**Quantum Technology and Quantum Phenomena in Macroscopic Systems**
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**Module-03**
**Lecture-38**
**Quantum Langevin Noise**

Hello, welcome to lecture 28 of the Course, this is lecture 7 of module 3. In this lecture we will discuss the classical counterpart of Langevin noise that is quantum Langevin noise and finally we will begin our discussion on the effect of quantum noise on an optical mode in a Fabry-Perot cavity. As Fabry-Perot cavity is the backbone of a cavity of the mechanical system. So, let us begin.

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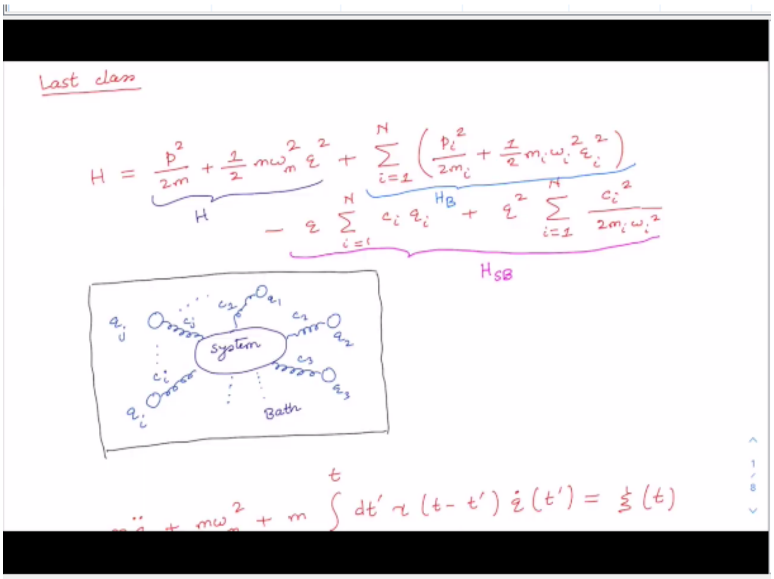


In the last class we continued our discussion on modeling a realistic situation regarding the interaction of the system harmonic oscillator with the surrounding. The surrounding was modelled as a collection of independent harmonic oscillator N harmonic oscillator each having different coupling coefficient or coupling constant.

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- $m\ddot{q} + m\omega_m^2 + m\int_0^t dt'\,\gamma(t-t')\,\dot{q}(t') = \underline{\underline{\xi(t)}}$

$$\xi(t) = \sum_{i=1}^{N} c_i \left\{ \left[ q_i(0) - \frac{c_i}{m_i\omega_i^2}\, q(0) \right] \cos\omega_i t + \frac{p_i(0)}{m_i\omega_i} \sin\omega_i t \right\}$$

- $\gamma(t) = \frac{1}{m}\sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i^2}\cos\omega_i t$ (Memory Function)

And our analysis ultimately led to an equation of motion for the damped harmonic oscillator and we got a memory function defined as this quantity gamma of t and on the right-hand side of the equation we get a parameter called Xi of t and this is so-called Langevin noise.

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→ The memory function is simplified by defining a function $\underline{\underline{J(\omega)}}$, the bath spectral density,

$$J(\omega) = \pi \sum_{i=1}^{N} \frac{c_i^2}{2m_i\omega_i}\,\delta(\omega-\omega_i)$$

And this memory function was simplified defining a function called bath spectral density denoted by the quantity J of omega.

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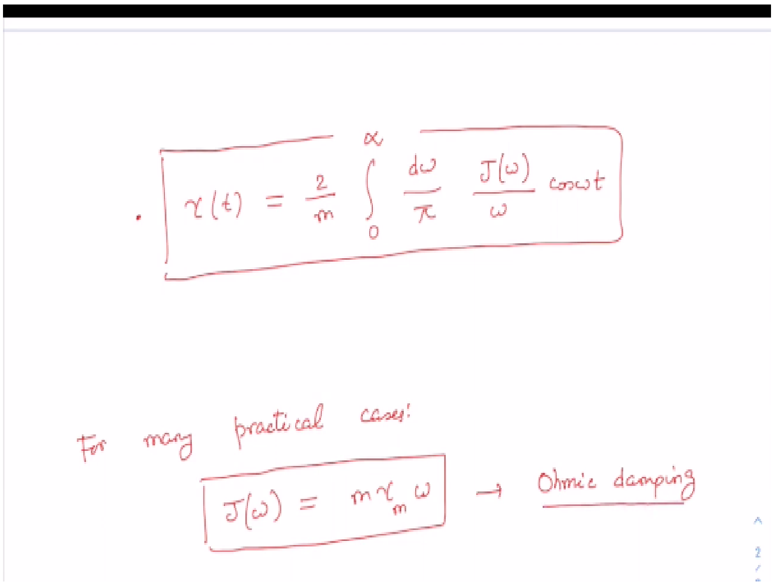


And using this, this memory function can be written in this particular form and in fact for many practical cases this bath spectral parameter can be taken as a mass into the mechanical damping gamma m into omega. This is known as the Ohmic damping.

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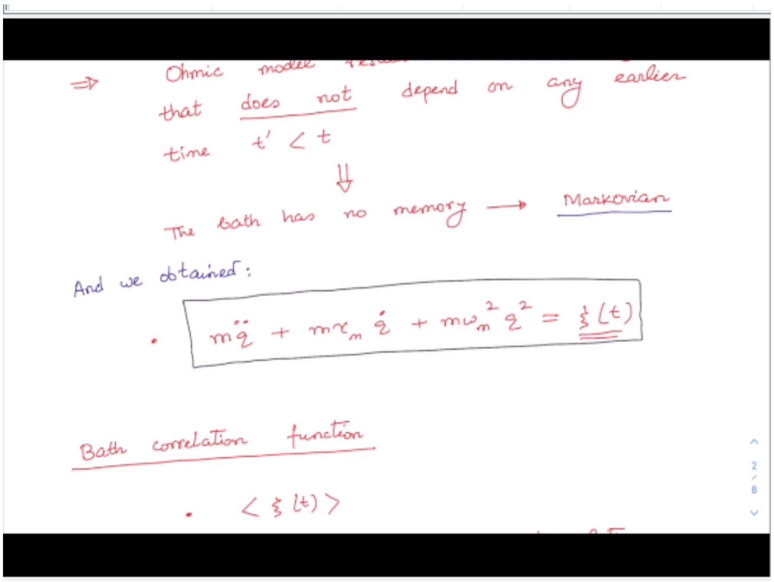


So, under Ohmic damping this memory function will take this form. In fact, Ohmic model results in a damping that does not depend on the previous history of the bath, it does not depend on any earlier time and therefore this process is known as the Markovian process and we obtain these equations here. Now on the right-hand side of this equation we have this Langevin noise and this is actually also called thermal force.

And this clearly shows that unlike the other case when it is 0 in that case transfer of energy can take place from the system harmonic oscillator to the surrounding but not from the surrounding to the system oscillator. So, here presence of this Langevin noise says that both process is possible. That means transport of energy from the system oscillator to the surrounding and back from the surrounding to the system oscillator and thereby the system will come into a thermal equilibrium.

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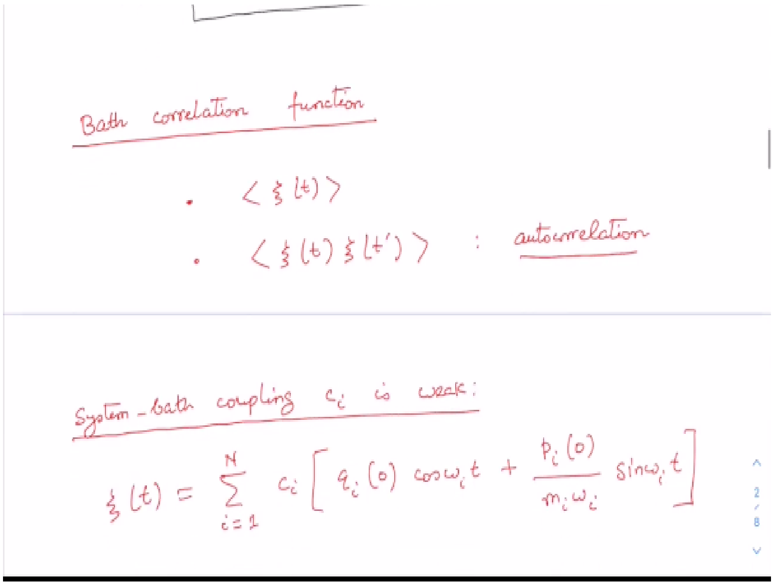


Then we went on to calculate the bath correlation function and this Xi of t as it is the Langevin noise, and it is related to the bath and we try to calculate the first moment that is average of the Langevin noise and its auto correlation function that is the second moment.

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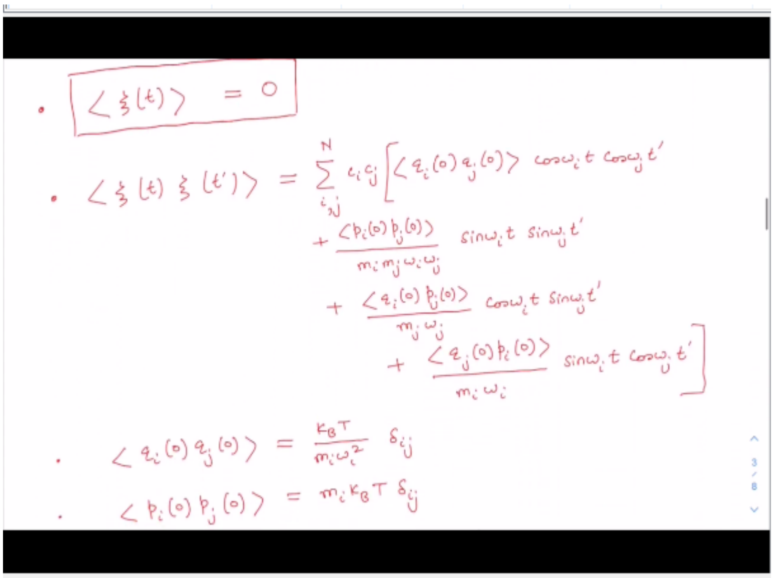


So, here what we assume that the system bath coupling is weak and then this Langevin noise expression become a little bit simplified and our calculation led us to the fact that necessarily the average value of the Langevin noise is 0. And the bath correlation function we calculated and in the process, we learned how to calculate the average value of various quantities.

