Quantum Technology and Quantum Phenomena in Macroscopic Systems

Prof. Amarendra Kumar Sarma
Department of Physics
Indian Institute of Technology-Guwahati

Lecture-37
Cavity Optomechanics: Langevin Equation

Hello, welcome to lecture 6 of module 3, this is lecture number 27 of the course. In this lecture we will continue our discussion on classical Langevin noise and then we will go over to the quantum counter part of it.

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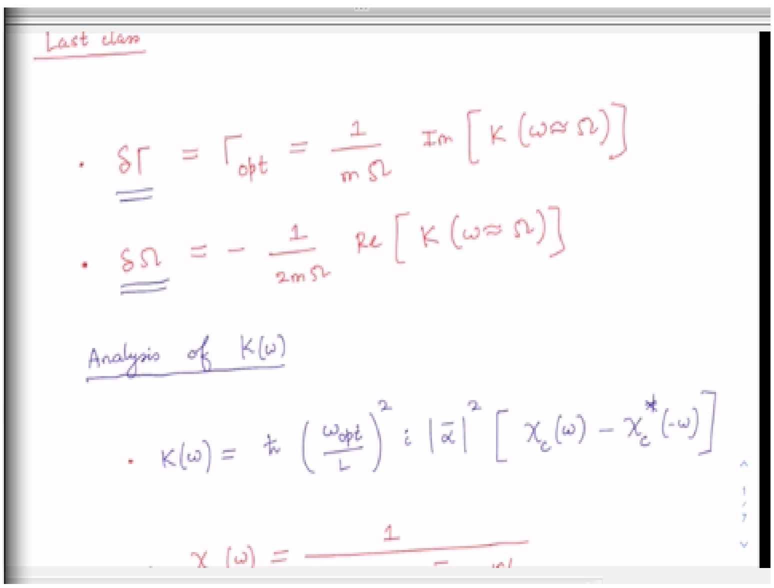

So, let us begin. In the last class we continued our discussion on classical regime from the previous one and we analyze the function K of omega. And we already know that the real part of the function K of omega is related to the frequency shift of the mechanical oscillator. On the other hand the imaginary part of the function k of omega is related to the extra damping. And this damping is induced due to the coupling to the light mode.

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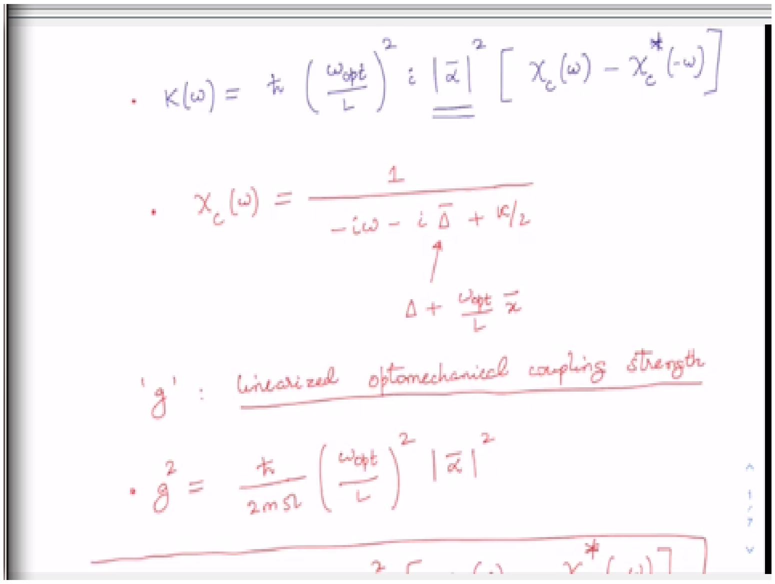

And this function K of omega it is a function of the intensity of the laser light as well as the susceptibility of the cavity field. Because we know the expression for the susceptibility and to also simplifying the notation by introducing a parameter called g that is the linearized optomechanical coupling strength.

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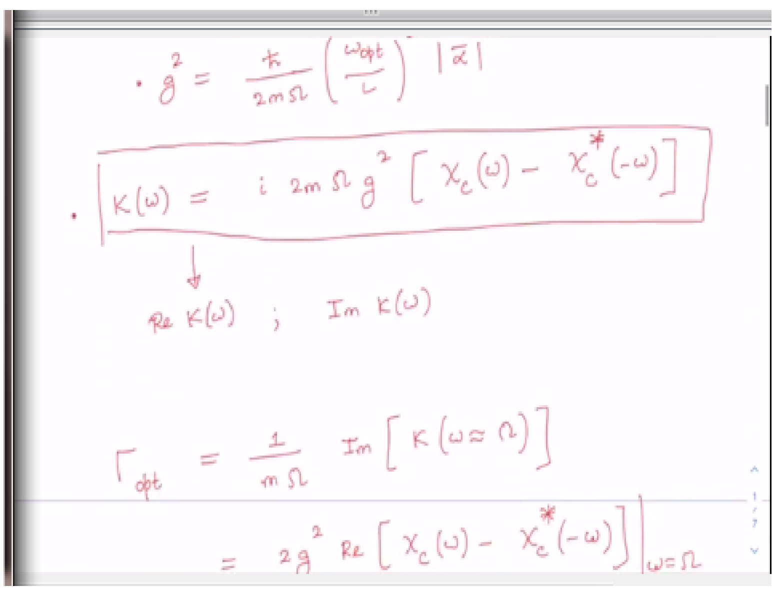

We can easily work out the explicit expression for K of omega and from there we can find out the real part as well as the imaginary part.

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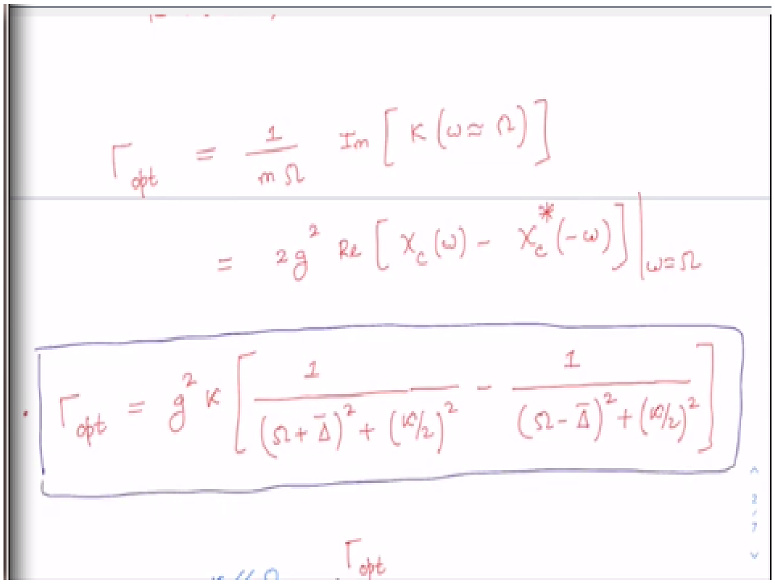

So, when we analyze it this expression can be worked out this is the extra damping or the light induced damping term.

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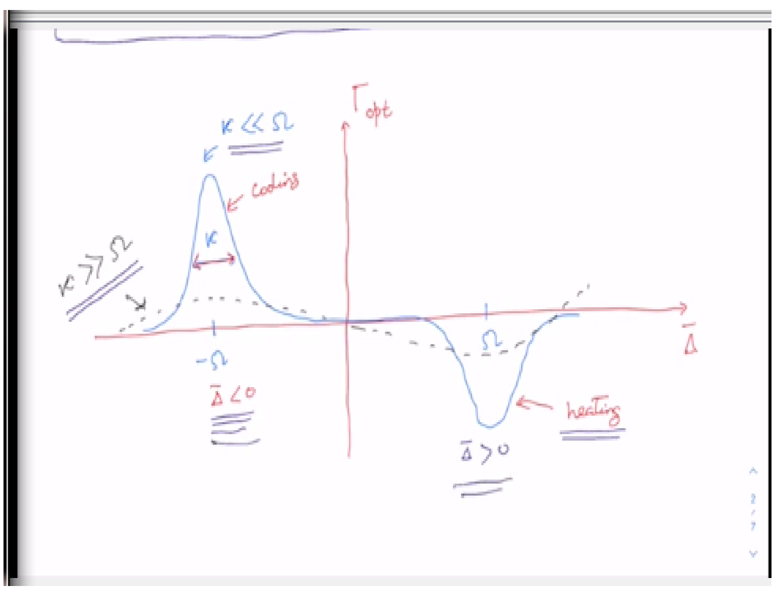

And when we plot the damping optical damping versus the detuning parameter we saw that for the regime when the cavity damping kappa is much, much larger than the resonance frequency of the mechanical oscillator. We get a anti-symmetric plot and we shows that for the detuning when it is negative delta less than 0, we have a positive damping. On the other hand if delta this detuning parameter is greater than 0.

We have a negative detuning, that means we will get the heating effect when the detuning is positive and we will get cooling effect when the detuning is negative. However these peaks the 2 peaks that we obtain are not resolvable much and the effect is also not that great. But if we go into the other regime when this cavity decay rate is much, much smaller than the resonance frequency of the mechanical oscillator, then these 2 peaks are highly resolved.

And this regime is known as the resolved sideband regime. And here we will get prominent cooling or heating effect depending on whether the detuning parameter is less than 0 or it is greater than 0.

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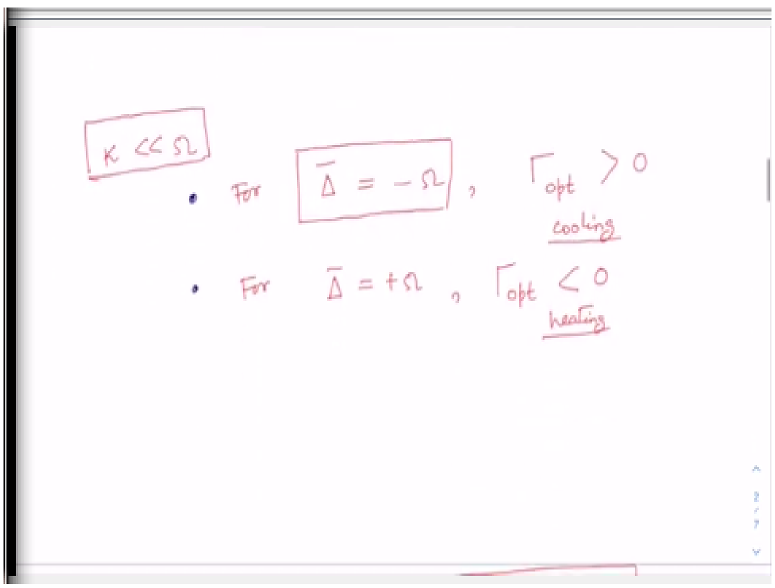

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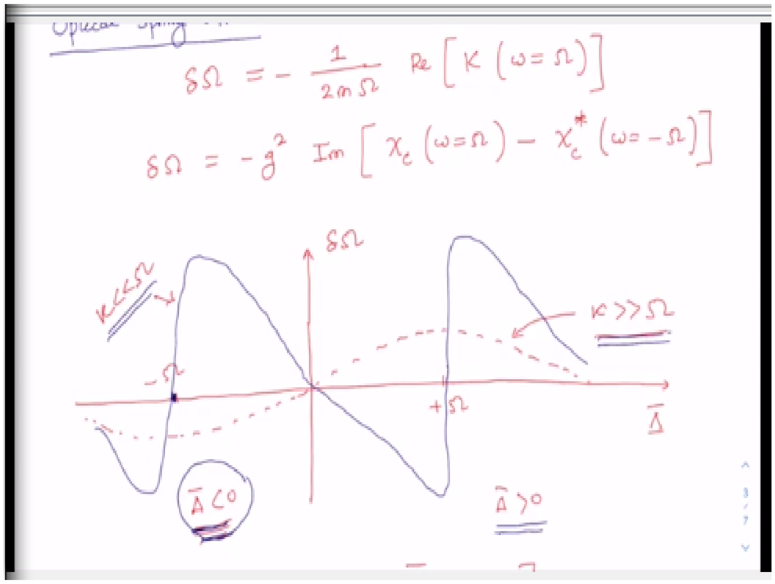

So, then we also discussed the optical spring effect. Here when the regime this kappa much, much greater than the resonance frequency of the mechanical oscillator, we get an anti-symmetric plot where we plotted the frequency versus the detuning parameter. And we saw that in the negative detuning regime when delta bar is less than 0, the frequency shift is negative on the other hand if delta is greater than 0 that means for positive detuning, the frequency shift is positive.

