Quantum Technology and Quantum Phenomena in Macroscopic Systems

Prof. Amarendra Kumar Sarma
Department of Physics
Indian Institute of Technology – Guwahati

Lecture – 35
Classical Regime – II: Classical Langevin Equation

Hello, welcome to lecture 5 of module 3. This is lecture number 26 of the course. In this lecture, we will continue our discussion on classical regime then we will start discussing the so-called Quantum regime. But before that we have to understand quantum noise, as quantum noise distinguishes the classical regime. In fact, to appreciate quantum noise better, we have to understand classical noise first; so, we will have a discussion on the classical noise in terms of the so called Classical Langevin Equation.

So, let us begin. **(Video Starts: 01:12)** In the last class, we started discussing the classical regime of cavity quantum optomechanics with the goal that it will later on help us in understanding the quantum regime better, in the classical limit, we replace the position operator x cap by the corresponding classical variable x, while the light field which is represented in a quantum regime by the annihilation operator a cap it is replaced by the coherent state alpha which is the most classical state of a harmonic oscillator.

And we all know that in the quantum regime the light which is an electromagnetic field behaves like a harmonic oscillator, the equation of motion will be in the form of x and alpha and we have written 2 equations for one for mechanics and other for the light mode. The mechanics is modelled by the mechanical oscillator is considered to be a damped harmonic oscillator and it is acted upon by the radiation pressure force.

And the light mode is represented by this equation which I explained in the last class as you can see that the first term on the right-hand side of this equation, this particular term takes into account how the light field is coupled to mechanics, the second term refers to decay of the amplitude of the photon and the third term takes the laser drive into account and where omega L is the laser frequency, the parameter alpha max here.

It indicates the fact that the value of alpha will settle down to the value of alpha max at resonance. So, then we got rid of the time dependence in this light mode equation by going over to the laser frame rotating laser frame and this is what the transformation we made, after making the transformation we get an equation with a new light variable, but for convenience we again represent the new light variable by alpha.

And then it resulted in this equation of motion for the light field, then we define the detuning parameter, as the difference between the laser frequency and the resonance frequency of the cavity and we get an equation for the light mode. And as we analyze it, that in the steady state when there is no coupling to the mechanics, then it turns out that indeed in the steady state where alpha takes the value of alpha bar.

It actually at resonance takes the value of alpha max as we described. And this is the intensity plot versus the detuning parameter in the steady state and we can see that this spectrum here has this full width at half maxima is given by this cavity decay rate kappa. Then, we try to solve these 2 equations for this steady state and we have linearized it around the steady state and to do that actually, we assume that the coupling between the system and the bath is weak.

And here by system we mean the mechanical system the mechanics the harmonic oscillator, the mechanical harmonic oscillator and by the bath we meant up driven optical cavity. So, basically the mechanical system is actually surrounded by the optical field and this is considered to be the bath and the coupling is considered to be weak and in that case we take that the value of alpha is slightly deviated from the steady state value by delta alpha.

While the mechanical displacement of the movable mirror is displaced from its steady state by a delta xt you can consider them as the fluctuation also. And the next we have taken certain steps to solve it first we solve it for the steady state and then we look for the first order parts of the equation of motion. Now, assuming that already we know what is the steady state value alpha bar and x bar we solve this equation of motion and then we just put x by replaced x by x bar + delta x and alpha by alpha bar + delta alpha.

And we obtain the equations for the fluctuations corresponding fluctuations for the mechanics as well as the light mode, light field. And we analyzed it and eliminating the idea was to eliminate the light field dynamics and then plug the solutions into the equation of motion for mechanics. And we have done that of course our idea was to see the response of the mechanical system to an external force.

So, therefore, in addition to the radiation pressure force, we have added an external term extra force term here because we are interested to know how the mechanical system is responding to this external force F here. And then going over to the frequency domain we discuss all these things in detail in the last class we solved we got the expression for the fluctuating field for the light field and we in the process we define the quantity called cavity susceptibility by these parameters.