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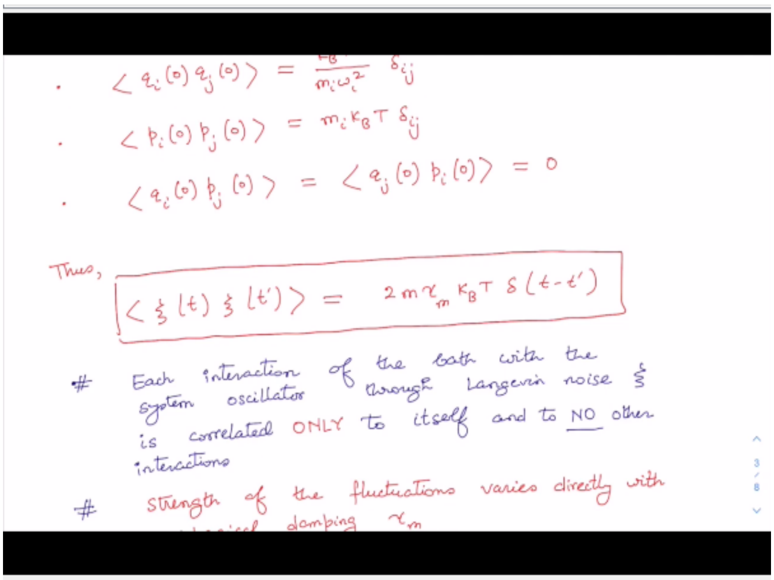


And it turns out that this second moment or the auto correlation function Langevin noise is a expression like this and physically it says that is interaction of the bath with the system oscillator via the Langevin noise is correlated only to itself and not to any other interaction and also the strength of the fluctuation varies directly with the mechanical damping parameter gamma m which is again related to the so-called fluctuation dissipation theorem.

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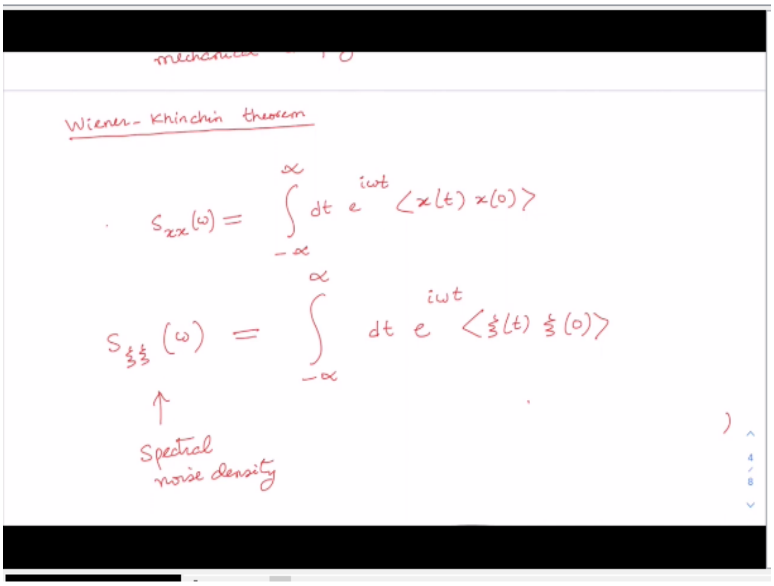


And we went to recall the Wiener-Khinchin theorem which we discussed in the context of movable mirror in an earlier class which basically says that the noise spectrum is nothing but the Fourier transform of the correlator and in analogy to this we have written down the noise spectrum for the bath and which is known as the spectral noise density and that is the Fourier transform of the second moment of the Langevin noise.

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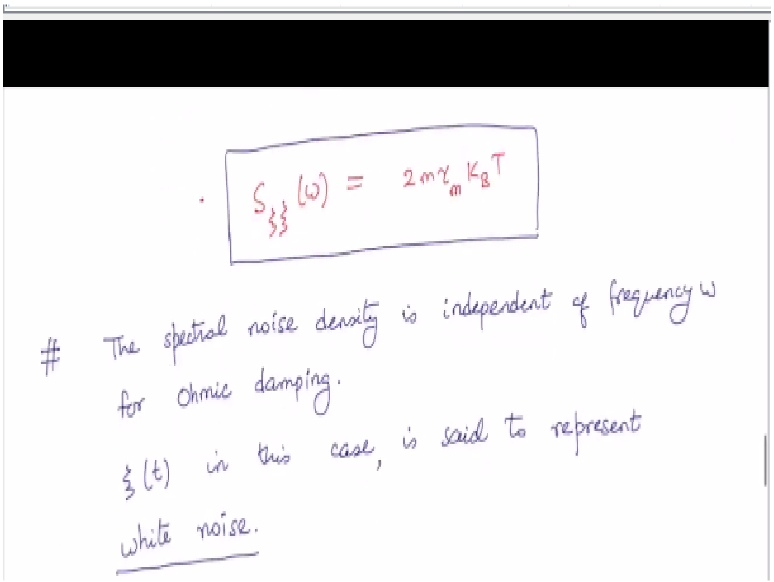


And we worked out that this for Ohmic damping this turns out to be a very simple expression and it is independent of the frequency for Ohmic damping and that is the reason that in the case of Ohmic damping the Langevin noise is said to represent the so-called white noise.

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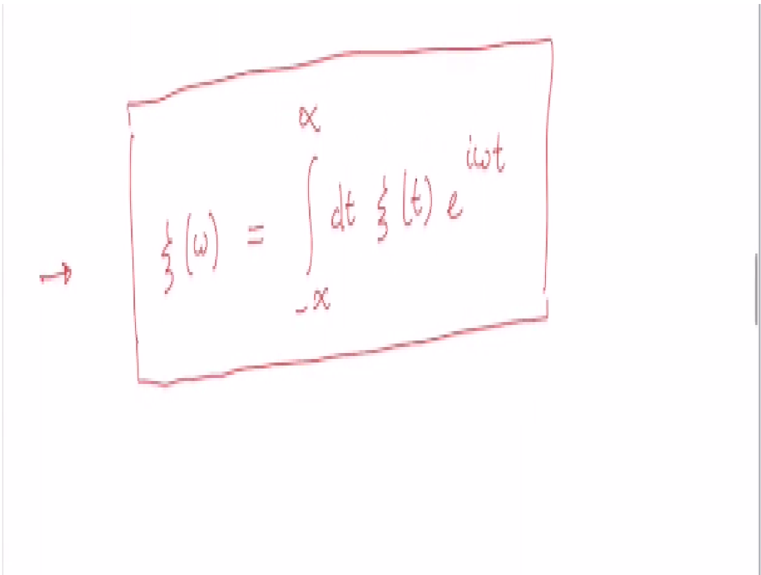


And this Langevin noise can be expressed in the frequency domain also just by taking the Fourier transformation.

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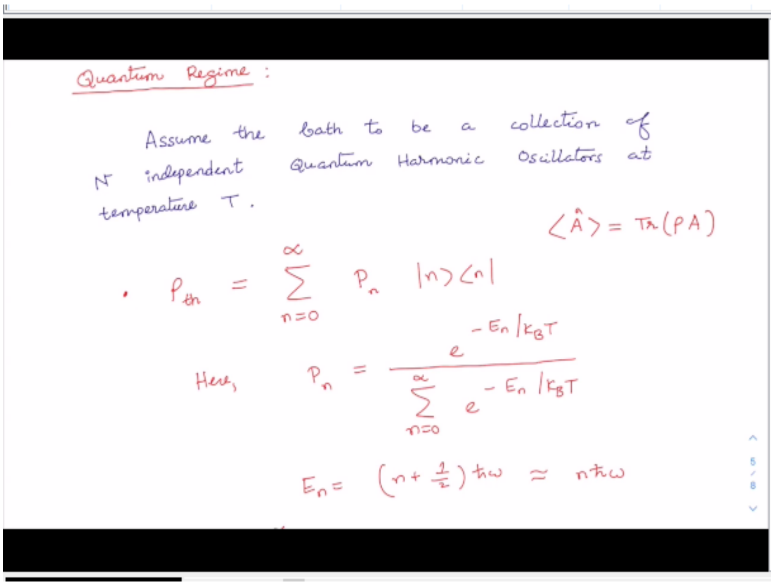


Then we started discussing the quantum regime. To discuss quantum regime, we now have to calculate the expectation value of various parameters, and because we are considering a collection of N independent harmonic oscillator at some temperature T. So, we written down the density operator for the system and that will be required because we know that when we take the average of any quantum optomechanical operator expectation value is given by the trace of the rho density operator into the operator. So, this has to be calculated so we worked out the thermal density operator for these thermal oscillators in the thermal state.