Which essentially mean is that for delta less than 0 delta bar that is the modified detuning parameter. If it is less than 0 then when the frequency shift is negative, that means the spring gets softer. On the other hand when the frequency shift is positive the spring get actually harder. And this has a problem because as regards the cooling is concerned. We know that delta less than 0 is the domain where cooling is observed.

And if we try to cool it harder I am talking about the regime where kappa is much, much greater than omega, if we try to cool the mechanical oscillator harder we will have to increase the intensity. And in that case the spring will get softer and softer and that will result in instability. This issue can be circumvented if we go over to the other regime that is the resolve sideband regime and here this optical spring effect can be avoided. So, this concluded our discussion on the classical regime.

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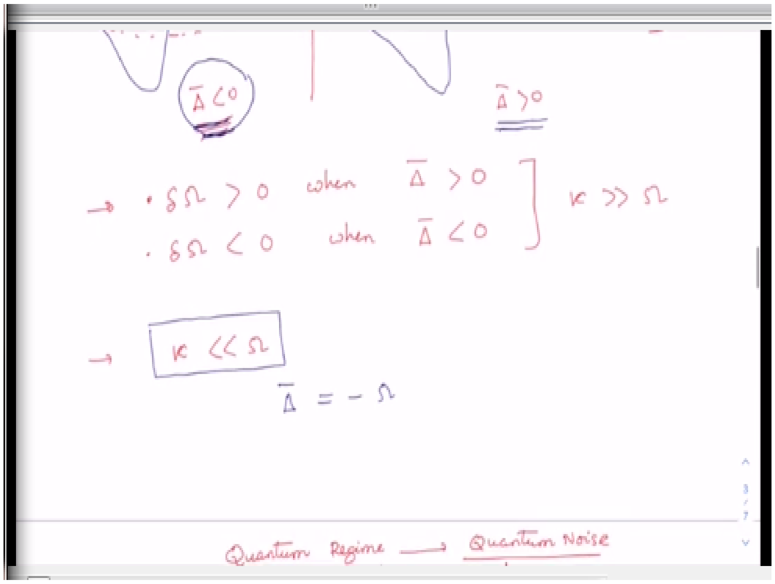

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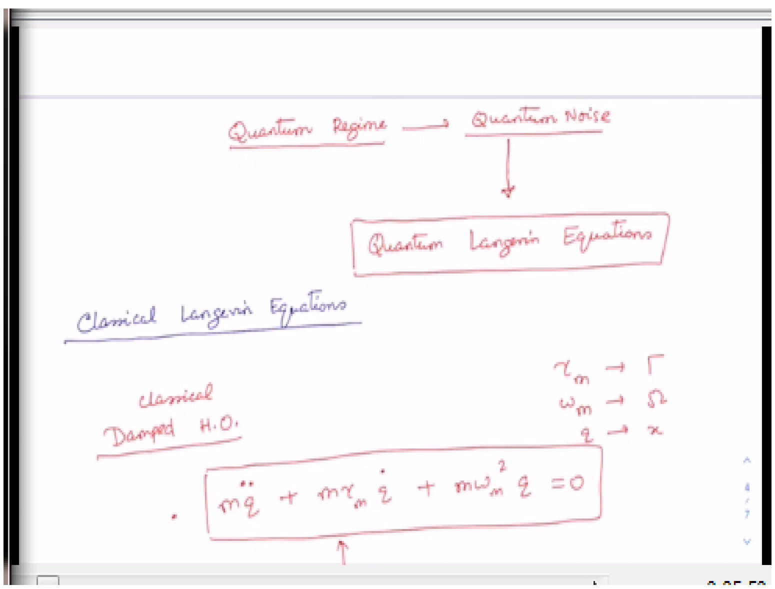

Now we are ready for discussion on the quantum regime. However, the quantum regime is different from the classical regime due to the so-called quantum noise. And quantum noises are generally discussed by various formalism and one formalism is quantum Langevin approach or the so-called quantum Langevin equation. In this course we are going to deal the issue by quantum Langevin equation to appreciate quantum Langevin equation we started discussing classical Langevin equation.

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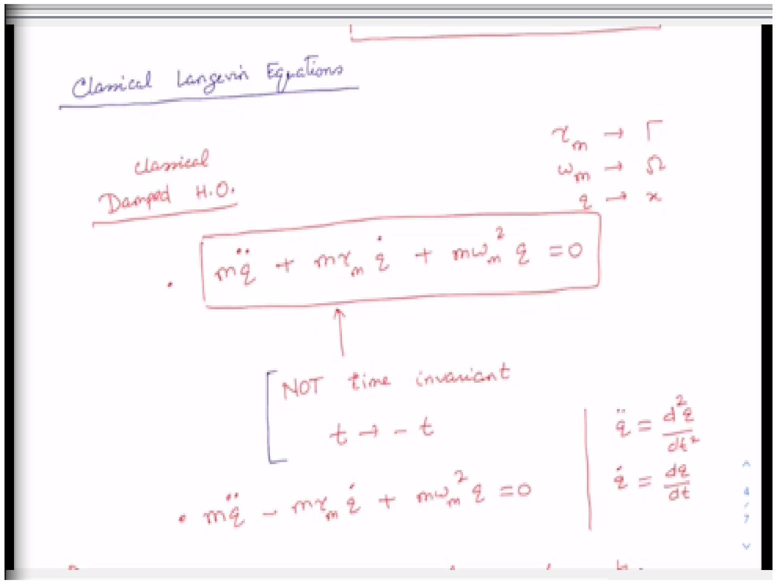

And to do that we begin with the usual classical damped harmonic oscillator model but this model is not exactly accurate. Because in thermal equilibrium as per the solution of this equation as I explained in the last class that the amplitude or the displacement of the oscillator will decay but that is not the case in thermal equilibrium because of thermal equilibrium the oscillation will not actually die down with time. And this is a fundamental reason behind it because this model as per this model is not time invariant.

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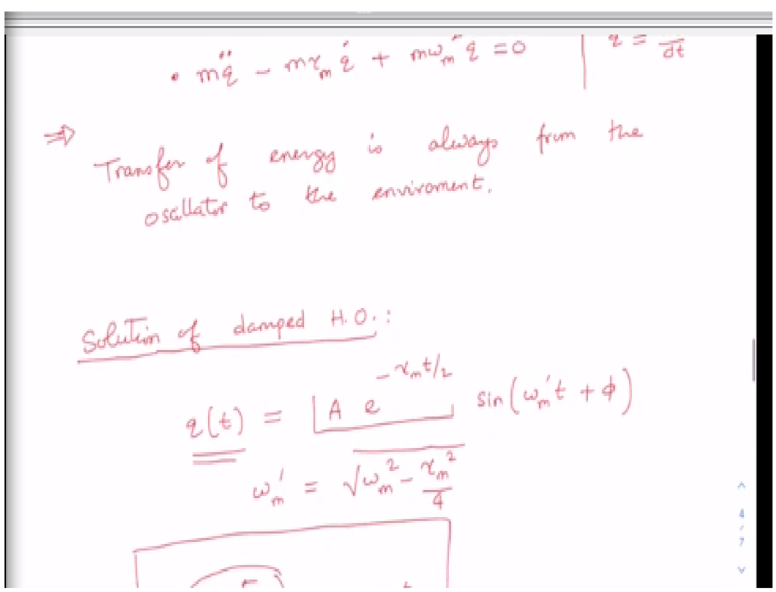

And physically it means that the transfer of energy is always from the mechanical oscillator to the environment, not from the environment to the mechanical oscillator. And if we want to be in equilibrium then both environment as well as the mechanical oscillator has to participate equally.

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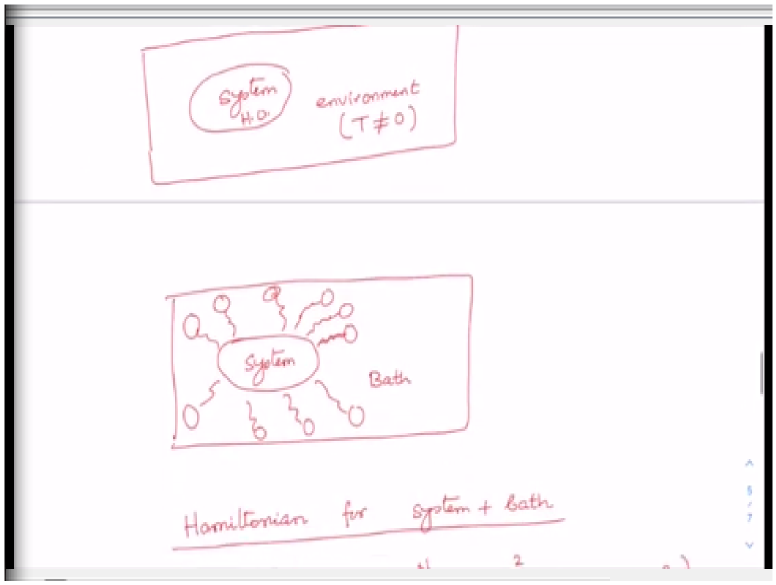

So, this issue is addressed by considering the environment which is also known as the bath, we model the bath as a collection of independent harmonic oscillator.

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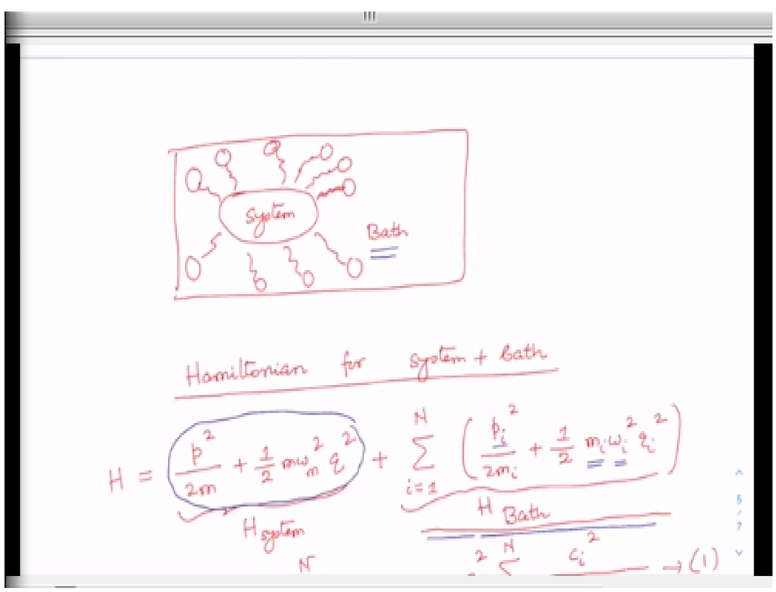

Here we consider it as a collection of n number of harmonic oscillator having different position variable and the momentum variable and frequency ith oscillator has frequency, omega it is mass as m i like that. And the system is simply a single harmonic oscillator we took, so this is the Hamiltonian photo system and this is for the bath.

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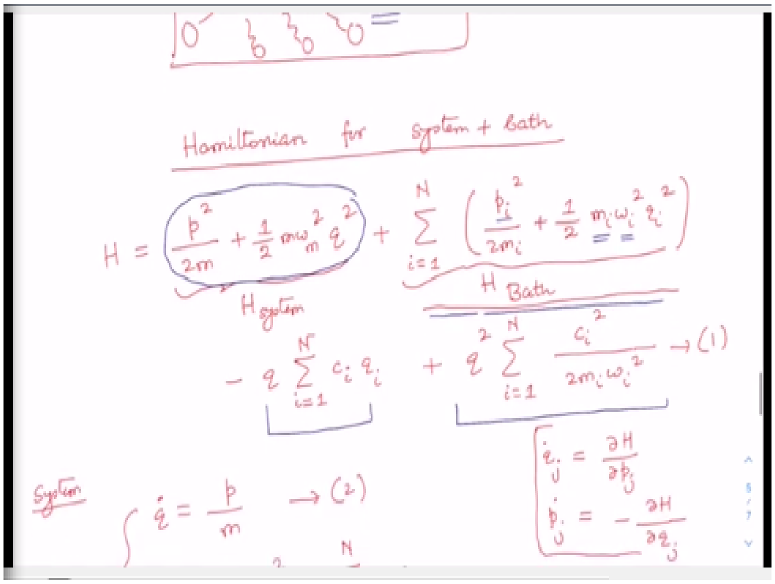

And this is the coupling between the system coordinate and the bath coordinate and this term is there just to compensate or cancel the frequency shift due to the coupling between the system and the bath.