And which basically gives us the response of the or basically shows us how the light field is getting modified due to the interaction with the mechanics. And the equation of motion for the mechanics is here it is written here and then we replaced the quantity delta alpha and delta alpha star which is the complex conjugate of delta alpha we put it here and eventually after using this Fourier transform relation.

We obtain an expression for delta x and this is basically the response of the mechanical system or the fluctuating quantity and we were able to write it in terms of a very compact equation and we have defined a parameter called K of omega and what it basically shows is that because of the presence of the interaction between the mechanics and the light, the mechanical susceptibility of the harmonic oscillator, mechanical harmonic oscillator is getting modified.

So, therefore, we are having an effective susceptibility parameter, then the next we went on to understand what is the meaning of this extra parameter and it turns out that our analysis finally shows that the real part of the susceptibility parameter is basically or this K of omega is related to the frequency shift of the mechanical extra frequency shift of the mechanical oscillator. And the imaginary part of this term K omega around the resonance frequency of the harmonic oscillator the mechanical oscillator is related to the extra damping that is suffered by the mechanical oscillator. **(Video Ends: 09:21)**

(Refer Slide Time: 09:22)

$$K(\omega) = \hbar \left(\frac{\omega_{opt}}{L}\right)^2 i \, |\bar{\alpha}|^2 \left[\chi_c(\omega) - \chi_c^*(-\omega)\right]$$

$$\chi_c(\omega) = \frac{1}{-i\omega - i\bar{\Delta} + \kappa/2}$$

$$\Delta + \frac{\omega_{opt}}{L}\bar{x}$$

So, now, what we are going to do we are going to analyze this term K of mega in some more details and then we will be able to extract certain quantitative information about the mechanical system. Here chi c that is the susceptibility cavity susceptibility and it is given by chi c of omega = 1 divided by minus i omega - i delta bar + kappa / 2, where delta bar this is the modified detuning parameter and this is assume equal to delta + omega optical / L and x bar this is a very small quantity. So, in many cases we may assume delta bar to be nearly equal to delta only.

(Refer Slide Time: 10:22)

'g' : linearized optomechanical coupling strength

$$g^2 = \frac{\hbar}{2m\Omega}\left(\frac{\omega_{opt}}{L}\right)^2 |\bar{\alpha}|^2$$

$$K(\omega) = i \, 2m\Omega \, g^2 \left[\chi_c(\omega) - \chi_c^*(-\omega)\right]$$

We can define another parameter which will simplify the calculation that is called optomechanical coupling linearized rather because we are now in the linearized domain. So, this is called linearized optomechanical coupling strength and you will see that this is going to be very, very important parameter and this is defined as g square = h cross divided by twice m capital omega, omega optical / L whole square, then mod alpha square, this is what is g square.

So, using these parameters g optomechanical coupling linearized optomechanical coupling strength we can rewrite the expression for this K omega as follows. That would be i into twice m omega g square chi c cavity susceptibility parameter evaluated at frequency omega minus its complex conjugate evaluated at frequency minus omega. Now, we can put the value of chi c and its complex conjugate evaluated at minus omega from this expression.

(Refer Slide Time: 11:55)

$$K(\omega) = i\, 2m\Omega g^2 \left[\chi_c(\omega) - \chi_c^*(-\omega) \right]$$

$$\downarrow$$

$$\mathrm{Re}\, K(\omega) \; ; \quad \mathrm{Im}\, K(\omega)$$

$$\Gamma_{opt} = \frac{1}{m\Omega}\, \mathrm{Im}\left[K(\omega \approx \Omega) \right]$$

And then we can find out from here we can find out the real part because this is a complex quantity we can find a real part of K of omega and imaginary part of K of omega a simple straight forward algebra we can do. And if we do that, then you will recall that this gamma optical that means, the optomechanical damping term that we obtained was this it is related to the imaginary part of K of omega and this is given by 1 / m of omega into imaginary part of this quantity K evaluated at the resonance frequency of the oscillator capital omega.

