y component is already taken to be zero and the z component is going to be zero because you see if it is not at a resonance the z component is non-zero, that is epsilon minus h cross small omega by 2 now at resonance this are equal.

So, therefore it would be zero okay. And the Hamiltonian would remain as it is in the same form that would be epsilon tilde dot this Pauli vector here sigma dot. So, as usual this Pauli vector is going to precesses around epsilon but the precession frequency in this case is easy to work out. Precession frequency, the Pauli vector precesses about this epsilon vector with frequency twice epsilon by h cross mod of that and because epsilon x component is h cross omega by 2 it has only x component.

So, therefore you can easily see that this precession frequency would be this capital omega which is we termed as the driving amplitude and this has a name and this is known as the so-called Rabi frequency, this is called Rabi frequency. In fact, what happens is this, let me explain it in a diagram. Let us say we have, okay first let me draw the coordinates, this is my x, this is my y and this is my z. So, as we now know that my epsilon vector is now epsilon tilde, in fact, epsilon tilde vector is now along x direction.

So, therefore the block vector is going to basically precess around this x axis. Let us say initially the 2-level system is in the ground state, that is, it is pointing along the negative z axis because you know the Pauli this vector has its component as sigma x sigma y sigma z now it is directed along z. So, it has only this z component and if it is directed along the negative z axis that is therefore it is in the ground state okay.

So, this is in the ground state okay, this is the situation we have. So, therefore this is also called block vector. So, we can plot only the expectation value of sigma which is we termed as block vector. So, this block vector is directed along the negative z axis initially and as time goes on what is going to happen is that this block vector will precess around this x axis in the y-z plane okay and then as time goes on it will precess in the upward direction.

Now at some time say t, the block vector is along this direction and. So, this here it has traversed an angle say capital omega t, that is what we will get after time t. Now if we wait for long enough time when omega t become equal to pi then automatically you will see that this means that this block vector will now will be directed along the positive z axis. So, when omega t is equal to pi either you wait for long enough time or you can wait for short enough time but if you increase that Rabi frequency.

But whatever it is if the product becomes omega t is equal to pi then the 2-level system will go from south pole basically or rather I would say will go from ground state to the excited state, all right. And omega t is equal to pi when this happens under this kind of a drive that is known as pi pulse all right.

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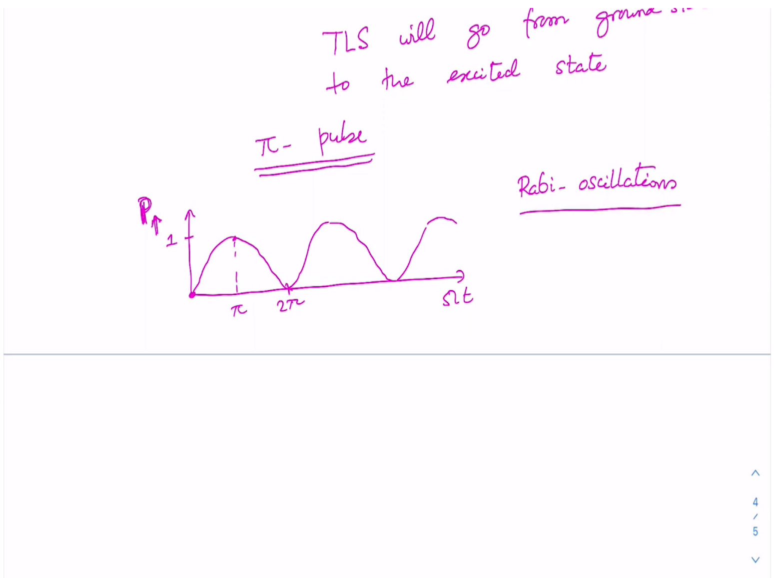

So, effectively speaking what is happening I can express these things in this way also initially we have started with say in the ground state. So, this is the probability of getting the 2-level system in the excited state. So, initially the atom is in the ground state. So, all the atoms are in the ground state. So, zero probability right, as time goes on when say x axis is omega t, when omega t is equal to pi the atom will go to the excited state, okay.

Now as time goes on this block vector will rotate, it will again keep on rotating and it will rotate along this direction and finally again it will go to the ground state. So, after another pi that means when omega t is equal to 2 pi the atom the, 2-level system will again go to the ground state and this process will go on and on. Of course, in this case we are assuming that there is no dissipation or there is no loss or any kind of things are there so therefore this oscillation will keep on.

And these oscillations are termed as Rabi oscillations and as you have already I say this oscillations is taking place at the frequency capital omega that is the Rabi frequency.