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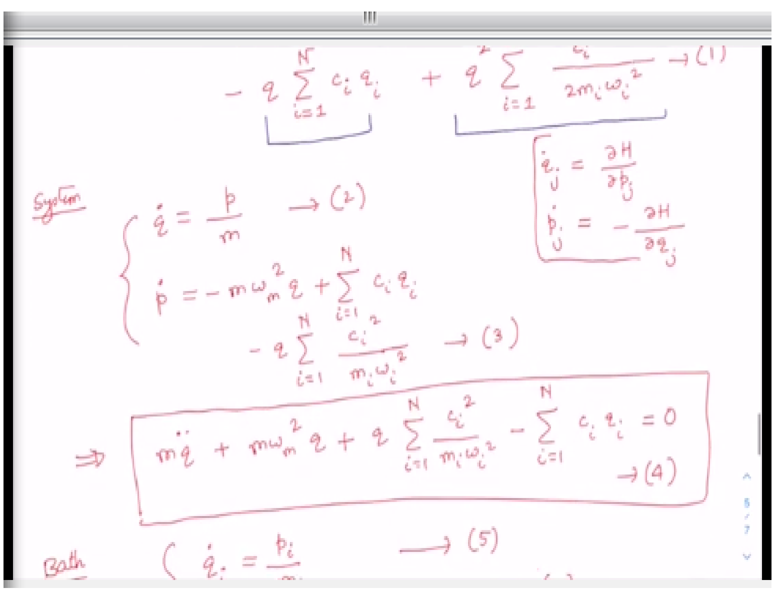

Then we went on to write down the equation of motion for the bath variable as well as the system variable.

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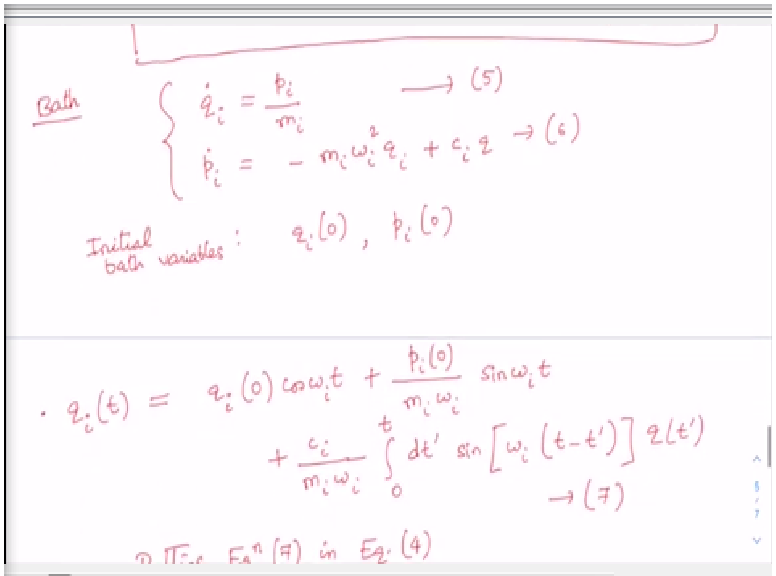

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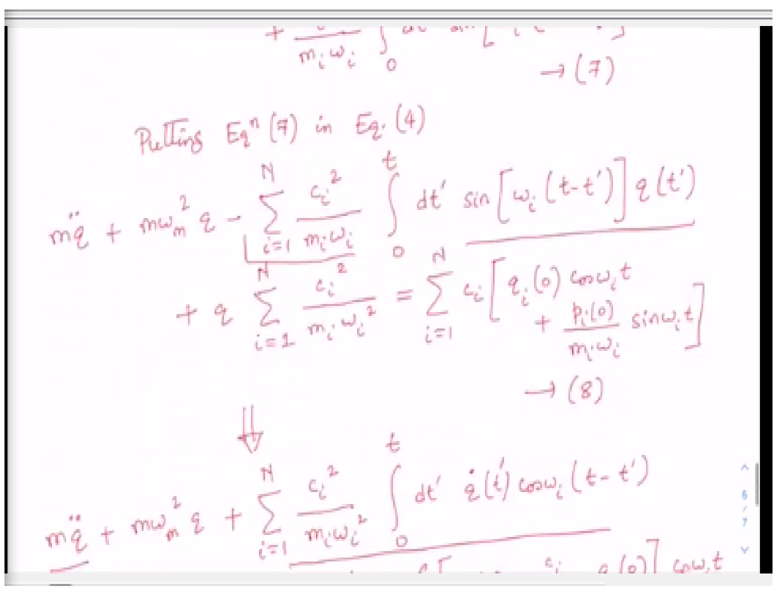

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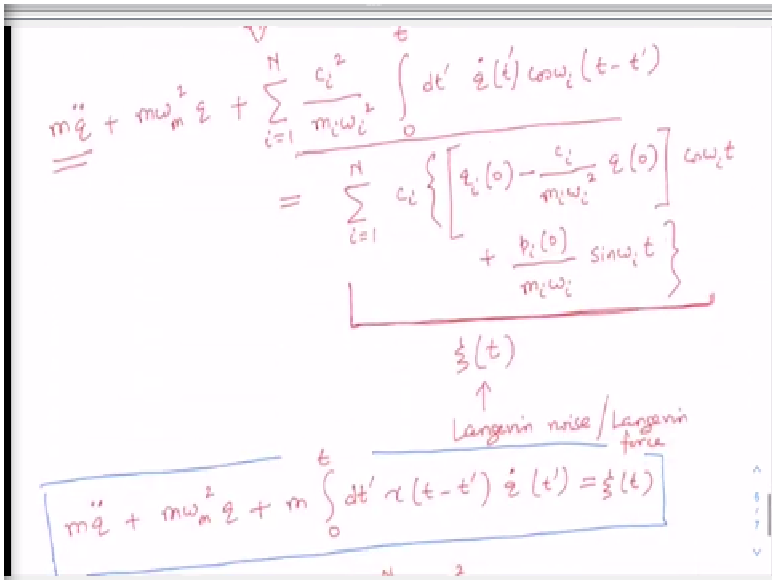

And we after doing the analysis we ended up into a very nice equation. And this model we finally got and it turns out that in the earlier model the right hand side was 0 but now we are having a non-zero term and this term is known as the Langevin noise or it is also called the Langevin force. And this term is arising because of the contribution from the surrounding or the environment.

So, this model nicely explains many of the classical statistical phenomena as regards this damped harmonic oscillator is concerned. And here we defined one function that is called, it is known as the memory function.

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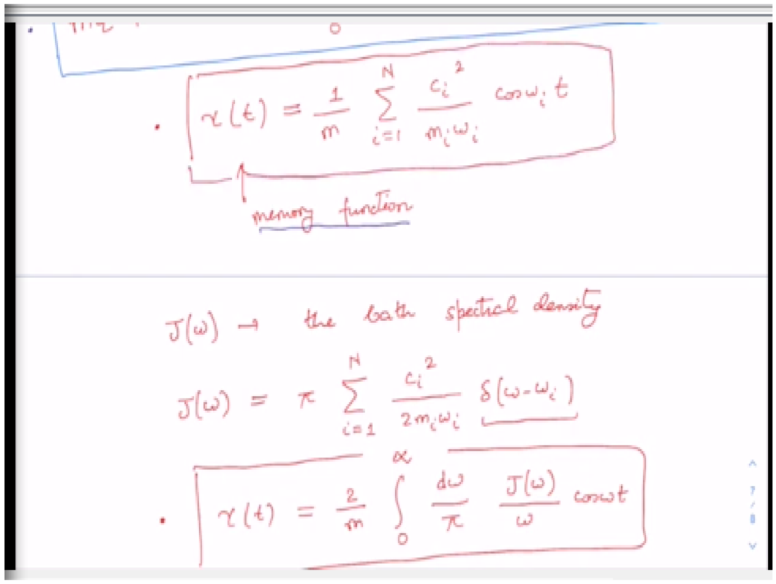

We can simplify this memory function and to do that let us define a function called denoted by J of omega and this is known as the bath spectral density. So, it is defined like this J of omega is equal to it would be pi into sum overall the N bar oscillators and it is c i square divided by twice m i omega i and delta omega - omega i. So, this is delta omega - omega i, this is the delta function.

And using this we can simplify this expression or we can rewrite it in interesting form. So, in terms of this bath spectral density we can write gamma of t the memory function as 2 by m integration 0 to infinity d omega by pi J of omega by omega cos omega t. So, this is exactly the same function as that of this one here. If you are not convinced let me quickly prove it for you, so let me start with this function.

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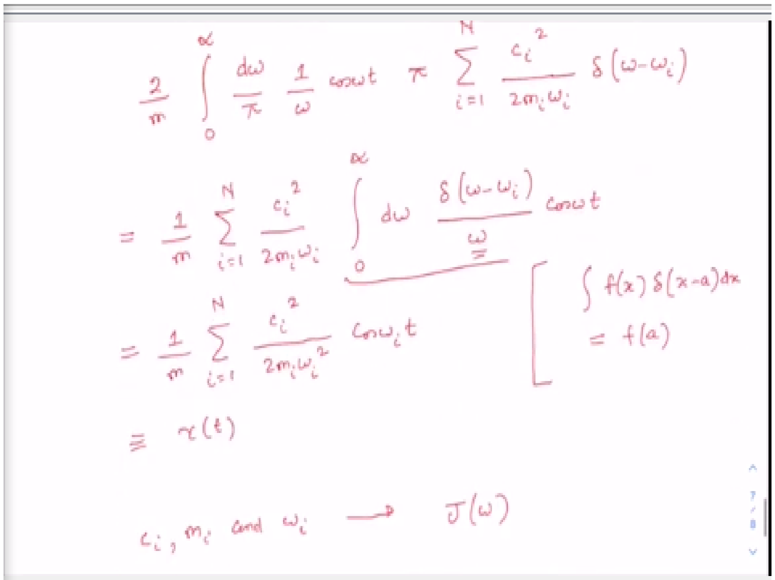

So, we have 2 by m integration 0 to infinity d omega by pi and we also have 1 by omega cos omega t and J omega, J omega let me now put the function J omega here, this one. So, that is pi into summation i = 1 to N c i square twice m i omega i and delta omega - omega i. This I can write as 1 by m summation i = 1 to n ci square divided by twice m i omega i and integration 0 to infinity d omega delta omega - omega i divided by omega and I have here cos omega t.

Now you know from your mathematical physics course or mathematics course this property of the direct delta function, suppose we have this function f of x and delta function delta x - a is there and this integration will give you f of a here. So, you can easily use that, using this property I can write it as 1 by m summation i = 1 to n. And I have c i square divided by twice m i, now this would become omega i because omega = omega i have to put. So, therefore I will have twice m i omega i square cos omega i t.

So, if you now you can see that this is exactly this function. So, this is exactly the same form that we obtained. So, you see that the unknown parameters this c i, m i and omega i these are unknown parameters, these are related to the bath. This can be expressed by using a single function J of omega the way we have defined this function J of omega **ok** here. So, this function is very important and for many and as I said it is called the bath spectral density.