(Refer Slide Time: 12:43)

$$\Gamma_{opt} = \frac{1}{m\Omega} \, Im\left[K(\omega \approx \Omega) \right]$$

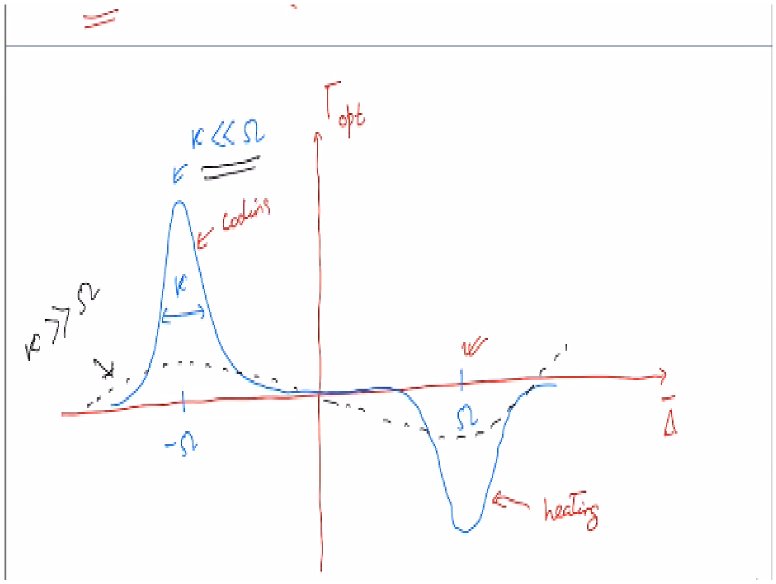
$$= 2g^2 \, Re\left[\chi_c(\omega) - \chi_c^*(-\omega) \right] \Big|_{\omega=\Omega}$$

$$\Gamma_{opt} = g^2 \kappa \left[\frac{1}{(\Omega+\bar{\Delta})^2 + (\kappa/2)^2} - \frac{1}{(\Omega -} \right.$$

And if we do the algebra, in fact, it is very straight forward to see from this expression that this is going to be equal to twice of g square and then it would be the real part because this is imaginary quantities there. So, we have to take the real part of this. So, this will be real part of chi of c of omega - chi of c susceptibility complex conjugate of susceptibility evaluated at minus omega in fact, of course, we have to take omega = capital omega.

If we do that and this simple algebra I urge you to do it otherwise, we will do it in the problem solving session you will get the expression for optomechanical damping parameter as this one, this would be g square kappa and a very nice expression you will get by simple algebra you will get it will be omega + delta bar whole square + kappa / 2 whole square and there will be another term that would be minus 1 divided by capital omega - delta bar whole square + kappa / 2 whole square. So, this is the expression for optomechanical damping. Now, we can plot this parameter is a function of the modified detuning parameter delta bar.

(Refer Slide Time: 14:20)