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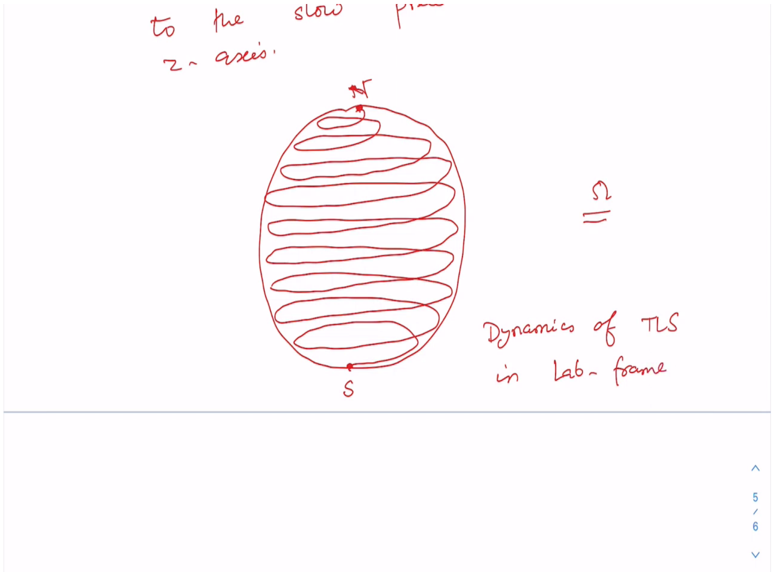

So, far we discussed the evolution of the 2-level system or in other words the block vector with respect to the rotating frame of reference. And there we saw that the block vector rotates about the epsilon vector, the block vector rotates about this epsilon vector with some precession frequency. So, this is what we have already seen and discussed.

Now the next question one can ask is what about the evolution of block vector as observed from the left frame? and let us discuss, this is not very difficult to understand what is going to happen. So, basically what we are going to do there we are going to find out the expectation value of this Pauli vector in the u v basis, that is our original basis. Now we know that u v is related to our new vector, this components u tilde v tilde by this equation e to the power -1 omega t by 2 and e to the power +i omega t by 2.

So, because of this when we calculate the expectation value of sigma Pauli vector in this basis then we see that this block vector is now going to rotate in the x-y plane with an angle omega t. So, the block vector will rotate or precess with angle omega t in the x-y plane. I will soon explain it actually in a diagram in the x-y plane. In addition because u tilde v tilde how it rotates we already know in the rotating frame.

In addition to the slow precision about the z-axis. In fact this slow precision already we learned there in this example here we see that this block vector originally in the downward direction it rotates in the y-z plane about this x axis or this epsilon vector there and that is a slow precision but this small omega here that is a fast rotation. So, overall this whole dynamics in the left frame will look like this.

Suppose we start in the ground state that means we start from the south pole in the block sphere and then there would be a very fast rotation in the x-y plane and this is going to, this block vector will go in the upward direction very slowly, it will be slow precision about the z axis and finally it will reach the north pole and this slow precision is going to takes place at the Rabi frequency omega as we have already known. So, this is going to be the kind of dynamics of 2-level system in lab frame, ok.

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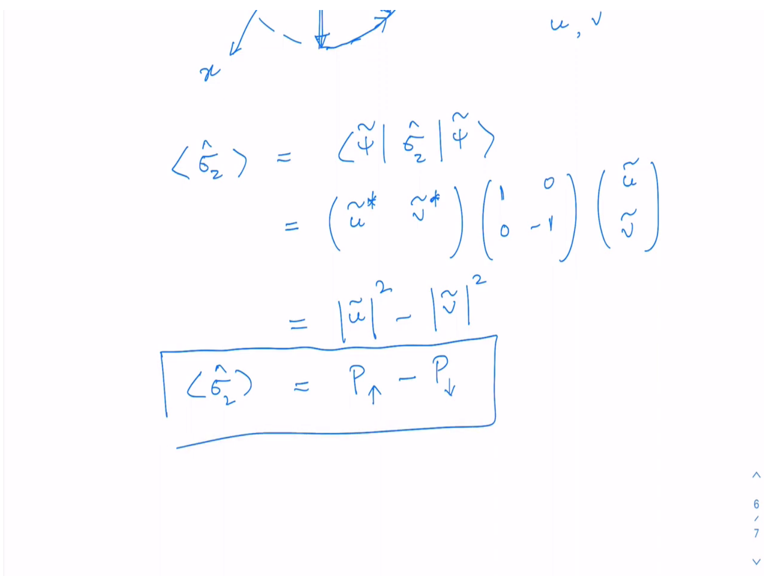

Now finally let us ask what is the probability to find the 2-level system in the excited state say at a given time. I mean can we get a quantitative expression, qualitatively we already know what we should get but let us see or what kind of behavior we get, already I discussed it. So, again going back to the rotating frame or working in the rotating frame of reference I can answer this question easily. Let us see.