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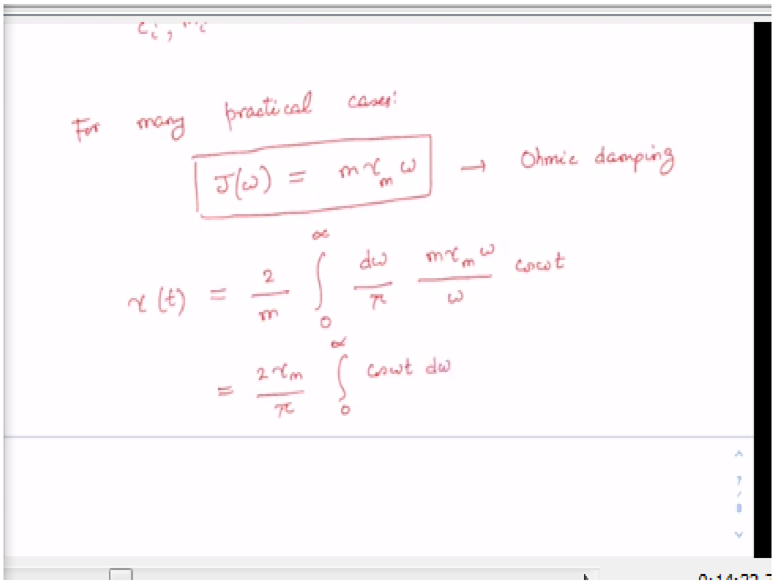

For many practical cases or realistic situations this function can be chosen as J of omega as m into gamma m into omega. And this is termed as Ohmic damping, it is significance will be clear very soon to you, it is called Ohmic damping. Now if I take my bath spectral density to be of this form then you will see that the memory function gamma of t it will take a simple form.

And this memory function is in terms J of 0 omega is this 2 by m 0 to infinity d omega by pi and here we have here J of omega, we are now choosing it as m gamma m into omega divided by omega cos omega t. And this I can write as twice gamma m by pi integration 0 to infinity cos omega t d omega. Now here we can apply a trick here.

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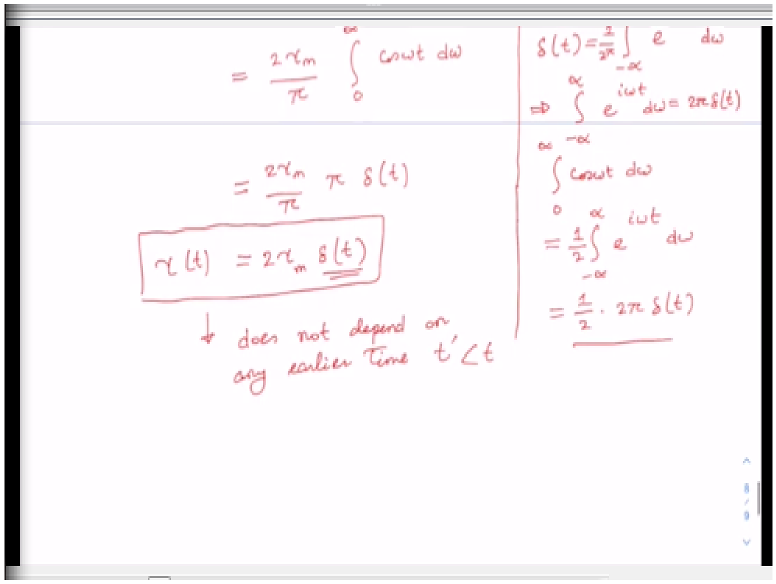

Because we know the so-called delta function say delta of t is defined as -infinity to plus infinity e to the power i omega t d omega in fact here 1 by 2 pi is also there. So, this means that I can now write it as -infinity to plus infinity e to the power i omega t d omega = twice by delta of t and this function 0 to infinity, this integration 0 to infinity cos omega t d omega who is I can write it as minus infinity to plus infinity. Because it is an even function I can write it as minus half of minus infinity to plus infinity and I can write here e to the power i omega t d omega.

And as you can see then I can have it is half into 2 pi delta of t. So, what I have here, I can express this whole thing as twice gamma m by pi and using this relation here, I have it as pi into delta of t or I will have it as gamma of m, in fact twice gamma m into delta t. Now you see this clearly shows that this memory function gamma of t is does not depend on any previous time or any earlier time.

So, it does not depend on it is instantaneous, you see because of this delta of t it does not depend on it is history or on any earlier time say t dash. So, t is always greater than t dash, this implies that the bath has no memory and this process is said to be Markovian. So, this is a Ohmic damping process. An omega dumping process bath contains no memory of it is earlier history and these processes are called Markovian process.

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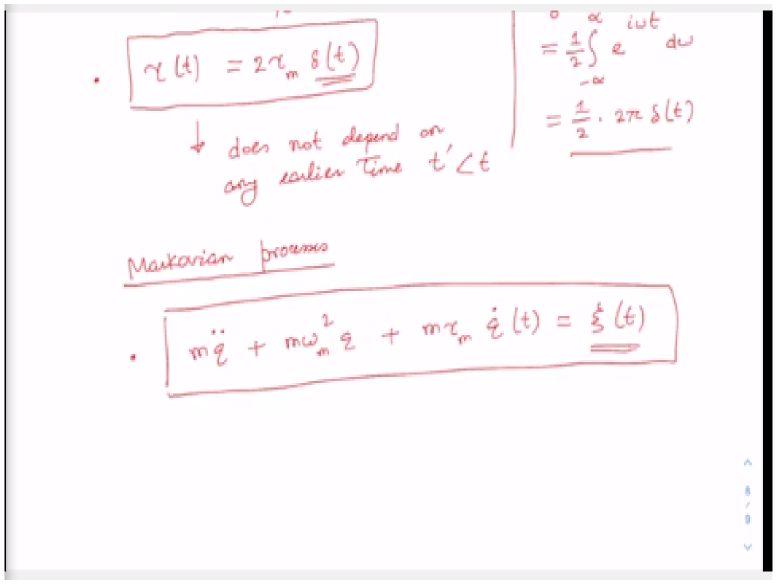

And Markovian processes are easy to deal with and mostly the systems we deal with are Markovian. And in this course we will focus only on Markovian processes only as regards the bath is concerned. So, now taking this as our memory function we can very easily rewrite our equation of motion for the damped harmonic oscillator. And under that process the damped harmonic oscillator equation would be m q double dot you can verify it, it will be m omega square q.

And here you will have m gamma m q dot t is equal to this is the Langevin noise or the Langevin force. So, this equation finally we have obtained and maybe some of you have seen this kind of equation in your in classical statistical physics or in statistical mechanics book. And now we will focus a bit more on this function or this Langevin noise or Langevin force because this is going to be very important and this will enable us to understand intricacies of quantum noise.

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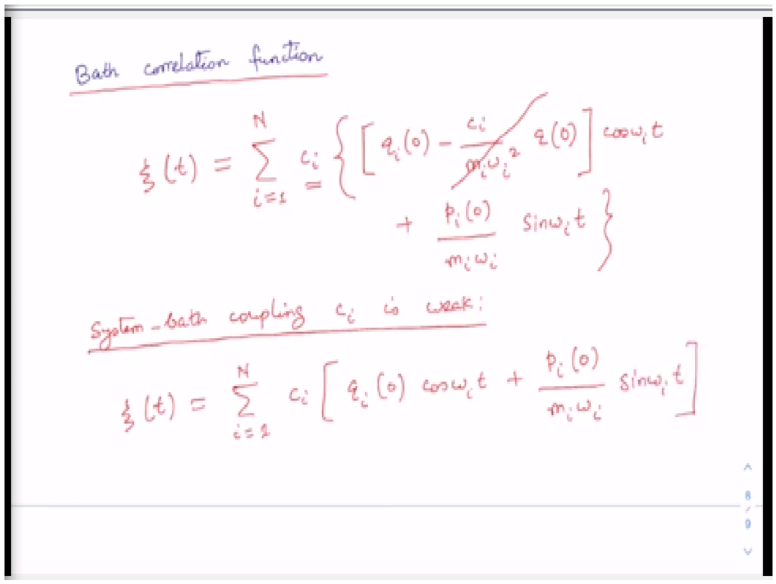

The parameter for the Langevin noise or Langevin force Xi of t we express it as follows it was sum over all N oscillators i = 1 to N c i we have actually written it previously, I am just repeating it again here q i of 0 that is the initial position of ith path oscillator, c i is the coupling, m i omega i square q of 0 that is the initial position of the mechanical oscillator cos of omega i t + p i of 0 divided by m i omega i sine omega i t, this is the expression.

Now if the system bath coupling c i is weak then we can ignore this second term here. And if we ignore this second term then we can write Xi of t the Langevin noise, this is we are doing under the approximation that system bath coupling is weak, coupling c i is weak. Then as you know that ci square would be further weak, then Xi of t we can write it as summation i = 1 to N, first term will retain, this would be c i, q i of 0, cos omega i t + p i of 0 divided by m i omega, i sine omega i t.

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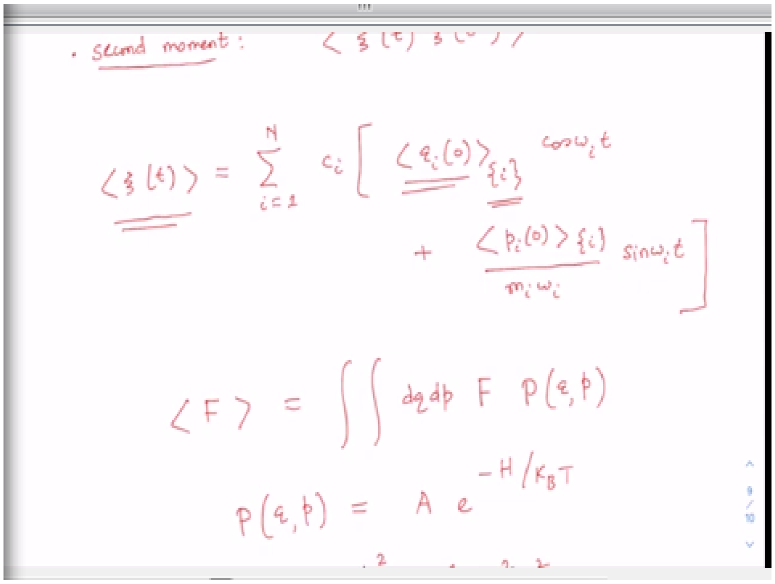

To understand the nature of this Langevin noise let us now calculate the first 2 moments. First moment would be the mean that is Xi of t and it would be average over the bath and the second moment and which is basically the autocorrelation function the Xi of t the Langevin noise at time t and Langevin noise at another time t dash. So, this will give us the correlation between these bath variables or Langevin noise at 2 different times.

So, now we are going to calculate these 2 very important quantities, averages of the Langevin noise and the second moment. So, to calculate Xi of t, if we take the average we now have the expression when the coupling is weak then it would be summation i = 1 to N c i and here I have q i of 0 I have to take the ensemble average of all the oscillators that is the meaning of the symbol here, I am taking the ensemble average and I will explain it further.

I have cos omega i t, then from this I will have p i 0 again it is the ensemble average divided by m i omega i sine omega i t. This calculation may look cumbersome but actually it is straightforward and I could have actually directly written down the answers what is Xi of t and what is the second moment. But I think it is better to do it in details because it will be very useful for you.

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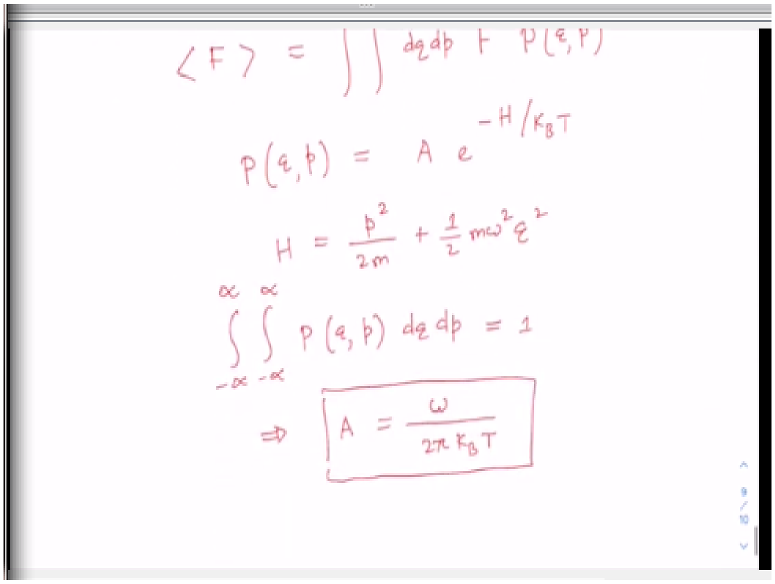

Now hopefully you know that the average of F variable say F, say average of a variable F in the phase space is given by this. It is integration over all the phase space variable q and p dq, dp and then we have this variable F and then we have this probability function P is a function of the variable q and p. And here this probability function we are taking it the so-called canonical ensemble here.