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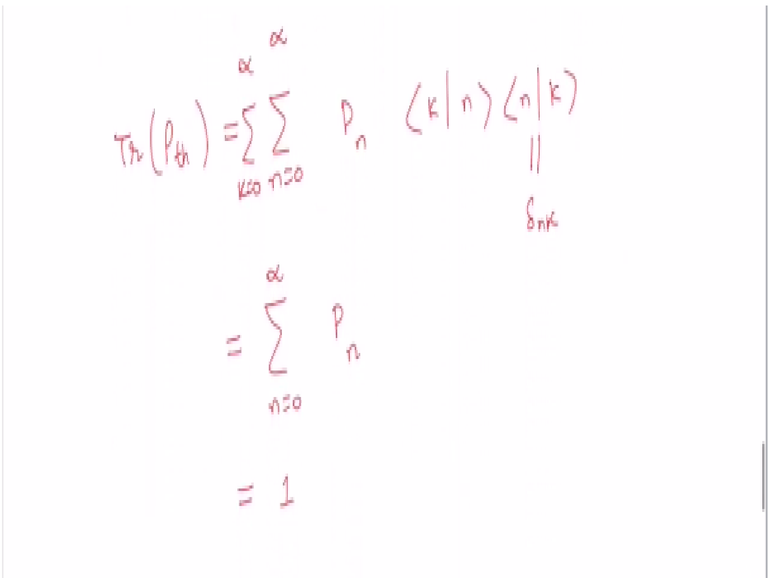

• Average phonon number:

$$\langle \hat{b}^\dagger \hat{b} \rangle = Tr\left[\rho_{th}\, \hat{b}^\dagger \hat{b}\right]$$

$$= \sum_{\ell=0}^{\infty} \langle \ell | \hat{b}^\dagger \hat{b} \sum_{n=0}^{\infty} P_n |n\rangle \underbrace{\langle n|\ell\rangle}_{\delta_{n\ell}}$$

$$= \sum_{n=0}^{\infty} n\, P_n$$

$$\langle \hat{b}^\dagger \hat{b} \rangle = \sum_{n=0}^{\infty} n\, e^{-E_n/k_BT}$$

And we calculated the average phonon number, because these are thermal mechanical modelling, it is a oscillator and in the thermal environment the quantum is termed as phonon there and we use the symbol b to represent this phononic oscillator.

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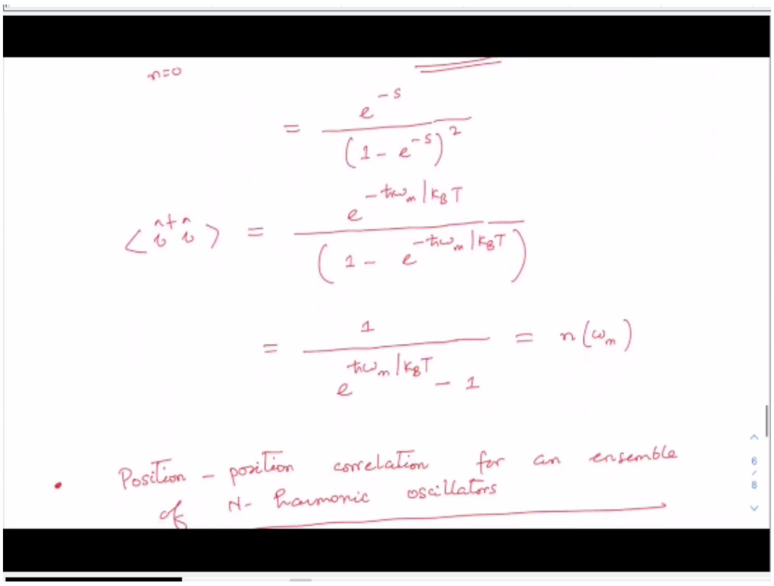
$$\Rightarrow \langle \hat{b}^\dagger \hat{b} \rangle = \frac{\sum_{n=0}^{\infty} n\, e^{-E_n/k_BT}}{\sum_{n=0}^{\infty} e^{-E_n/k_BT}}$$

$$= \frac{\sum_{n=0}^{\infty} n\, e^{-n\hbar\omega_m/k_BT}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega_m/k_BT}}$$

$$\sum_{n=0}^{\infty} e^{-ns} = 1 + e^{-s} + e^{-2s} + \cdots = \frac{1}{1-e^{-s}}$$

$$\sum_{n=0}^{\infty} n e^{-ns} = -\frac{d}{ds} \sum_{n=0}^{\infty} e^{-ns} = -\frac{d}{ds}\left(\frac{1}{1-e^{-s}}\right)$$




And we calculated the average value of phonon number which is the usual this is the familiar expression we obtain.

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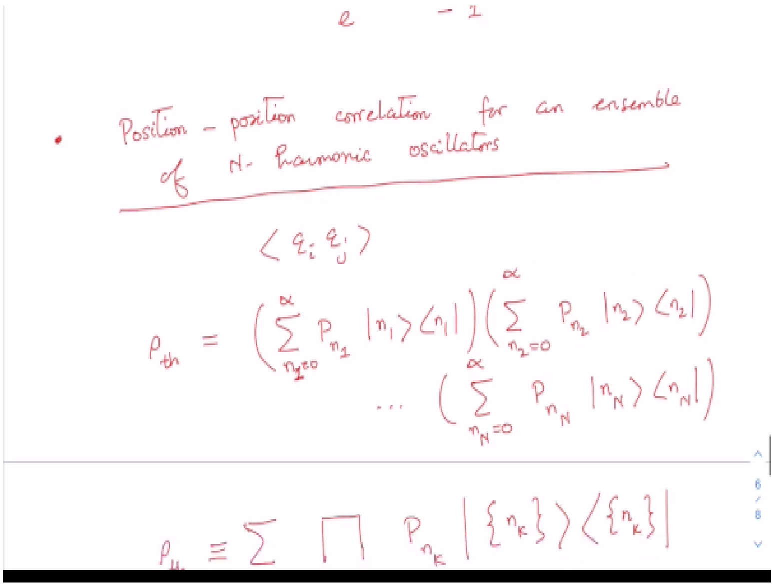

where $\{n_k\} = (n_1, n_2 \cdots, n_N)$

$$|\{n_k\}\rangle = |n_1\rangle|n_2\rangle\cdots|n_N\rangle$$

$$\langle q_i q_j\rangle = \mathrm{Tr}\left(\rho_{th}\, q_i q_j\right)$$

$$= \sum_{\{n_p\}} \langle\{n_p\}|\, \rho_{th}\, q_i q_j \,|\{n_p\}\rangle$$

$$= \sum_{\{n_p\}} \langle\{n_p\}|\sum_{\{n_k\}}\prod_k P_{n_k}|\{n_k\}\rangle\langle\{n_k\}|q_i q_j|\{n_p\}\rangle$$

$$= \delta_{\{n_p\}\{n_k\}}$$

$$= \sum_{\{n_k\}}\prod_k P_{n_k}\langle\{n_k\}|\,q_i q_j\,|\{n_k\}\rangle$$

$$\langle q_i q_j\rangle = \sum_{\{n_k\}}\prod_k P_{n_k}\langle\{n_k\}|\,q_i q_j\,|\{n_k\}\rangle$$

Note $\{n_k\} = (n_1, n_2, \cdots n_N)$

$$|\{n_k\}\rangle = |n_1\rangle|n_2\rangle|n_3\rangle\cdots|n_N\rangle$$

And then we try to calculate the position-position correlation for an ensemble of N harmonic oscillators and we finally obtain this particular expression. Now we are going to build up our next analysis from here only thing let me remind you that you should understand this notation here it is bracketed n k represents this one. So, it is a collection of all n 1, n 2 up to n N number of items we have there and this ket represents the direct product of various number state corresponding to the oscillators. Let us now write the position operators this q i, q j. These are operators.

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$$q_i = q_{i0}\left(b_i + b_i^{\dagger}\right) \quad , \quad q_{i0} = \sqrt{\frac{\hbar}{2m_i\omega_i}}$$

$$\left[\; x = x_{zpf}\left(a + a^{\dagger}\right) \qquad x_{zpf} = \sqrt{\frac{\hbar}{2m\omega}} \;\right.$$

$$q_j = q_{j0}\left(b_j + b_j^{\dagger}\right)$$

$$q_{j0} = \sqrt{\frac{\hbar}{2m_j\omega_j}}$$

Let me write q i is equal to I am not using this operator sign but you please understand that I am not talking about operator because now we are in the quantum regime. So, q i = q i 0 into bi plus bi dagger. This is the position operator for the ith oscillator and here q i0 is equal to this is the zero-point fluctuation, this is h cross divided by twice m i omega i, m i is the mass of the oscillator ith oscillator and omega i is the corresponding resonance frequency.

Now by the way please recall that we have utilized a similar thing earlier when we have this displacement operator x. In this we wrote earlier in the context of when we discuss about harmonic oscillator, this is x zero-point fluctuation and we there we used a + a dagger. Now x zero point this fluctuation was defined as h cross divided by twice m omega. The same thing here we are doing it and we can then also write for q j. For q j we have q j0 for the jth oscillator. b j + b j dagger and here q j0 = h cross divided by twice m j omega j. Now we have to calculate this particular quantity.

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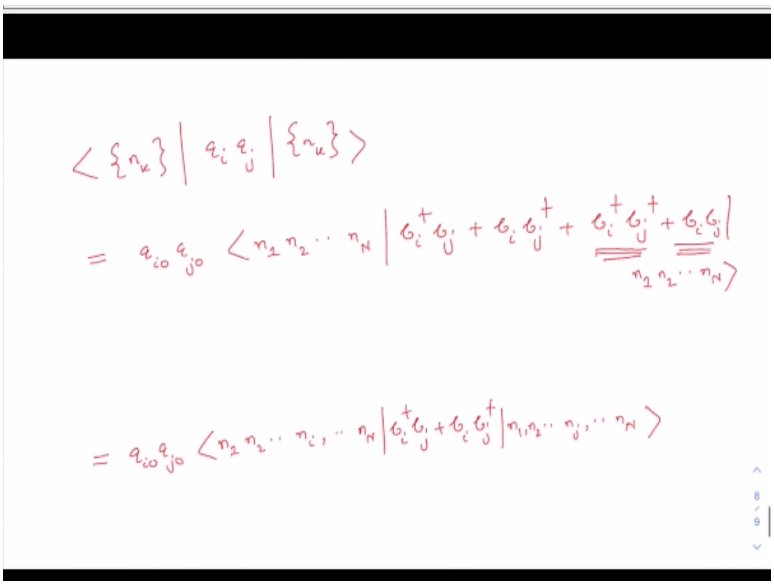


So, let us do that, first of all let us calculate this bracketed bra of this number states q i, q j and this ket basically we are calculating the scalar product for this product of this operators q i q j. Now if I put q i q j I have q i0 q j0 and this is actually I can write it as n 1, n 2, these are the direct products up to n N in short hand notation q i, q j if I break it up so I will get 4 terms, so those would be b i dagger b j + b i b j dagger + b i dagger b j dagger and we will have b i, b j.