So, this is what my block sphere is. So, initially this block vector is say in the ground state okay then it starts rotating and the question that I want to access what is the probability that it would be found at a in the excited state after some time t? This is what we want to find out. So, to do that now I can take the help of this diagram which already we learned in the rotating frame of reference. So, after some time t suppose this is the angle the block vector makes, that is omega t.

So, we know that this expectation value of sigma z, okay. Expectation value of sigma z if we calculate it in the u tilde v tilde basis that means psi tilde say sigma z psi tilde. We can work it out like this. So, this would be say u star v tilde star sigma z is 1, 0, 0, -1 and then you have u tilde v tilde and it is very trivial to find out that what you will get is u tilde mod square minus v tilde mod square.

So, which is basically the probability of finding the 2-level system in the excited state and this would be the probability of finding it in the ground state because sigma z you know these eigenvalues are 1 and -1. So, this is what we get okay.

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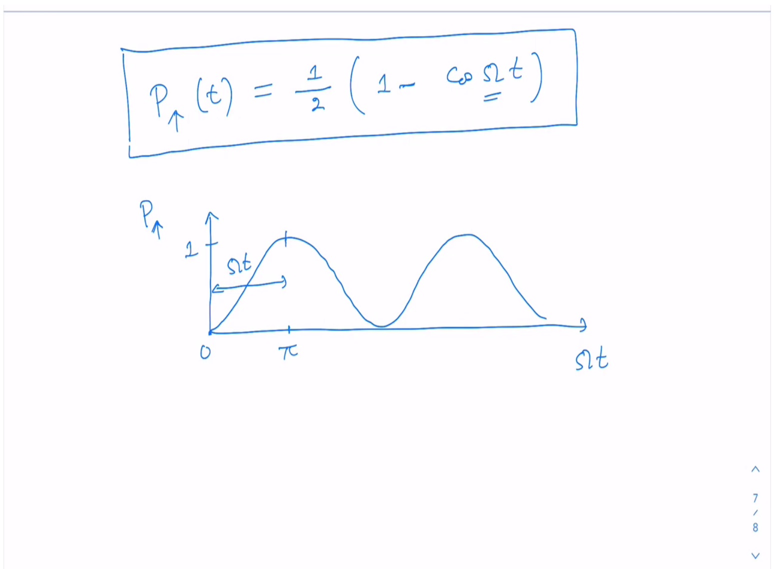

Taking this as the hint now another thing I know that at any given time the total probability has to be equal to 1. So, I can express using this I can find my P up that is the probability of finding the system in the excited state would be from here I can write it as P up here that is probability of finding it in the excited state using this expression here I can write it as 1 minus P up here. So, therefore it would become 2 P up -1.

So, immediately you see P up I can write as half 1 plus expectation value of sigma z, all right. Now if you look at this diagram, we have started, in fact, if you see this diagram what is sigma z? Sigma z is basically is the projection of block vector, it's basically projection of block vector along the z direction. So, as I have started from the ground state. So, you can guess that sigma z should be equal to minus cos omega t in this particular case okay because see we have started with the ground state.

So, sigma z at time t is equal to 0 is -1. So, sigma z should be minus cos omega t. So, putting this expression there, so I have P up probability of finding the system in the excited state would be half 1 minus cos omega t. So, you see we now obtained a very compact equation for probability of finding the system in the excited state at a given time t. So, we have got an expression as a function of time and it is you know depend, it is basically dependent on this Rabi frequency and immediately it is very easy to plot it and this plot already I talked earlier qualitatively.

So, if I plot P up, the probability of finding the system in the excited state and here its x axis say omega t then we started in the ground state and then as time goes on, so, this kind of oscillatory behavior we are going to get. So, this is what I have, this is omega t and this is say omega t is equal to pi then we will get the system in the excited state after pi then that is or we termed it as pi pulse then after 2 pi the system will be again in the ground state and so on.

So, this oscillation is known as Rabi oscillation. In this lecture we have learned most of the fundamentals of a generic 2-level system. In the next lecture we are going to consider a 2-level atom interacting with a classical field and this will give us more insight into the working of 2 level atoms and its physics. And this will be very useful later on when we are going to discuss interaction of quantized radiation or quantized light with a 2-level system, thank you.