Therefore the probability function would be this, it would be some constant into e to the power -Hamiltonian over the energy divided by K B T. In fact here H is equal to it is a harmonic oscillator we are dealing with, this is H = p square by 2m + half m omega square q square and this probability if you integrate it over the whole phase space. And you know that this integration must give you = 1. Because the total probability must be = 1 over the when you integrate it over the whole phase space.

And using this actually you can find out this in constant A and it is very easy to just use the Gaussian integral formula, then you will get it as omega by 2 pi K B T. Then we are going to use this later on as you will see. Now the average of the initial position say this quantity because to calculate this average of this Langevin noise we need to calculate this ensemble average of the initial position of the ith bath oscillator as well as the corresponding the initial momentum of the bath oscillator.

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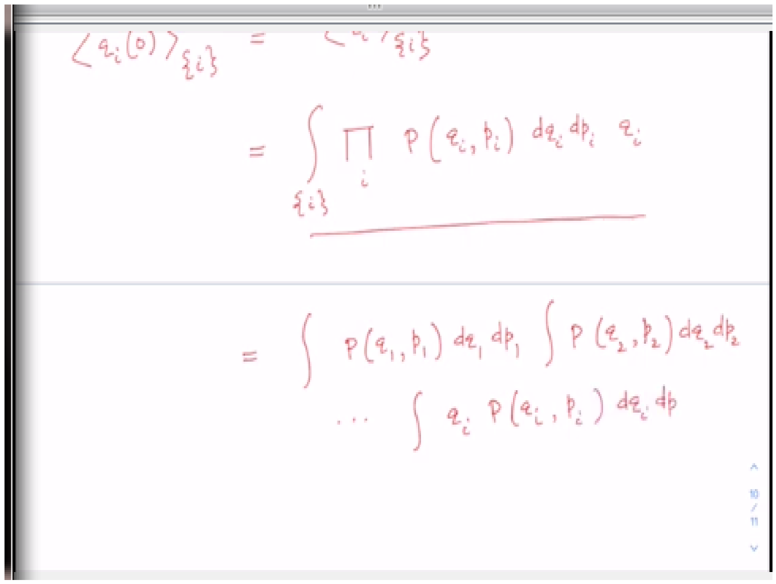

First let me show you how to calculate this quantity q i initial position of the bath oscillator in fact the ensemble averages. So, rather than writing it this 0 here I will just write it as q i, this is what we are now going to calculate. Now this would be integration over the whole phase space but we are going to calculate it for all the oscillators, I will explain it. And this is basically the multiplication, ok I will explain it just let me first write it.

P, this is the probability function q i, p i and we have here dq i, dp i and of course we have also q i, q i is the variable. Now let me explain this particular term, so we have many variables and suppose we have first variable is say q 1 and the corresponding probability function for that is a function of q 1, p 1. And then we have dq 1, dp 1 and then we have probability q 2, p 2 and then we have dq 2, dp 2 and so on. But we are now interested in working out if the variable we have is q i. So, we have this probability function q i, p i and we have dq i, dp i.

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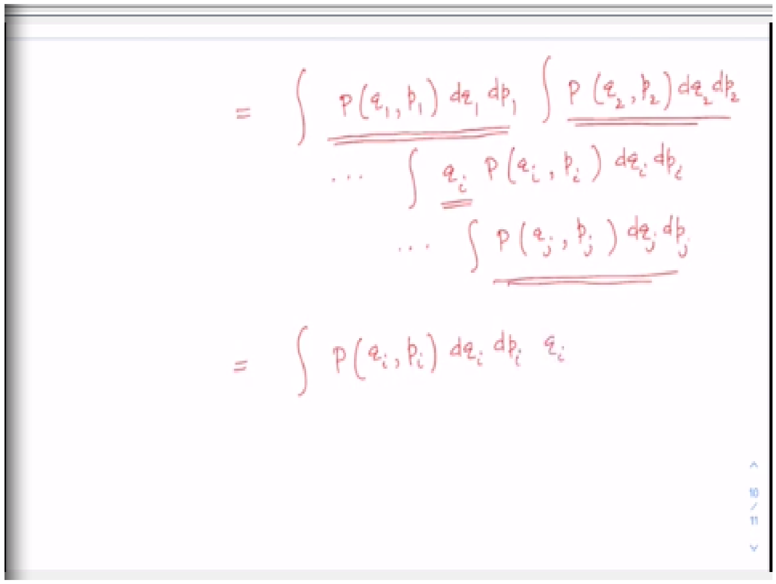

And then we also if we can have another say Jth variable, J oscillator so q j, p j, so dq j, dp j but we need to calculate only the q i this thing oscillator ensemble average for this particular oscillator. So, as you know that this particular term this is the probability, total probability with respect to the variable in the phase space this would be equal to 1 corresponding to q 1, p 1 variable.

And similarly for q 2, p 2 this would be equal to 1 because the total probability as I think I wrote it here that this total probability, yes, this one. It has to be equal to 1, so all the terms will get normalized and you will be left out with only one corresponding to 1 variable say q i. So, I hope you are getting the notation here, so this would be this probability function of the variable for the oscillator ith oscillator q i, p i and you have here dq i, dp i. And this is the integration we have to work out.

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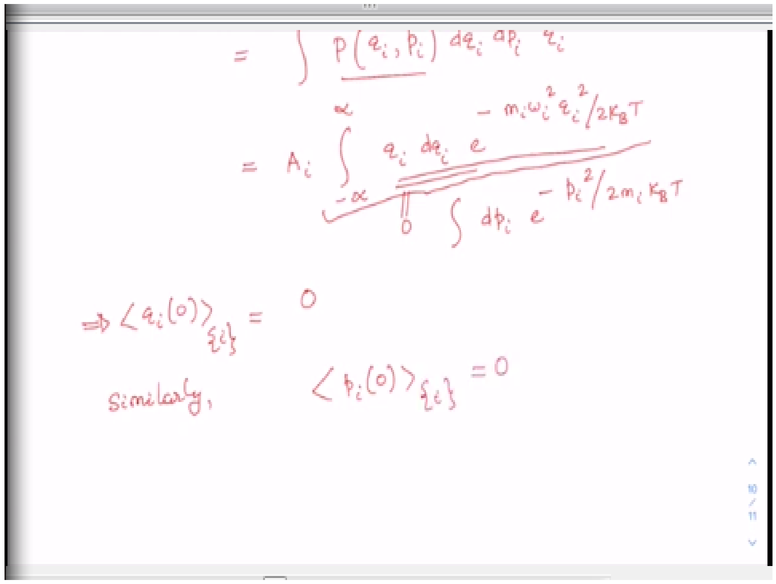

And then I have here A i and integration minus infinity to plus infinity, I know what is this function. So, we have here say for q i, dq i and e to the power -m i, omega i square, q i square this is from the Hamiltonian function, I have here twice K B T. Let me again show you here the Hamiltonian function I just put it here for the corresponding ith oscillator and then the for the momentum part I have here dp i e to the power -p i square divided by twice m i K B T.

Now as you can see that this integration, this integrant here is odd function, ok so therefore this is going to be 0. So, overall I get that this implies that the average value of the bath position. So, this would be if you take the ensemble average you are going to get 0. So, in the similarly you can show that the corresponding momentum ensemble average for the momentum is also going to be 0. Now because we have both these average quantities this is 0 and this is 0.

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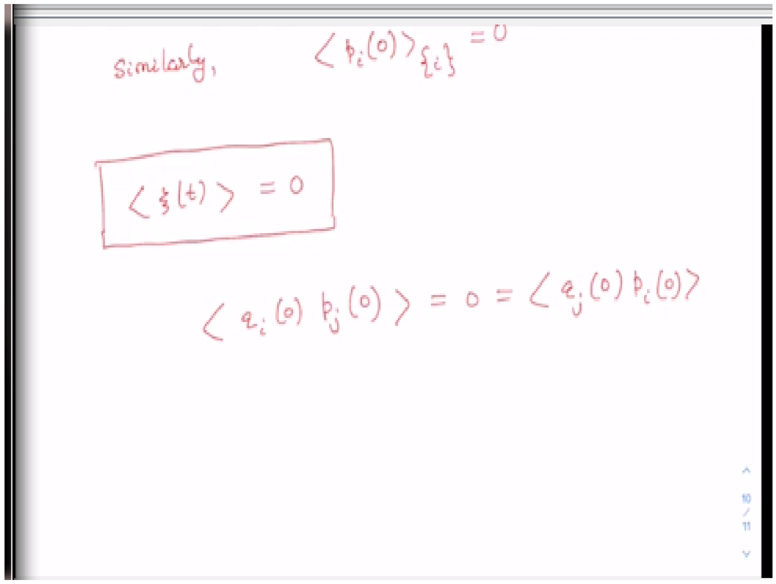

So, quite clearly the average of the Langevin noise Xi of t = 0 and this is also expected physically because this is fluctuation after all, so this is equal to 0. Now we should be useful for us using the ordinance of the intrigrands it can easily show that and it is easily understandable that you will get say q i of 0, p j of 0 and that would be equal to 0. And this is actually the same as for the 2 different oscillators say ith oscillator and jth oscillator.

If you calculate this particular quantity, averages of the products here for the jth oscillator and the ith oscillator their corresponding momenta and position, so you will get this as 0. And that is because of the integrand is same odd integrand. So, this would be useful for us.

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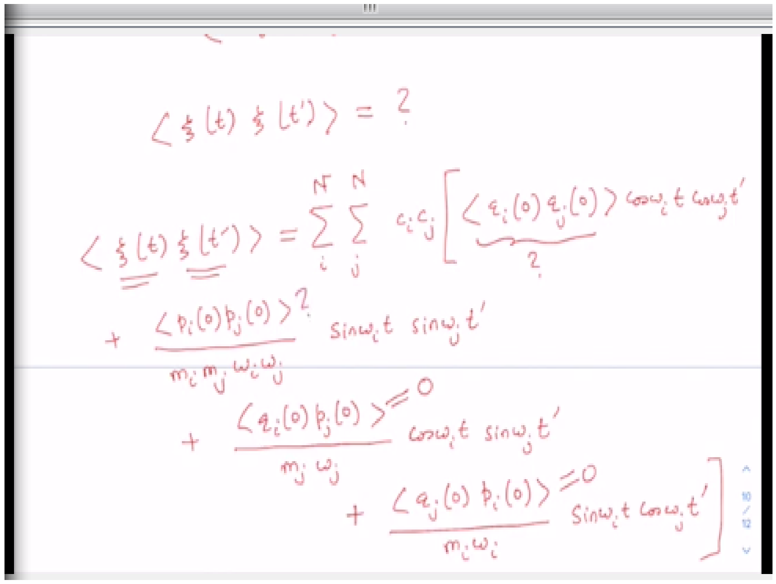

Now what about this particular quantity that is the second moment Xi of t and Xi of t dash, let us work it out now. Now you can actually put down the expression for Xi of t and Xi of t dash and take the product and if you take the average let me write down what you will get. You will get there will be 2 sum for the ith oscillator and the jth oscillator. You are taking Xi of t it is taking over the sum over the variable i and Xi of t dash you are taking the sum over the variable j, so that is why I am combining both these things in this expression ij say going.

Otherwise I can also write it here I have i and then I have j going to N and then I will have c i, c j and here I will get several terms, let me write down all the terms q i 0, q j 0. And here you will have cos omega i t cos omega j t dash you just have to take the product and then the average. Then you will have p i of 0, p j of 0 then you take the average, then you have m i, m j omega i omega j sine omega i t sine omega j t dash.

And you will have term q i 0, p j 0 and you have to take the average divided by you will have m j omega j ok, and you have cos omega i t sine omega j t dash plus the last term would be q j 0 p i 0 and take the averages divided by m i omega i sine omega i t. Then you will have cos, you will have here it would be cos omega j t dash. So, these are the terms you will have.

Now already our life is simple because we know that this is equal to 0. So, therefore this particular term would be 0 and similarly this will be 0. So, we will be left out to just calculate this particular quantity and this particular quantity. It can be very easily calculated let me show one calculation.