And then on the other side you will have n 1, n 2 up to N number of oscillators. So, we will have the corresponding quantity for that and you will see that the contribution from these 2 terms, this term as well as this term is obviously going to be 0. Just recall to understand it we can calculate say n b b or n b dagger b daggers, you can do that also. Then here you will get it as your simple calculation will give you it as n - 1 into n - 2 and you will have here n n - 2 and these are orthogonal.

So, this would be 0, so these 2 contributions would not be there because of these 2 terms. So, I will simply have q i0, q j0, n 1, n 2 say n i, n N and here you will have b i dagger b j + b i b j dagger and the other side you will have say n 1, n 2 let me take it. So, this anyway you have understood. This one let us say I have n j up to n N.

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$$= q_{i0}\, q_{j0} \Big[ \langle n_1, n_2, \cdots (n_i - 1), \cdots n_N | \sqrt{n_i}\sqrt{n_j} \,| n_1, n_2, \cdots (n_j - 1), \cdots n_N \rangle$$

$$+ \langle n_1, n_2, \cdots (n_i + 1) \cdots n_N | \sqrt{n_i+1}\,\sqrt{n_j+1} \,| n_1, n_2 \cdots (n_j+1), n_N \rangle \Big]$$

$$= q_{i0}\, q_{j0} \left[ \sqrt{n_i n_j} + \sqrt{(n_i+1)(n_j+1)} \right] \delta_{ij}$$

So, now let us proceed further. So, let me we calculate the term by term. So, I have q i0 q j0, I have here n 1, n 2. So, here let me first consider this particular term, so b i dagger b j, so b i dagger when it this operator operates on the bra n i here it will operate on only the ith oscillator. So, it would become because it is b i dagger operating on this, this would become n i - 1 and then you will have up to n N.

And this will give you square root of n i similarly when b j operates on n j you know you will get n j square root of n j and here you will have n 1, n 2 and so on up to say it will n j would become n j - 1 and you will have n N. So, this is what you will get from the first term. Now let us consider this term and in the similar way here you will see that this would become n 1, n 2 and this ith operator b i annihilation operator when it operate this is you will get it as n i would become n i + 1 and you will have n N here.

And this would become square root of n i + 1. Similarly, now b j when it operates on n j it would become square root of n j + 1 and in this side you will have n 1, n 2 and you will have it as n j + 1 and n N. So, therefore I can write the whole thing as q i 0, q j0 square root of n i n j + square root of n i + 1, you will have n i + 1 and n j + 1 and then I will write it as delta ij because all the other terms will not contribute.

As you see this one for example n i - 1 and n j - 1 they would become orthonormal it will get normalized provided i = j, if i is not equal to j then they will vanish, because scalar product with n 1 because this is just a number n 1 and n 1 will give you 1, n 2 and n 2 will give you one scalar product but n i – 1 and n j – 1 will give you one provided i = j. Similar is the logic for this particular second term also. So, therefore we will end up with this particular expression.

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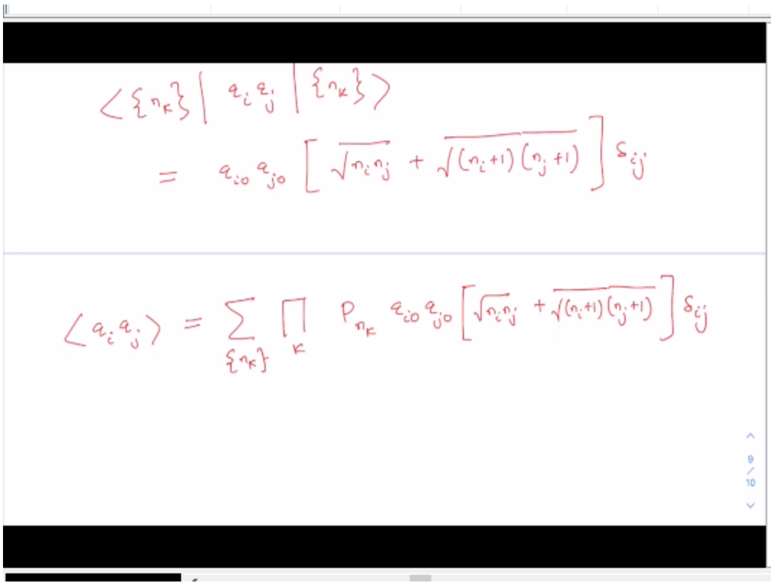


So, therefore we have evaluated let me write it again here what we got is we have evaluated this particular quantity and we have q i, q j scalar product we have worked out and this is we got q i0 q j0 square root of n i n j + square root of n i + 1 into n j + 1 delta ij. Now let us proceed further ultimately, we have to work out the expectation value of this product of these 2 operators and this is equal to we have summation.

So, let me show you the expression once again here. So, this we have to have let me write here again we have summation over all this n k's and this is product and P n k q i0, q j0 this already we worked out. So, just let me copy from here this same term we are having here let me put it, let me do it, let me write here. So, q i0 q j0 plus square root of n i n j + square root of n i + 1 n j + 1 and delta ij.

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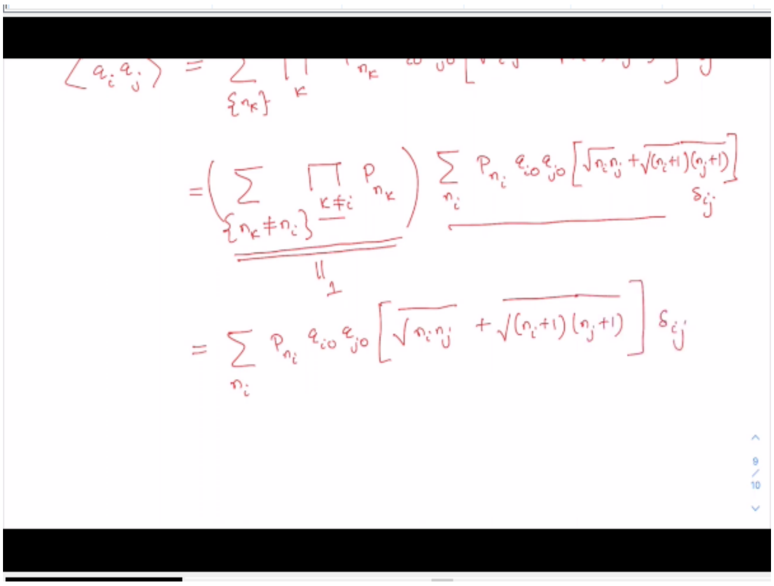

Now let us do one thing. Let us look at separated terms where k is not equal to i, so we will separate it terms like this, say when n k is not equal to n i and that means k is not equal to i here I have P n k this term and concentrate only the n i terms here and when n k = n i, so we have here P ni and I have q i0, q j0 and this term already I have square root of n i n j + square root of n i + 1 n j + 1 delta ij.

This mathematics may look little bit cumbersome but it is actually not that difficult but you will get a very useful result and it is important to know how to do this calculation. Now you see this particular term will give you simply 1 because for example any corresponding probabilities you have terms like this P 1 summation P 2 because of this product you will have all these probabilities would be equal to 1, this would be equal to 1, this would be equal to 1, this would be equal to 1.

So, overall from here it will be unity the whole thing and therefore I will be left out with only this term. So, let me write here you will have summation n i P ni q i0 q j0 and square root of n i n j + square root of this. Again, let me write here and n j + 1 delta ij.

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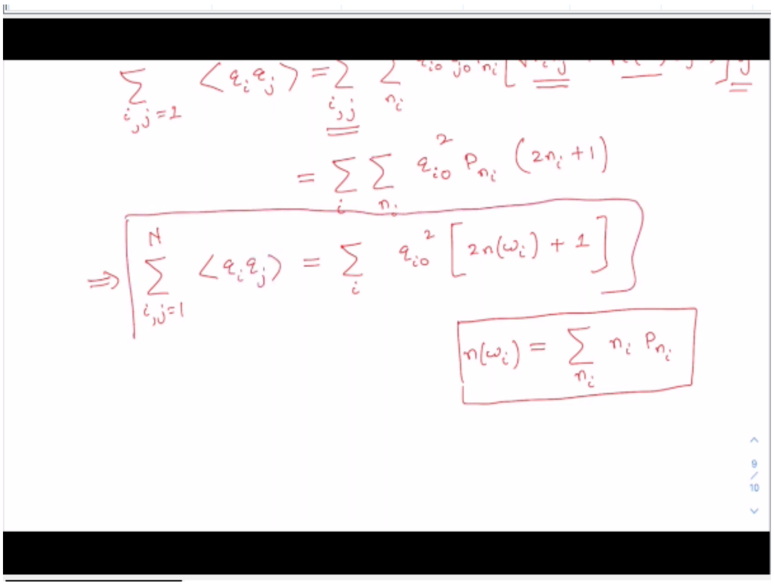


Finally let me now sum it over all the oscillators finally if I sum up that means if I sum up over the variable ij for all the oscillators N oscillators, I have q i q j. So, this would be I can write it as i j and here I have n i and it is this particular term, let me put it here. I have to write it again. So, I will have q i0 q j0 square root of n i n j plus this one and delta i j fine.

So, this you can easily evaluate because of this kronecker delta I put i = j then I will be left out with only one summation. This is basically 2 summations are there. Now we are having one summation and then this summation over n i variables n i and then q i0 square and I will have I think I miss this term here that is P ni has to be also there. So, let me put it here. So, I have P ni or rather let me write it this way.