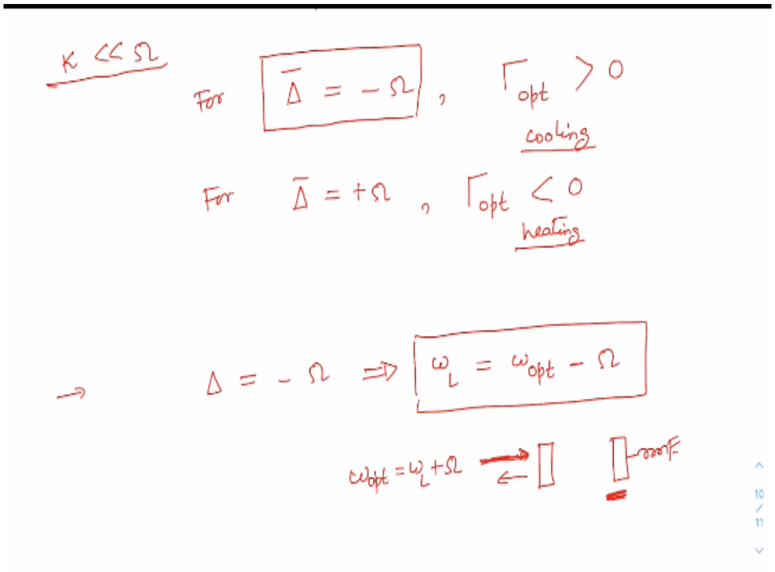


And if we plot it we are going to get a plot like this. So, you can actually use a computer in some parameter but typically the plot would look like this. So, on the x axis it is delta bar and here we have gamma optical there is a damping parameter. So, the plot will be something like this it will be somewhat like this. Alright, this is a symmetric in a way and symmetric here, this is what we are going to get and this peak could be at minus omega and here it would be a plus omega and full width at half maxima.

So, we are getting basically 2 peaks, one is in the upward direction other is in the downward direction as you can see and this full width at half maxima is given by this kappa. So, we are getting sharp peak and this is the plot when this cavity decay rate kappa is much smaller than the resonance frequency of the oscillator. On the other hand, if say kappa is much greater than the resonance frequency then you are going to get a plot something like this.

So, you will get this is for this particular one is when this cavity decay rate is much larger than the resonance frequency. So, we will get dotted one for kappa much larger than the resonance frequency omega and we will get very sharp peak for kappa much less than the resonance frequency omega. Now, let me explain what a physically mean and these 2 plots particularly this one is very important for us kappa much smaller than omega. You can see that it is very clear that I am going to discuss the case kappa much less than omega.

(Refer Slide Time: 16:34)



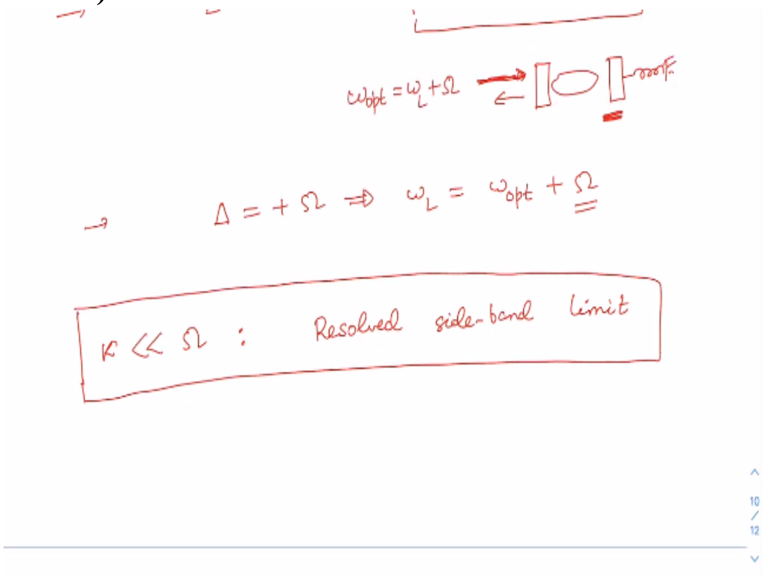
Here, if you see from this plot that for delta bar is equal to minus omega, we have positive damping. So, here as you can see this gamma optomechanical damping this is greater than 0. So, which effectively means that we are going to get cooling because the mechanical oscillator will get cold when the detuning parameter, when we can have delta bar is equal to minus omega. On the other hand, in this regime here for so, we will get cooling for delta is equal to minus omega and for delta bar is equal to plus omega.