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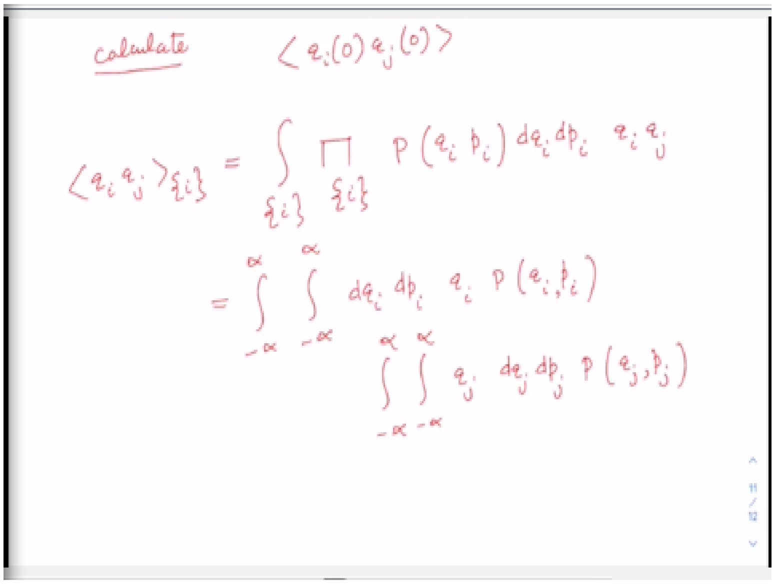

So, let us calculate this quantity say let us calculate in the similar way you can calculate it would be q i of 0 and q j of 0, so let us calculate it. So, I am going to find out the ensemble average, so I have here it has integration over the whole phase space for all the oscillators, then all the variables, all the oscillators I considered. In the similar way I have here q i say p i I have variable dq i, dp i and here I have q i q j.

So, this would be actually if I break it up then I have 2 functions minus infinity to plus infinity, minus infinity to plus infinity dq i dp i I have variable number ith oscillator is there corresponding only I have here q i this probability function q i p i. And I have also the jth oscillator for that I have here q j, so it would be dq j dp j and the corresponding probability function is p of q j, p j.

So, this is what I will have integration is from minus infinity to plus infinity. And all the other variables of course would get normalized as I give you the logic little bit ago. Now here quite clearly this will vanish actually, whole integral will vanish if i is not equal to j.

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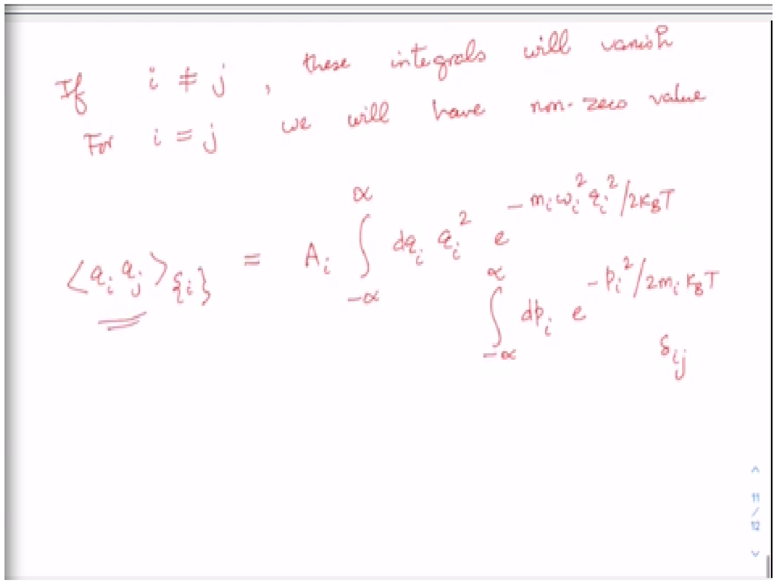

These integrals will vanish and we will get non-zero number, we will get a non-zero value if i = j will have non-zero value. So, therefore let us take i = j non-zero value. So, therefore we will write this ensemble average q i q j and to denote it I will have only one function that is because now i = j, so I have here dq i q i square minus infinity to plus infinity e to the power -m i omega i square q i square divided by twice K B T.

And other one it is integration minus infinity to plus infinity dp i e to the power -p i square divided by twice m i K B T. By the way I have taken i = j, so to note that this is the Kronecker delta remember that I am taking now i = j, if i is not equal to j then this will go to 0. So, this is the meaning of the presence of this Kronecker delta here. Now you can solve this integration very simply just to remind you about the Gaussian integral.

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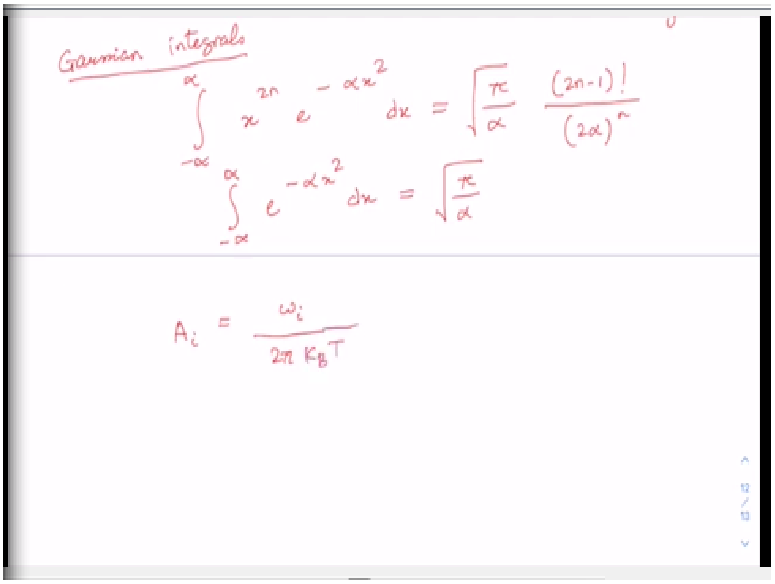

You can use the Gaussian integrals and hopefully you know but let me write here says from minus infinity to plus infinity x to the power 2n e to the power -alpha x square dx this is equal to root over pi by alpha, you have twice n - 1 factorial 2 alpha to the power n or it is also useful e to the power -alpha x square dx minus infinity to plus infinity that would be root over pi by alpha. You can utilize this integral formulas and then if you put A i = omega i by twice pi K B T then you will be able to obtain this expression.

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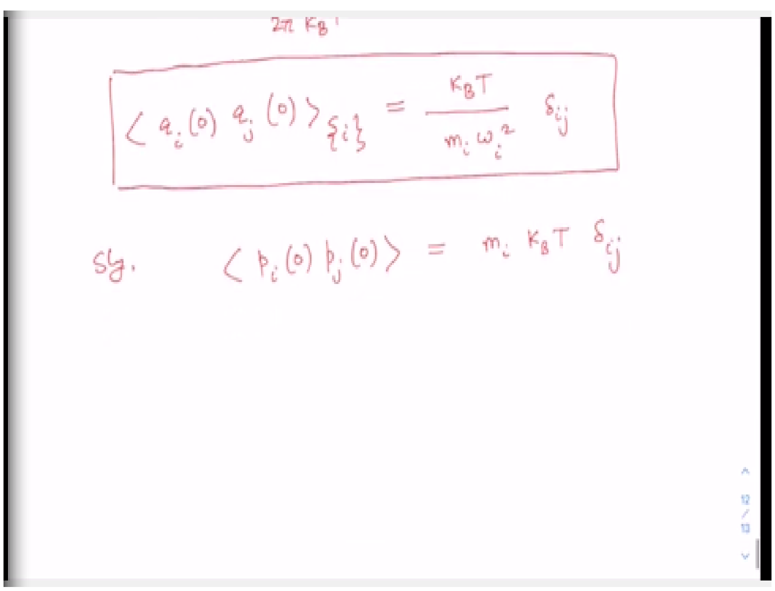

This is very straight forward calculations you can do, some of the calculations we will do in the problem solving session as well for your practice purpose. So, you will be able to get it as K B T divided by m i omega i square and also to emphasize that we are here i = j only then only this would become a non-zero quantity, so we will get a non-zero value. Similarly you can show that exactly similar calculation you can do for the other one for the momentum variables ensemble average for the product of the p i p j that would be equal to mi K B T. Similarly here the Kronecker delta ij. Now having these expressions as I said that now we just need to find out these 2 quantities only here. So, we have found out these 2 quantities.

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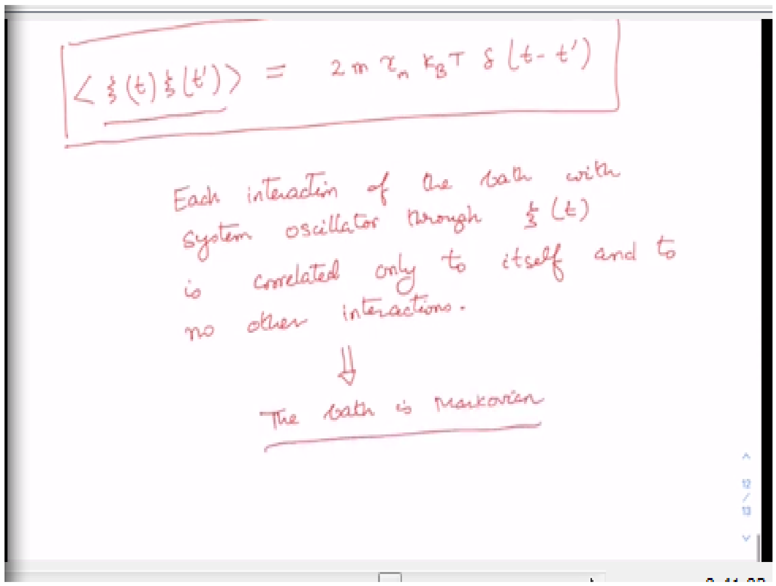

Now if we put it then finally after some manipulation or algebra which will do it in the problem solving session, hopefully we will show it the detailed calculations. You will get this final expression for the moments or the autocorrelation function for the Langevin noise and it has huge physical significance which I am going to explain it would be equal to twice m gamma m K B T delta t - t dash.

So, this is what finally we obtain. This autocorrelation function means that each interaction of the bath. Let me write here each interaction of the bath with the system oscillator through the Langevin noise through Xi of t is correlated only to itself and to no other interaction. What it means is that this means that the bath is Markovian.

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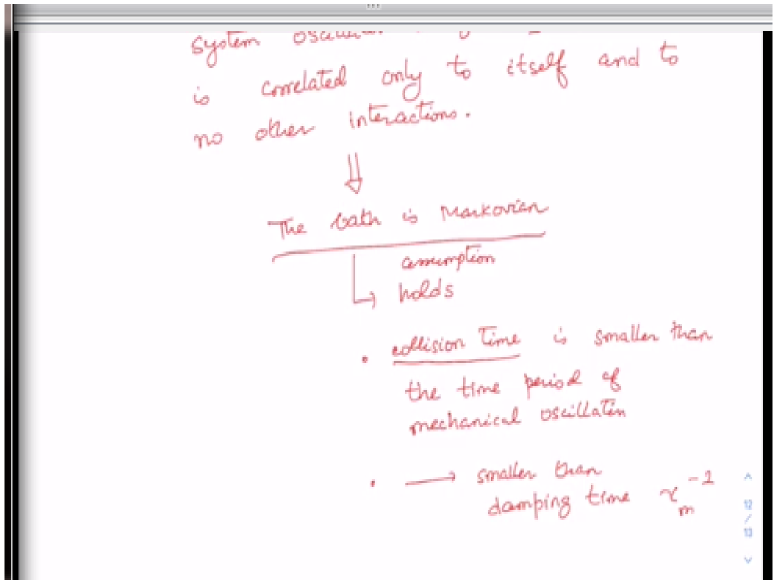

However, this assumption holds when the duration of each collision of the bath oscillator with the system oscillator is smaller than the time period of mechanical oscillator. So, collision time is smaller, smaller than the time period of mechanical oscillations as well as it also it should be smaller, smaller than the damping time. So, damping time, so this you let me write it properly, this is the collision time is smaller than the damping time and damping time is given by inverse of gamma m.

Another thing this term twice m gamma m K B T, this is the measure of the magnitude of the fluctuating thermal force. And the strength of the fluctuation as you can see from these terms that it varies directly as the mechanical damping gamma m. And of course this is a manifestation of the fluctuation dissipation theorem that we discussed earlier.