Then I have this term P ni also, I have to write. So, I have here P ni. Now because I take i = j, I will have q i0 square then I have P ni, so from here I will get ni and from here I will get ni + 1. So, this will give me 2 ni + 1. This I can further write as this I can write it as summation i q i0 square and this would be twice n of omega i, I will explain how I am getting it. So, this is what I will have.

Here this n of omega i is nothing but the average number of phonons for the ith oscillator and that is ni P ni, I think this you can recognize, I have just utilized this one. So, this is the expression I get. Now therefore this is a very important result I obtain ij when I takes the summation q i q j. This is what I got.

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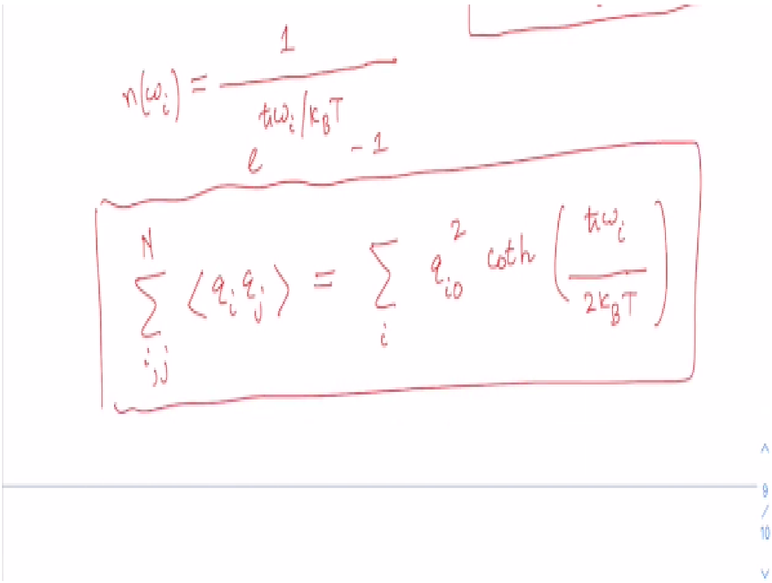


This one I can further simplify because I know that n of omega i this already we worked out and this is equal to e to the power h cross omega i by K B T – 1. And if you put it here and do the analysis you will get it as this quantity position, position correlation term would turn out to be very straightforwardly you can work it out I encourage you to do it otherwise maybe we can do it in the problem-solving session, it is a simple algebra, you will get coth hyperbolic h cross omega i by twice K B T. So, this is what we obtain. We will now discuss the quantum Langevin noise.

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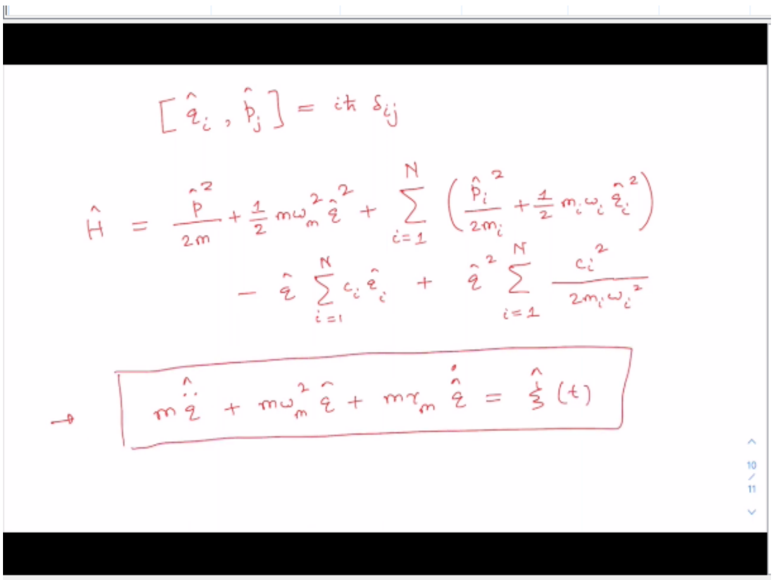


In the similar line as we did for classical Langevin noise, now both the system and the bath oscillator variables are now quantized and they follow this commutation relation say q i p j, this commutation is equal to ih cross delta ij, the Hamiltonian for the system and bath combined together is now written in this form. So, this is exactly the same as you will see except that now the position variable and the momentum variable are replaced by the corresponding operator.

So, this is the system part we have P square by twice m that is the kinetic energy then I have this potential energy of the system oscillator, m omega m square q cap square operators then these baths are considered to be a collection of N quantum harmonic, independent quantum harmonic, oscillators with corresponding momentum variable say p i for the ith oscillator, m i is the mass of the ith oscillator plus this potential energy term half m i omega i q i cap square and then the interaction between the system and the bath oscillator is given by this particular term.

This is exactly what we wrote for the classical case as well and then we have this term is added there to take into account to cancel the effect of shifting of the frequency of the system oscillator. So, you have this term c i twice m i omega i square. Now using Heisenberg equation of motion we will again get equation of this form exactly the classical equation we get but now in the operator form m q double dot + m omega m square q + m gamma m q dot that is equal to Langevin noise. Now this Langevin noise term is an operator.

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$$\hat{\xi}(t) = \sum_{i=1}^{N} c_i \left[ \hat{q}_i(0) \cos\omega_i t + \frac{\hat{p}_i(0)}{m_i \omega_i} \sin\omega_i t \right]$$

$$\hat{\xi}(t) = \hat{\xi}^{\dagger}(t)$$

$$\hat{\xi}(t)\,\hat{\xi}(0) \neq \hat{\xi}(0)\,\hat{\xi}(t)$$

It is written in the operator form where this is same as in the classical case only it is the variables are now replaced by its operator. So, assuming that the coupling between the bath and the system to be weak then I can write this as this one q i cap cos omega i t + p i cap 0 m i omega i sine omega i t. Now it can be very easily verified that this quantum Langevin noise operator is Hermitian. So, Xi of t = Xi dagger of t and also you can show that because of the commutation you will see that Xi of t Xi of 0 is not equal to Xi of 0 and product of Xi of 0 Xi of t.

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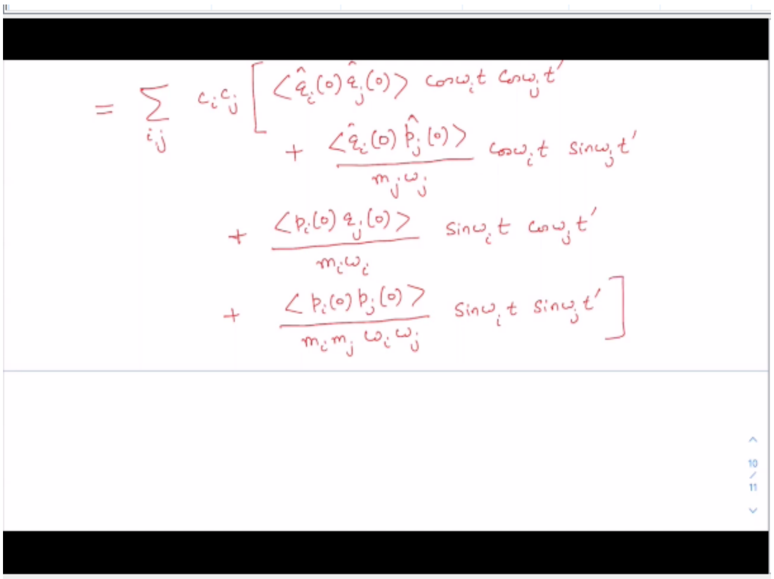


Now we can calculate the auto correlation function for the quantum Langevin noise that is we have to calculate Xi of t Xi of 0 and this would be in the similar line as in the case of the classical case let me first write the whole term i j then I have here c i c j and we will have q i of 0 q j of 0 the expectation below the product then I have cos omega i t cos omega j t dash. I will have 4 terms.

So, let me write all of them and I will have expectation value, all these are operators, I am not writing the operator's sign but you please understand that these are all operators. So, I have q i0 here p j of 0 here divided by m j omega j cos omega i t sine omega j t dash and then I have plus p i 0 q j 0 divided by m i omega i sine omega i t cos omega j t dash and then plus p i of 0 p j of 0 divided by m i m j omega i omega j sine omega i t sine omega j t dash. So, this is what I will have. Now just a while back we have calculated these quantities.

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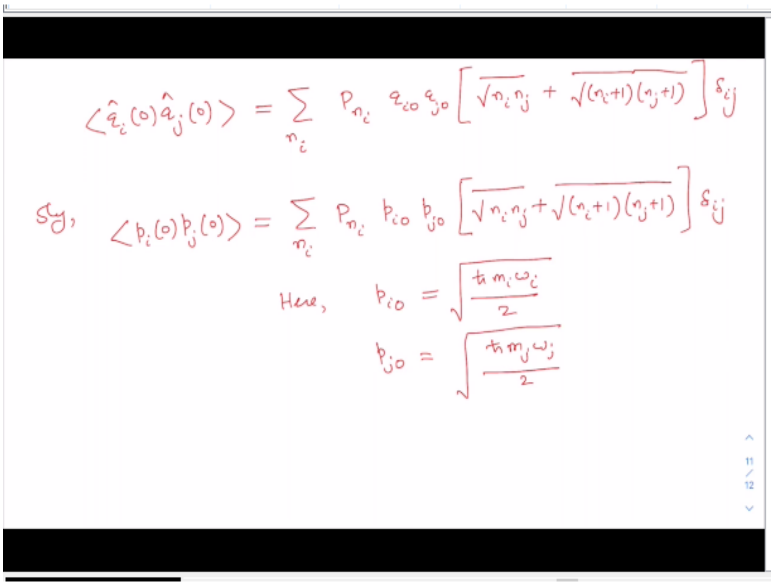


Say q i0, these are operators again we calculated q i0 average expectation value of this product of these 2 operators that we calculated as summation over n i P ni q i0 q j0 and we have it is square root of n i n j + square root of n i + 1 n j + 1 delta ij and similarly you can show that you will get for this p i0 p j0 that would be equal to summation n i you have P ni p i0 p j0 and this is the same it is square root of n i n j + square root of n i + 1 n j + 1 and you have delta ij. However here this p i0 similarly for p j0 that would be h cross m i omega i by 2 square root or p j0 = h cross m j omega j by 2 square root.