So, here this one we are going to get heating because we have negative detuning because gamma optical that is your negative damping. So, this here will have heating the mechanical oscillator will get heated up and on the other hand here we are going to get the cooling effect. Now, let us understand it actually what is going on, it is very simple, because if we come up with say delta our laser frequency, laser has a frequency such that our delta rather than delta bar let me just take delta because the correction is very small.

So, if say delta is equal to minus omega then it means, because delta you know that is equal to the difference between the laser frequency and resonance frequency of the cavity. So, that would be omega optical minus capital omega. So, if the laser frequency is such that this is you know this is less than the resonance frequency of the cavity, in that case quite clearly the laser light cannot enter into the cavity because it is not satisfying the resonance condition, it will go here and it will hit the front mirror and then it will get reflected.

However, what happens is there, If suppose there are some kind of vibration is there, that means, some phonons are there say it is cantilever. So, the light field then can this light field can absorb a phonon and thereby, so, in that case you see if we will have omega optical is equal to the laser frequency is omega and then if it can extract 1 phonon from the cantilever, then this laser light will be able to enter into the cavity and the laser frequency will be enhanced. So, therefore, the photon that is entering into the cavity will be now blue shifted as it has gained energy from the mechanical oscillator resulting in cooling of the cantilever.

(Refer Slide Time: 19:57)

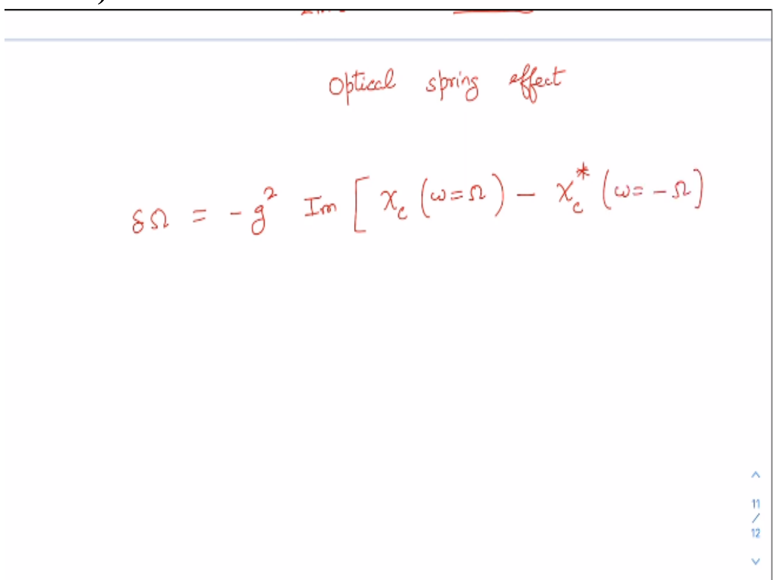


On the other hand, when we have this delta is equal to plus omega you can do the similar analysis, then photon will be able to enter into the cavity if it dump extra energy, because you see be, it is plus omega then your omega L is equal to omega optical plus omega. That means, now, if the resonance condition has to be satisfied, then the photon has to dump this the laser light that is impinging on the front mirror has to dump this extra energy.

And if it dumps the extra energy to the cantilever, then it will be able to enter into the cavity and circulating light will be there inside the cavity and in the process, the mechanical oscillator will get heated up and phonon will now become red shifted in frequency. So, one thing you should, I forgot to tell you please note that these domains when kappa is much, much less than omega in this case, we can have to sharp peaks and we can you know these peaks can be resolved and this is known as the resolved side band limit. So, this is very important.

So, the domain where kappa is much less than omega, this is popularly known as resolve side band limit. In most optomechanical systems generally people work in this limit, because it is very useful for applications. Now, let us discuss about the implication of the real part of k omega as pointed out earlier, it is related to the frequency shift of the mechanical oscillator.