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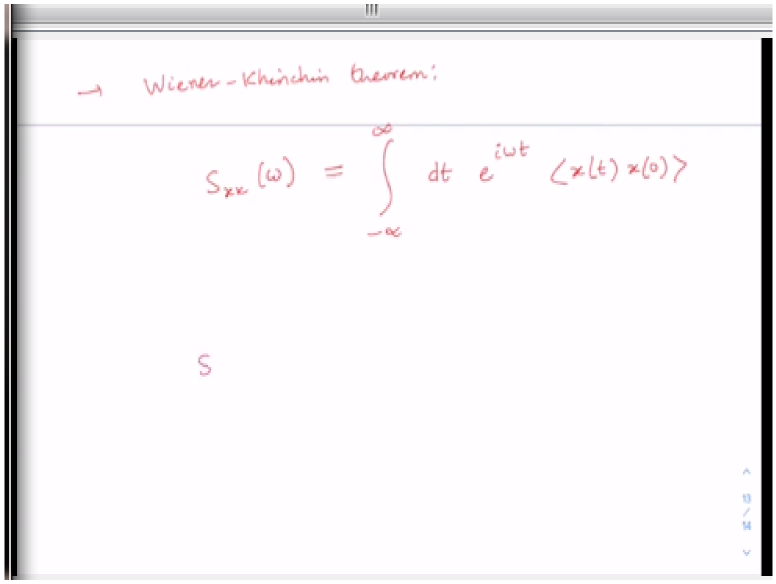

Now if you recall the so-called Wiener-Khinchin theorem that we discussed earlier in the context of displacement variable x of the movable mirror**.** So, let me write here the Wiener-Khinchin theorem once again Wiener-Khinchin theorem that we discussed this is S xx of omega which represents the noise spectrum.

And this noise spectrum was the Fourier transform of the correlator the displacement at time t and the product of the displacements at 2 different times. And in the context of Langevin noise Langevin noise we can similarly define a quantity S Xi Xi of omega which it is termed as spectral noise density.

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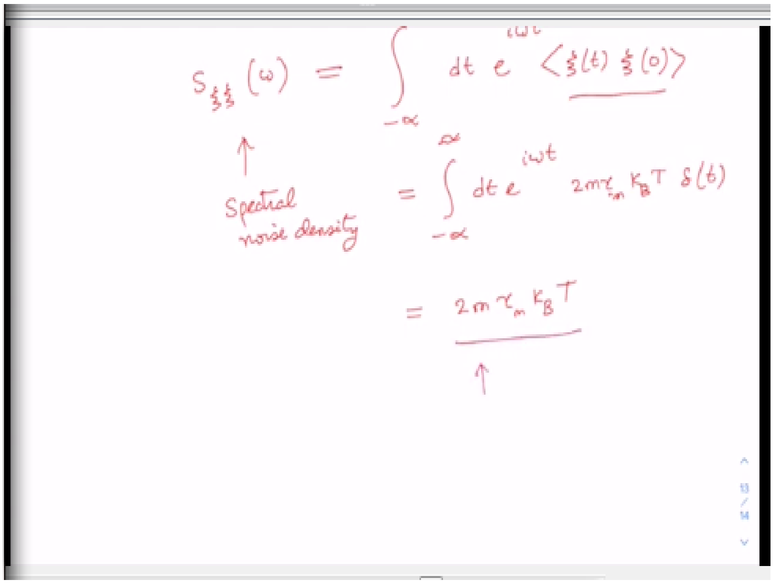

And this is the Fourier transformation of the correlator of the noise at 2 different times Xi of t and Xi of 0 and you have e to the power i omega t dt, this is the Fourier transform of the correlator. And we can now put the expression for this already we know, that would be dt e to the power i omega t.

So, what we know is we have this expression just let us put t dash = 0 here and then we can write it as twice m gamma m K B T and this is delta t. Now if we apply the dirac delta function property then it will be simply this integral will be twice m gamma m K B T. So, from here you see that this spectral noise density is independent of frequency omega.

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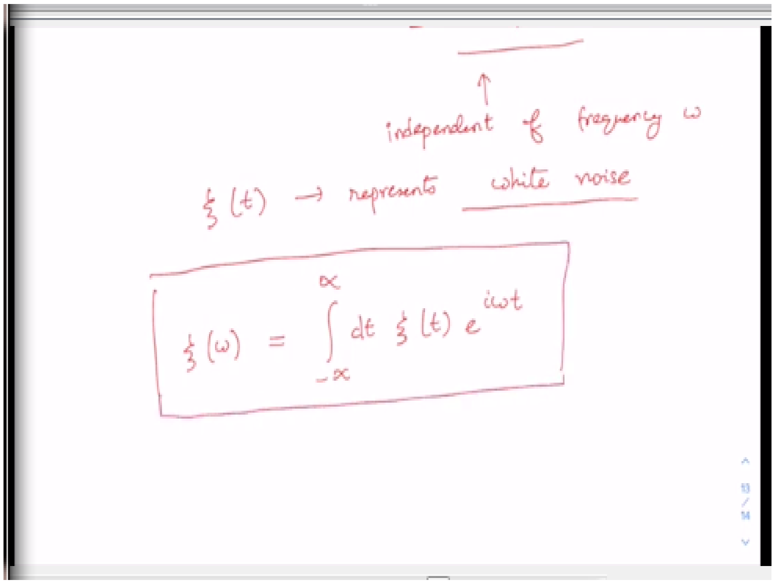

So, this term as it clearly shows that this is independent of frequency omega and this is often it terms as say Xi of t it represents what is called white noise because of the frequency independence. By the way we can express the Langevin noise in the frequency domain as well by using the Fourier transformation. So, Xi of omega = minus infinity to plus infinity it would be simply the Fourier transformation of Xi of t and this is going to be useful for us because as you know that it is many times useful to work in the frequency domain.

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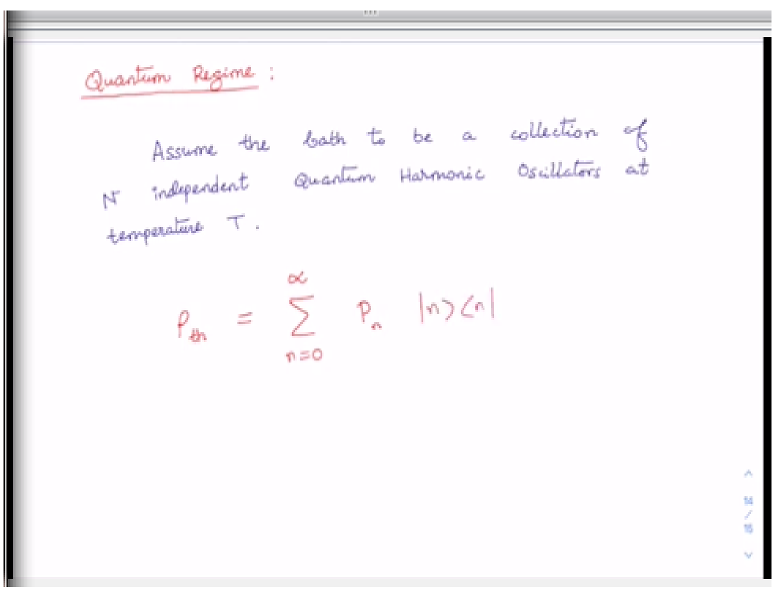

Now let us discuss the quantum counter part of classical Langevin noise. Again let us consider the bath to be a collection of independent, this time quantum harmonic oscillators at some temperature T. We know from our earlier classes that the density operator for the thermal state of these oscillators are given by this expression in the number state basis, rho thermal that is the density operator for the thermal state would be sum over n = 0 to infinity. This is probability P n and this is the Fock (46:47) state that is how we can write the density operator.

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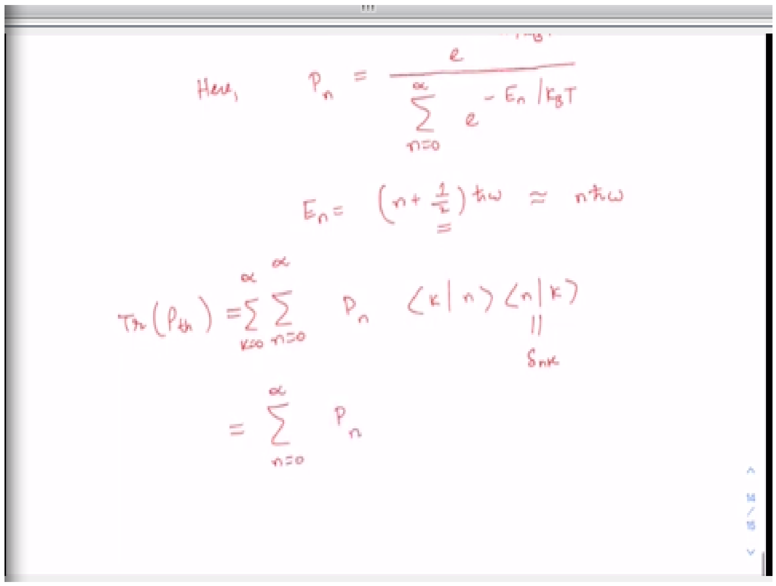

Here this P n the probability function is e to the power -E n, E n is the energy of the harmonic oscillator and it is divided by summation n = 0 to infinity e to the power -E n by K B T. And you know that E n = n + half h cross omega, that is the energy for the harmonic oscillator. And generally we do not worry about this constant term and we write it simply as n h cross omega. You can quickly saw that the trace of this density operator should be equal to 1 and it is indeed equal to 1.

As you can see if I take the trace of this density operator it will be say n = 0 to infinity and P n and here I have n in fact I should put here say K = 0 to infinity and here I have k and I have n k. And because this is equal to Kronecker delta n k and using this immediately you can see that I can write it as n = 0 to infinity, I will simply will be left out with this P n.

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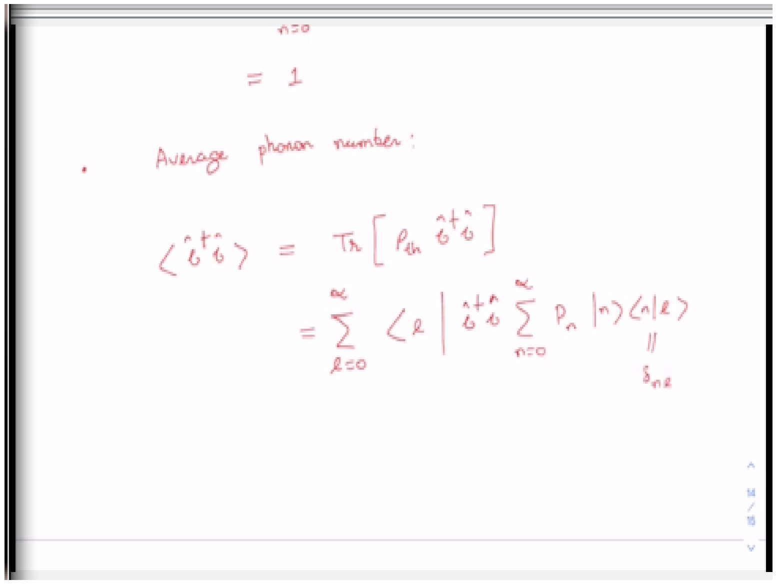

And this is the sum over all the probabilities and that is equal to 1. Now the average phonon number because these are thermal oscillators, so quanta let us consider them to be phonons. So, average phonon number can also be calculated very easily, so let me denote the phonons by the annihilation operator b, so phonon number would be b dagger b, the symbol A is reserved for photons.

So, b dagger b the expectation value would be we have to calculate this quantity trace of rho thermal b dagger b let me quickly show you, so this would be say sum over l = 0 to infinity l there is a bra here then I have b dagger b and rho thermal is n = 0 to infinity P n, n, n l, ok. Again this is your Kronecker delta, delta nl.