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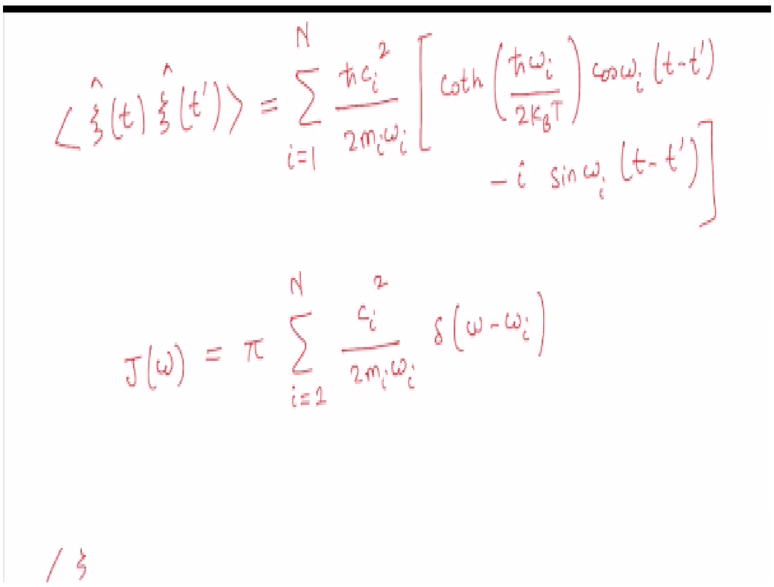


Now using these relations we can work out the autocorrelation for the Langevin noise, quantum Langevin noise here because these are now operators, quantum operators and if the calculations are done you will see that you will get it as summation i = 1 to n h cross c i square divided by twice m i omega i and here you will have cot hyperbolic h cross the reduced Planck's constant h cross omega i divided by 2 K B T, K B is the Boltzmann's constant and here you have cos omega i t - t dash - i sine omega i t - t dash.

So, this is what you will get. Again, as like in the classical case we can define the so-called spectral density function J of omega and that is exactly in the similar way we have it is pi into summation over all the oscillators and I have c i square divided by twice m i omega i delta omega - omega i.

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$$\langle \hat{\xi}(t)\hat{\xi}(t')\rangle = \frac{\hbar}{\pi}\int_0^{\infty} d\omega\, J(\omega)\left[\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\omega(t-t') - i\sin\omega(t-t')\right]$$

Now we can rewrite the autocorrelation function for the Langevin noise in terms of the spectral density function as this Xi of t Xi of t dash. This should be equal to you can just put it and you will get it as h cross by pi 0 to infinity you please look into the classical case once again and you will get it; it will be  0 to infinity d omega j omega cot hyperbolic h cross omega by 2 K B T cos omega t - t dash - i sine omega t - t dash. So, this is what you will get.

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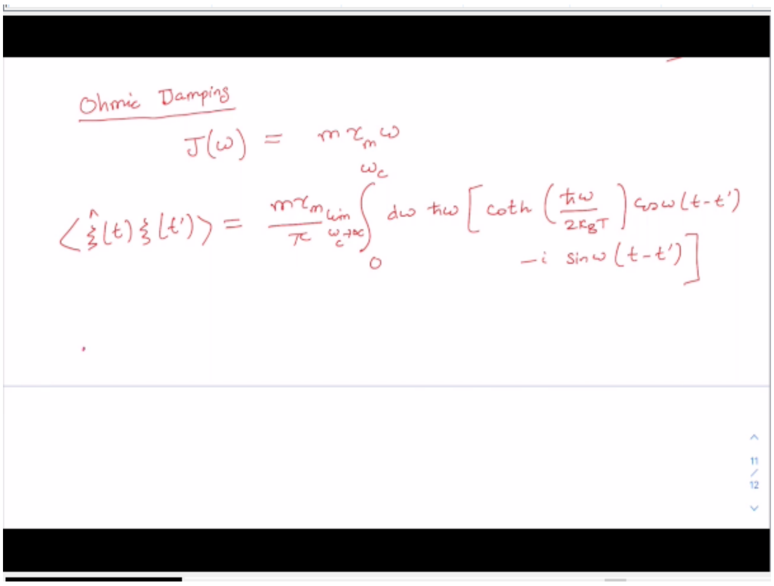


Now as in the classical case here also let us go for Ohmic damping, because we are interested in Markovian process. So, for Ohmic damping if we consider Ohmic damping where this spectral density function is given as mass into gamma m omega we obtain the autocorrelation function expectation value of Xi of t Xi of t dash that would be equal to m into gamma m divided by pi integration d omega h cross omega coht hyperbolic h cross omega by 2 K B T cos omega t - t dash - i sine omega t - t dash.

Now here this integration limit is from 0 to generally we write of infinity but to be precise it is there is some cut-off frequency is there for Ohmic damping that is omega c and it is generally taken in the limit say omega c tends to infinity.

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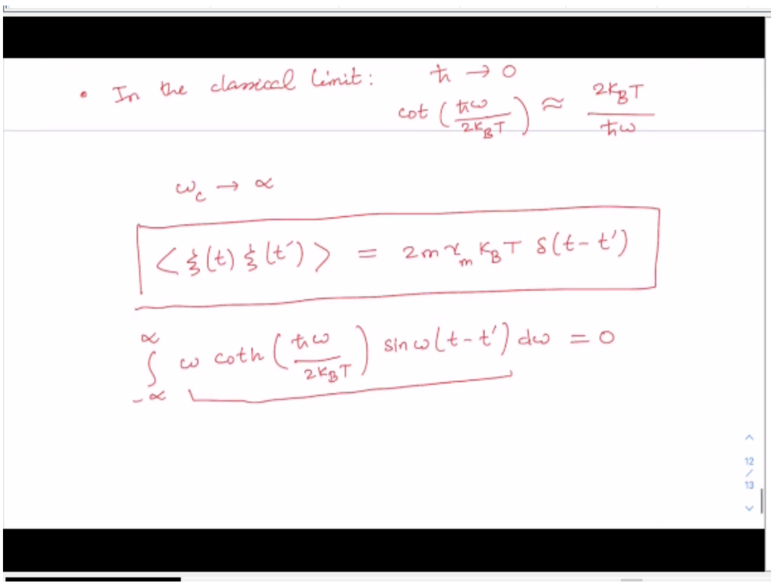

Now in the first approximation in the classical limit you know that in the classical case h cross tends to 0 and temperature is non-zero, t is non-zero and cot hyperbolic h cross omega by twice K B T. This can be approximated to be as you can actually show it, it would be 2 K B T divided by h cross omega and if we set omega c at infinity then you can very easily show that in the classical limit this autocorrelation function for the Langevin noise would be again will regain what we got in the classical case. That is twice m gamma m K B T delta t - t dash.

So, this is what we obtain and in fact we must get it. Now there is a actually another interesting form of this second moment of the quantum Langevin noise which you will often encounter in research literature and it is very easy to get it what is done there, you just have to utilize these facts and you can immediately rewrite it, you know that minus infinity to plus infinity omega cot hyperbolic h cross omega 2 K B T sine omega t - t dash. This integration is in the frequency domain, so you have here d omega. So, this would be equal to 0 because overall this integrand this would be odd function.

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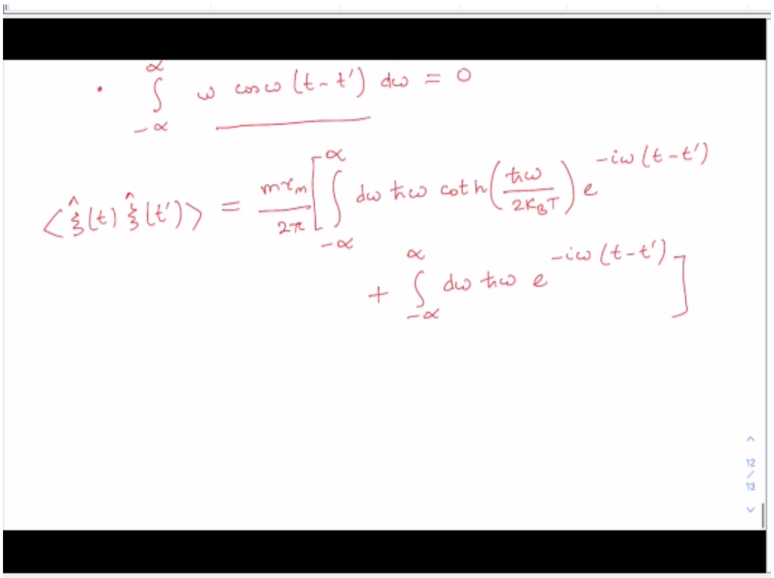


And another fact you can utilize is say minus infinity to plus infinity omega cos omega t - t dash d omega = 0. If you utilize these 2 properties then this autocorrelation function for the quantum Langevin noise can be written in a slightly different form that would be this. You will have m gamma m divided by 2 pi, you see here the limit is from omega say will you will put it as infinity then 0 to infinity we can replace it by a half of that is why this 2 term is coming; here we had m gamma m by pi so I will have a half term here because now I am taking the limit from minus infinity to plus infinity.

So, that I can utilize these 2 properties there and what I will have is this, you will have it as it is very simple and straightforward you can verify it. d omega h cross omega cot hyperbolic h cross omega by 2 K B T then rather than writing here you have this cos. So, I will write here e to the power - i omega t - t dash because the sine part will give me 0 that is why I can do it and for the second term that means this particular term in the similar way I can utilize this particular property and then I can write it as minus infinity to plus infinity d omega h cross omega e to the power - i omega t - t dash and this is what I will get.