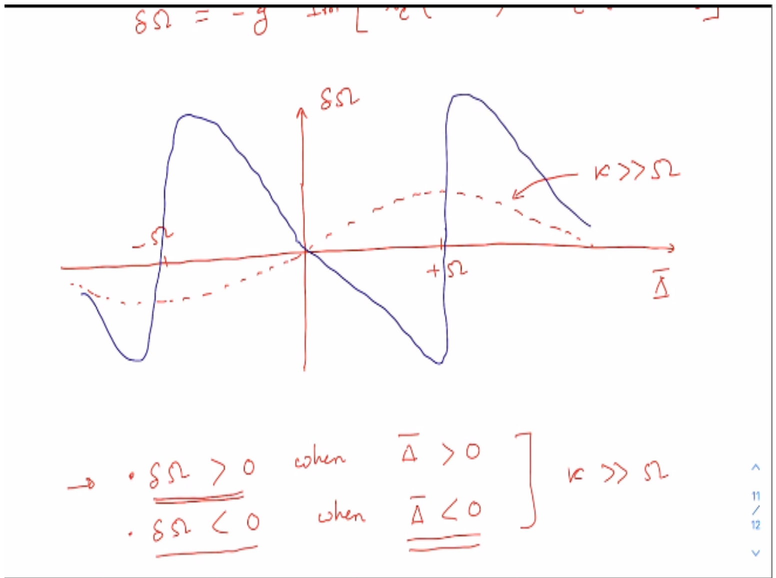
(Refer Slide Time: 21:56)



So, we show earlier this expression that frequency shift delta omega is equal to minus 1 by twice m omega, omega here is a resonance frequency and it is real part of K evaluated the resonance frequency. So, this is basically we know as the optical spring effect. Now, plugging the value of K of omega we can get these expressions very easily because K of omega which we have written earlier here from this expression you see, you will have if you put that here delta omega will be equal to minus g square.

And this would be imaginary part of chi of c cavity susceptibility evaluated at plus capital omega then minus of its complex conjugate evaluated at negative of this capital omega the resonance frequency. You can easily work out the details doing the straightforward algebra.

(Refer Slide Time: 23:28)



And then it is possible to get a typical plot for the frequency shift delta omega as a function of the detuning parameter delta bar let me just draw it here for various cases. So, this is my delta omega the frequency shift versus delta bar. Now, in the regime when the cavity decay rate kappa is much much larger than omega the resonance frequency the plot will look like this it would be antisymmetric and it would be something like this.

So, we will have here it is for plus omega and this would be for minus omega. And here I am plotting it for the case when kappa is much larger than capital omega the resonance frequency. Now, as you can see, from this plot, 2 things are clear one is that this is antisymmetric and shift in the frequency is positive that is greater than 0 when we our detuning parameter, delta bar is greater than 0.

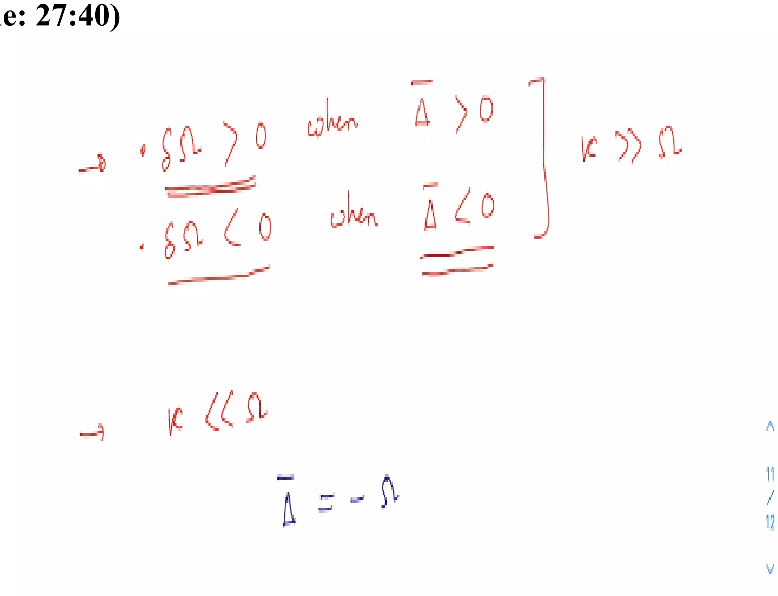
On the other hand, this delta omega the frequency shift is negative when we have this detuning parameter is negative. And this is for the case when we are in the unresolved side band regime because, as I said here that kappa less than much less than omega is the so called resolved side band regime and this is the opposite of the regime. Now, physically what it means is that that the mechanical spring in this case as you see here delta omega here, delta omega is greater than 0.