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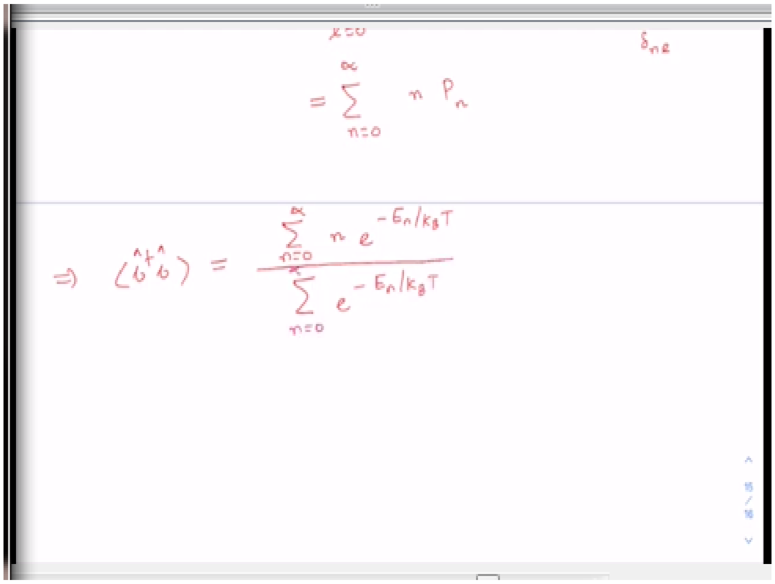

So, if I use it, so I will immediately get the expression n = 0 to infinity and I will have n P n, so this is what we have to work out. And P n already we know, so let me write here. So, b dagger b expectation value is summation in fact expression for p n is known to us. So, we will get this as summation n = 0 to infinity n e to the power -E n by K B T divided by e to the power -E n by K B T. This kind of things we have actually worked out elsewhere also but let us quickly do it.

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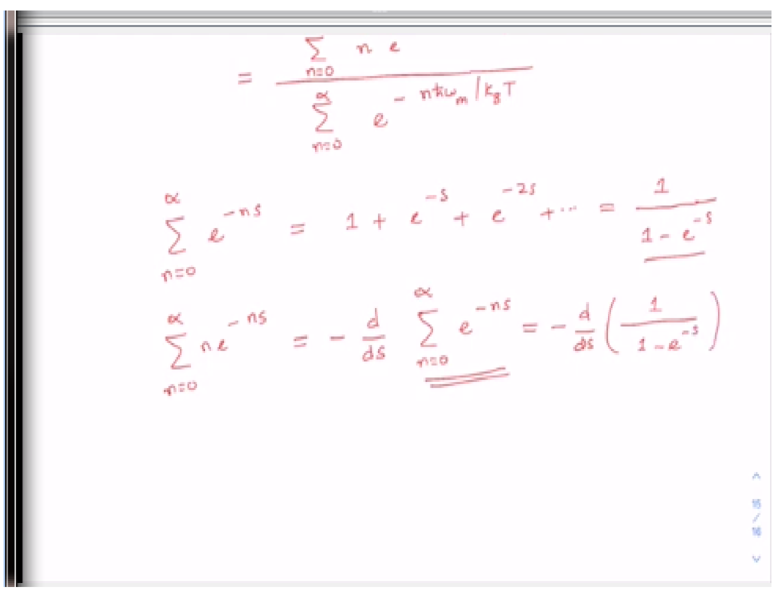

So, this would be summation n e to the power it is harmonic oscillator, so we have nh cross omega m by K B T and here it is summation here n = 0 to infinity, n = 0 to infinity e to the power -nh cross omega m by K B T. Now this can be quickly worked out, just to remind you that this is you can write it as e to the power -ns, if you take summation from n = 0 to infinity then you will have 1 + e to the power -s, + e to the power -2s + so on and this is equal to 1 divided by 1 - e to the power -s.

And also this expression we can write n into e to the power ns, n = 0 to infinity we can write it as minus derivative of d of ds you can verify it, it is very simple e to the power ns, n = 0 to infinity and this should be therefore -d of ds. This already we know this term, so that is this one, so 1 divided by 1 - e to the power s.

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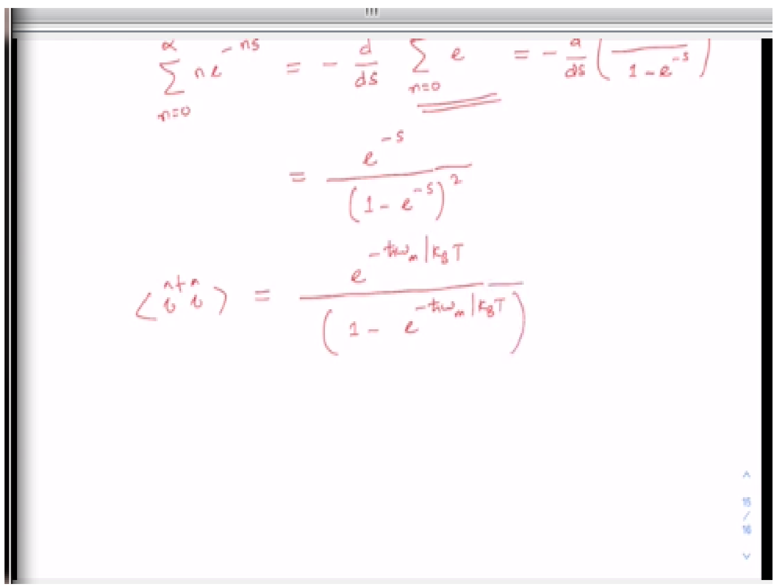

And this will give e to the power -s divided by 1 - e to the power -s square. And if you use it then you will get b dagger b, the usual expression you will get and this would be e to the power -h cross omega m by K B T divided by 1 - e to the power h cross omega m by K B T.

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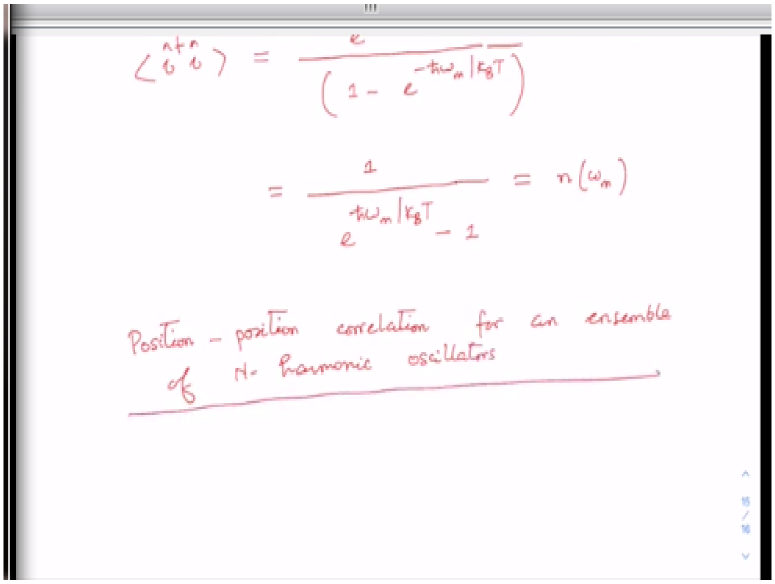

Or we can write it in this form also, it will be 1 divided by e to the power h cross omega m by K B T - 1. This is the average number of phonons in this thermal oscillator. Now this is important let us now calculate the position-position correlation function. Position-position correlation this we have done in the classical context, now here we will do it for the quantum case position-position correlation for an ensemble of n harmonic oscillators, let us do it.

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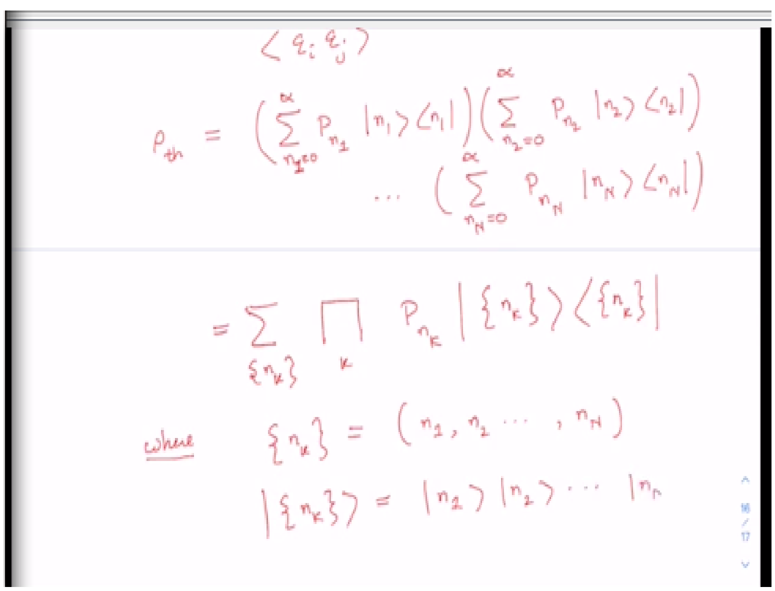

So, to do that basically what I want to calculate is this quantity q i q j and take the average. Now we will need to, because it is quantum case we will need to know the density operator overall. So, that would be because there are n number of harmonic oscillator and all these are independent harmonic oscillator for say oscillator number 1 which says n 1 number of phonons there.

So, n 1, n 1 that is the density operator for that, for the other second oscillator let us say it has n 2 n 1 go from 0 to infinity, n 2 goes from 0 to infinity. And then you will have P n 2, n 2, n 2 here and this way you will have up to N number of oscillator. So, you will have P n N you will have n N then this n N. This looks cumbersome, so we can write it in a certain notation and then we will write it as sum I will explain the terms here.

I can write it as this in bracketed term n k, then this is the multiplication and we have P n k and I will write it as n k and I will have this bra of this guy n k. Let me explain what I mean by this bracketed term. Here n k means that I am having n 1, n 2 up to n N. On the other hand this ket of bracketed n k symbolizes the fact that we are talking about products of all these n 1 direct product of n 1, n 2 up to n N.

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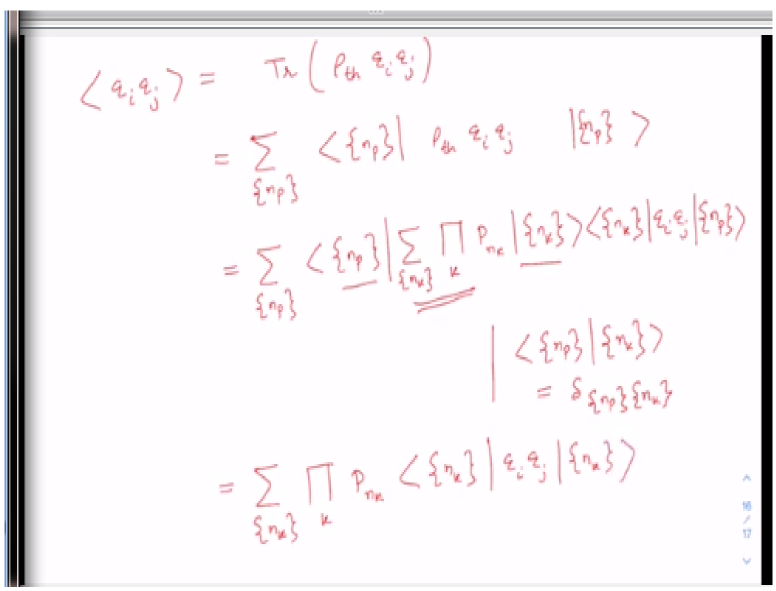

Now we can work out the expectation value of the product of q i q j that would be equal to trace of rho thermal into q i q j, this I can write it as sum over say n p, This is n p and then I will close this by ket n p and we have in between rho thermal q i q j. I already have the expression for rho thermal, so I will just use it here this would be n p rho thermal is summation over n k. This is the multiplication P n k and we have n k ket n k then the bra of n k.

And we have q i q j and it is finally we have n p. Now we can use the fact that this scalar product of n p and n k or rather, ok, that is what we have this you see, this we can take outside. So, if I take the scalar product of n p n k that would be equal to delta n p n k that is the Kronecker delta and you can use the property of this Kronecker delta. Then we can write this final expression, so that would be sum over n k, then we have P n k here, then this would be n k q i q j n k. So, we will now build up things from here in the next class.

Let me stop here for today, in this lecture we have completed our discussion on classical Langevin noise. We have worked out the first and the second moments of Langevin noise in the classical context. Then we started discussing quantum counterpart of it by considering the bath oscillator as a collection of independent quantum harmonic oscillators. In the next class we will continue our discussion on quantum noise, so see you in the next lecture, thank you.