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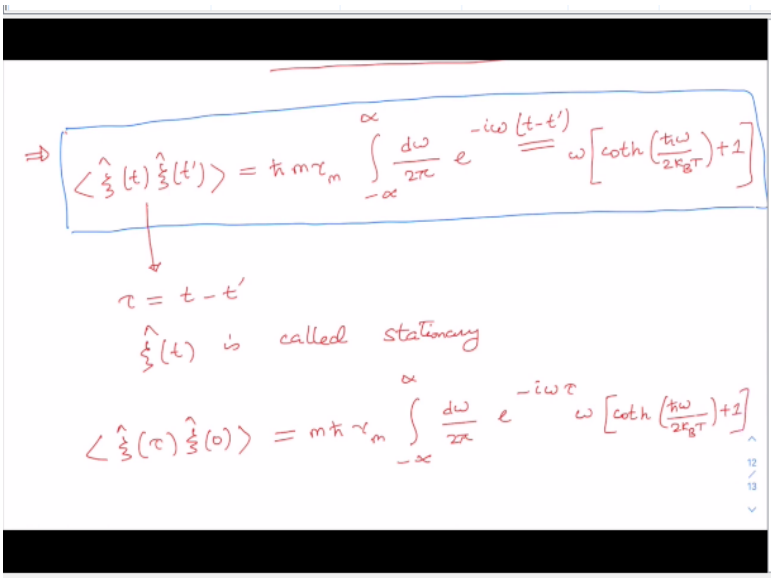


Or in short I can write it this very useful expression which you will encounter in many, many research literature or research paper it would be simply h cross m gamma m minus infinity to plus infinity d omega by 2 pi e to the power - i omega t - t dash omega coth hyperbolic h cross omega by 2 K B T + 1. I am just simplifying this expression and this is very popularly and very frequently used the second order moment or the auto correlation function for the quantum Langevin noise.

You can see that the autocorrelation function depends only on the time difference tau say time difference let me denote it by tau that is tau = t - t dash as you can see from this term here. So, because it depends on the time difference only therefore this quantum Langevin noise Xi of t is called stationary and therefore we can in fact write it in this form also this autocorrelation function Xi of t Xi of 0 that would be equal to m h cross gamma m integration minus infinity to plus infinity d omega by 2 pi e to the power - i omega tau omega cot hyperbolic h cross omega by 2 K B T + 1.

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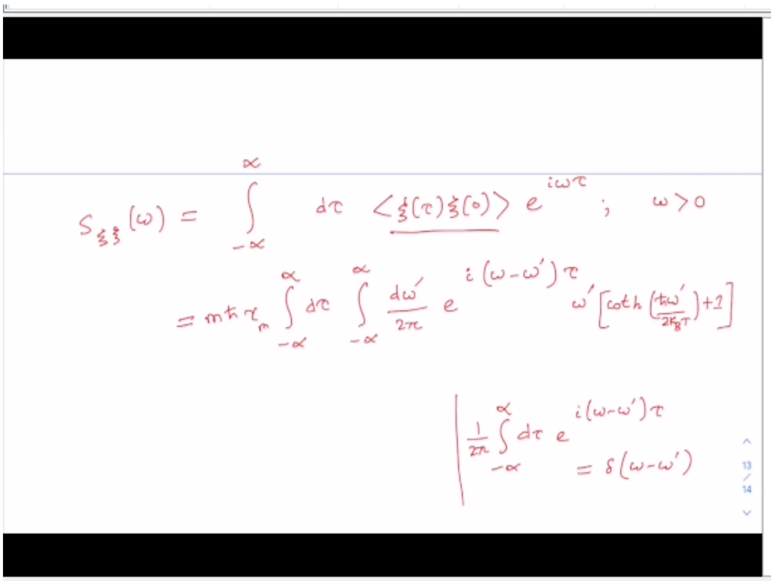


Now the Fourier transform of this autocorrelator gives us the so-called spectral noise density in the quantum domain. That would be S Xi Xi omega that is the Fourier transform of the autocorrelator, so Xi of tau Xi of 0 and the Fourier kernel i omega tau we are evaluating it at for frequency say omega greater than 0 and integrating over time; integration limit is from minus infinity to plus infinity.

We can quickly evaluate it is very simple so let us do that. We will just put the expression that we have derived for this autocorrelator that is m h cross gamma m integration minus infinity to plus infinity d tau and here I have minus infinity to plus infinity. Let me use the frequency variable say omega dash d omega dash by 2 pi e to the power i omega minus because now I have here this - omega is here.

So, doing it replacing by omega dash. So, this is what I have then here I will have omega dash I will have cot hyperbolic h cross omega dash by 2 K B T + 1. This can be further simplified because you can recognize that what I have here is this Dirac delta function minus infinity delta function I will have if you look at it this expression I have a term like this integration d tau 1 by 2 pi e to the power - i omega - omega dash tau. This is nothing but the direct delta function delta omega - omega dash.

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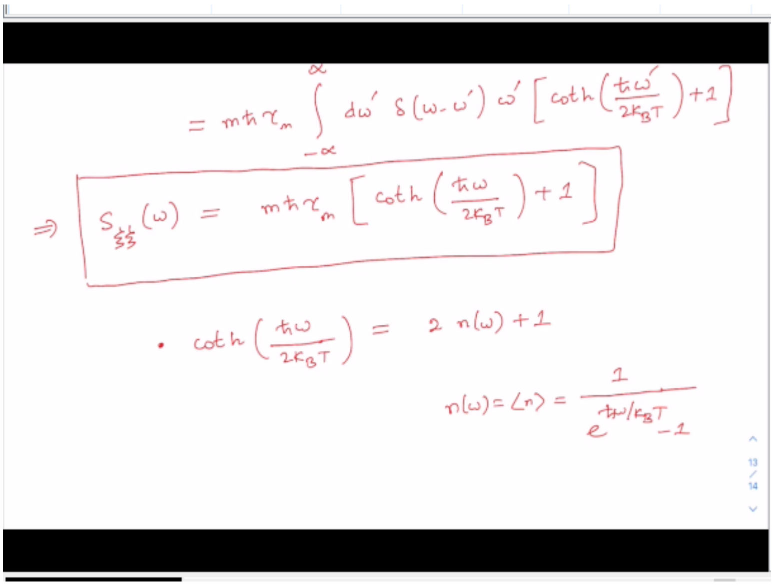


So, I can utilize it, if I utilize it then I will be able to write the whole thing as m h cross gamma m - infinity to + infinity, you will have d omega dash delta omega - omega dash omega dash I have here cot hyperbolic h cross omega dash divided by 2 K B T + 1. Now using the property of the Dirac delta function I will get the spectral noise density for the quantum Langevin noise would be m h cross gamma m cot hyperbolic h cross omega divided by 2 K B T + 1.

So, this is what we obtain. Some time back we actually show that if you have worked out that this cot hyperbolic h cross omega by 2 K B T, I can write it in terms of the average number of phonon as 2 into n omega + 1 and just recall that n omega is nothing but the average number of phonon and it is given by 1 by e to the power h cross omega by K B T – 1. So, if you utilize it then you can show that this function is nothing but cot hyperbolic h cross omega by 2 K B T **is** can be expressed in this form.

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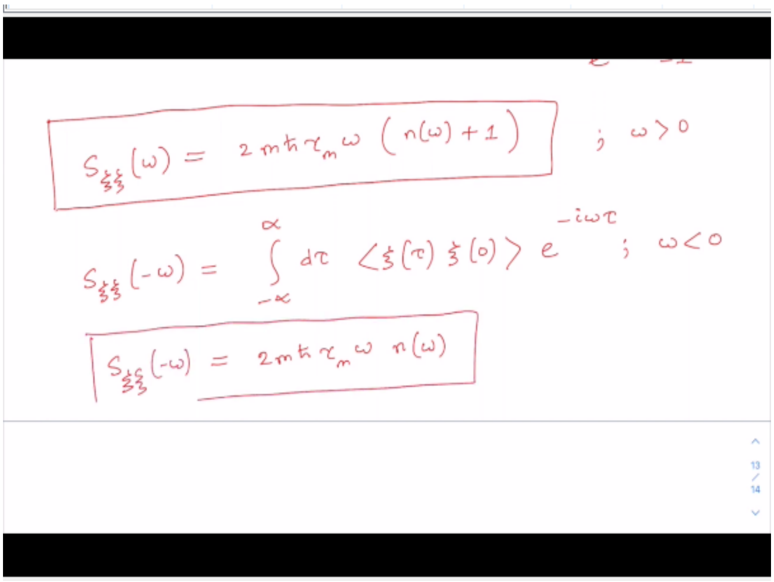


And using this we have the spectral noise density that would be in terms of the phonon number I have twice m h cross gamma m omega. So, this I got it. So, I think I missed something here, here I have this term omega is also there because I have used the Dirac delta function. So, that is how this omega term is coming and I will have here n of omega + 1. So, this is what we obtain for when omega is greater than 0. On the other hand, I can also work out what is the spectral noise density at minus omega and in that case, it would be integration will be minus infinity to plus infinity d tau the Fourier transform of the correlator.

I have here and because it is omega is less than 0 in this case. So, I have minus rather than plus I have minus i omega tau. In this case I am having omega less than 0 and please do this calculation very straight forward the way we have done it. You can show you will get twice m h cross gamma m omega. Now you are going to get n of omega not this +1 would not come. So, this is what you will get S Xi Xi at minus omega.

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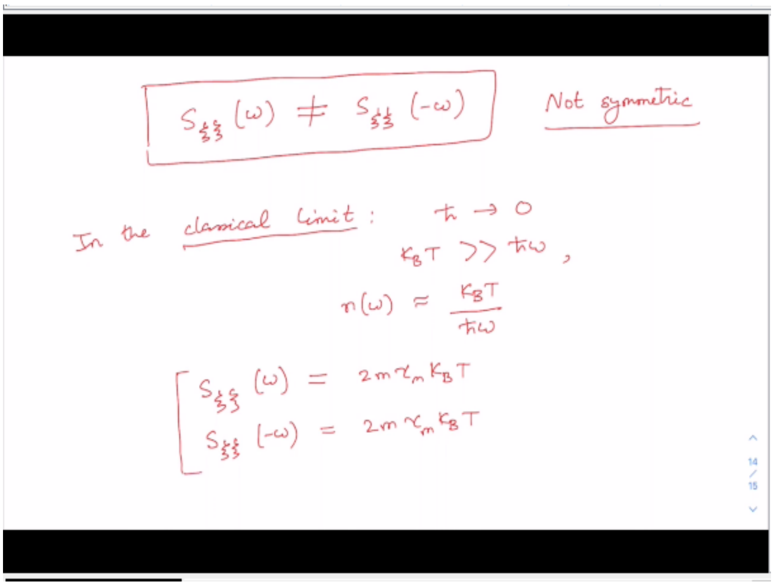


So, it is clear that this spectral noise density in the quantum regime is not symmetric. This is what the first thing we obtained in what is different from the classical regime. So, this spectral noise density function is not symmetric. However, in the classical limit you can show because you know that in the classical limit your h cross tends to 0 and your this K B T is much, much greater than h cross omega.

And in that case, you can have this average phonon number can be written as K B T divided by h cross omega and utilizing this you can show that this spectral noise density function in the classical limit would be twice m gamma m K B T and if it is evaluated at minus omega this will also give you twice m gamma m K B T. So, in the classical domain this spectral noise density function would be symmetric.

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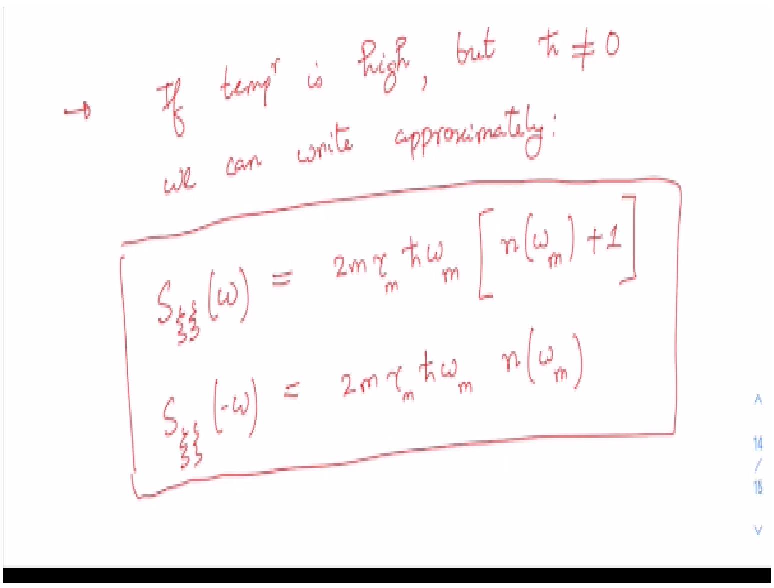

One thing we can do here if say temperature is very high but you are still in the quantum limit say but your h cross is not equal to 0 then we can write very approximately and this is going to be useful later on approximately this spectral noise density function, quantum noise density function as twice m gamma m h cross omega m.

Now this n of omega is evaluated at the mechanical frequency omega m + 1 or the resonance frequency of the mechanical system oscillator and S Xi Xi at the frequency minus omega this would be twice m gamma m h cross omega m you will have n evaluated at omega m. Now let us come to our Fabry-Perot cavity which is the main setup for cavity optomechanical system and let us see how the quantum noise affects an optical mode.

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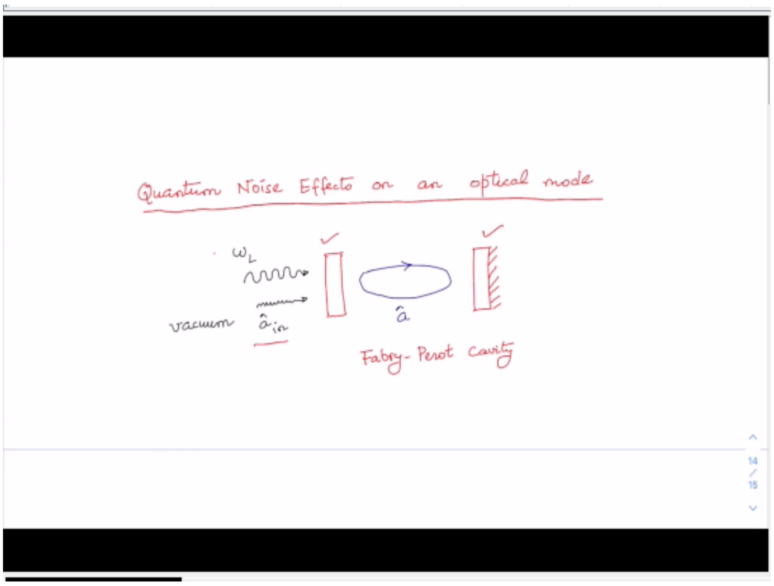


So, we are going to study optical noise effects on an optical mode. This actually comes under the so-called input output theory as well, You know that the optical mode undergoes damping inside the Fabry-Perot cavity and the electromagnetic fluctuation from vacuum outside the cavity inject quantum noise into the cavity and in the case of cavity optomechanics the cavity is often driven by a single mode laser having frequency say omega L.

We will consider these environmental effects for a single sided cavity as depicted here in this schematic diagram where one of the mirrors is perfectly reflecting. Say this one is perfectly reflecting while the other one is weakly transmittive. We will begin our analysis with a Hamiltonian that I am now going to write in the Heisenberg picture.

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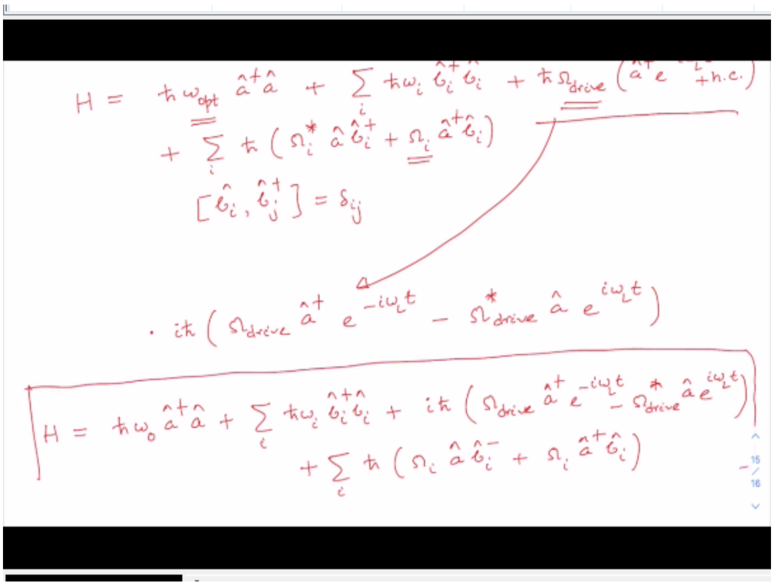


Let me first write down the full Hamiltonian H = h cross omega optical a dagger a. This is the Hamiltonian which denotes the energy of the optical mode in the cavity, where omega optical is the resonance, cavity resonance frequency. Then we have another term that is summation over all bath oscillators, h cross omega i, b i dagger. b i. So, this term describes the energy of the bath oscillators in fact these are electromagnetic oscillators.

This vacuum we are now modeling it as a collection of independent harmonic oscillators and it surrounds this optical cavity only with the constraint that it has to has is that b i, this commutation relation has to be satisfied b i b j dagger should be equal to delta ij and then we will have another term that is it is now driven by a laser from outside. So, that is taken into account by h cross omega drive this is the driving amplitude.

And we have this term a dagger a photon is created inside the cavity and the laser frequency is omega L and it should be Hermitian. So, therefore there is a term Hermitian conjugate and many times this particular term is people write it in this form also. Say i h cross omega drive because here the drive amplitude is considered to be a real quantity, but it may be a complex quantity because it has a phase part so people write it in this form also, omega drive a dagger e to the power - i omega L t - omega star.

This omega drive, it is complex a e to the power i omega L t. So, this is another form and in fact it is the most general form and maybe let us consider this particular form rather than this one here we will come to that and there will be another term that would be summation over all the bath oscillators h cross omega i star a b i dagger + omega i a dagger b i. So, this particular term actually refers to the fact that there is a coupling between the system and bath.

And the strength of the coupling between the cavity mode and the ith bath oscillator mode is given by omega i. So, this is the usual coupling and we will now going to analyze it but as I said let me rewrite the whole Hamiltonian in this form and this is what we are going to analyze. Rather than writing omega opt, so let me just put h cross omega 0 here that is the optical cavity resonance frequency, h cross omega 0 a dagger a + summation h cross omega i b i dagger b i and then let me now take the general form here.

That is i h cross omega drive a dagger e to the power - i omega L t - omega drive star a e to the power i omega L t and finally we have summation h cross omega i. So, please do not get confused with omega terms here, this is omega drive and this is the omega i, omega refers to the coupling between the optical cavity and the bath oscillator. So, what this particular term says that when a optical cavity is getting annihilated and this is getting resulted in the creation of a bath mode or opposite process can also happen.

So, this is basically is happening due to the coupling between this bath oscillator mode and the optical mode. So, we are now going to analyze this particular Hamiltonian and this **is** will give us lot of insight about what is going on as regard quantum noise effect in an optical mode is concerned. Let me stop here for today. In this lecture we discussed about the quantum Langevin noise, we have worked out the autocorrelation function for the quantum Langevin noise.

And also, we saw that the quantum spectral noise density is not symmetric unlike its classical counterpart and finally we started discussing on the effect of quantum noise on an optical mode in a Fabry-Perot cavity and will continue this very important discussion in the next class. So, see you in the next class, thank you.