So, therefore, the mechanical spring constant is now enhanced. So, therefore, the spring is getting stiffer or harder due to the light induced effect. On the other hand, for delta bar less than 0 in this regime the frequency shift is negative. So, therefore, the spring constant is getting reduced that means the spring is getting softer as delta omega is less than 0. And it has certain issues, because of this as regards the cooling is concerned because we already know that for optomechanical cooling, we work in the regime delta less this negative detuning, delta less than zero.

Just let me again remind you, this is the regime here you see, we work in the regime where delta bar is less than 0 and we get the cooling effect. So, if we are interested in cooling when if we want to cool the mechanical spring harder than the spring as a result of because this delta omega, this frequency shift will also increase if we increase the intensity of the laser, and because of that the spring will get more softer and this is actually going to lead into the instability.

So, this is not a good thing. But this issue can be circumvented or avoided, if we go over to the resolved side band regime that is where we have kappa is say much, much less than the resonance frequency. And in this case, the plot is a little bit slightly complicated but still, let me try to plot it. So, this would be something like this, it will be again here, let me draw it here, it would be antisymmetric and you will have a peak here. And it will pass through it, there will be almost kind of a singularity here and it will be like this, something like this.

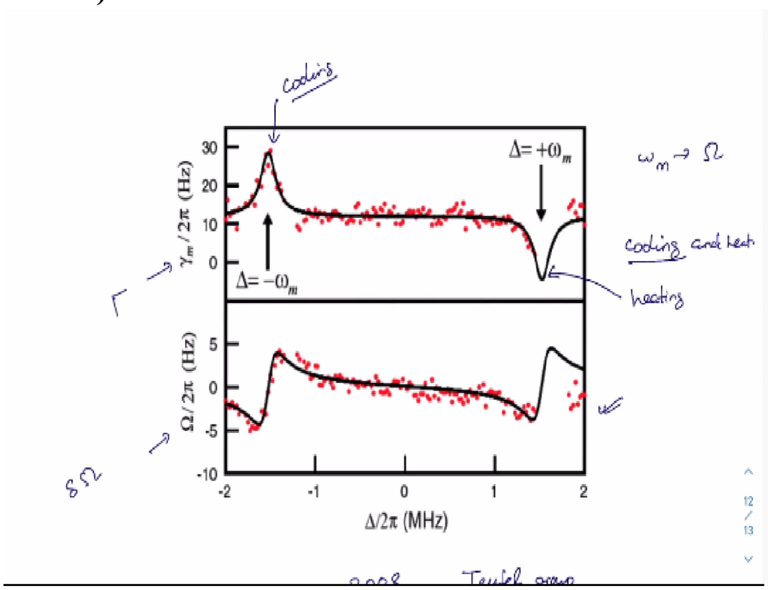
(Refer Slide Time: 27:40)



So, here you see that we can see, we can choose this resonance frequency here, delta in this case, we can choose the detuning parameter exactly at the resonance frequency here at the negative of the resonance frequency and when then the cooling will be in this case strong. At the same time, we will be able to avoid the softening of the spring and will no longer have an optical spring effect here.

So, in fact, that is the reason or the motivation behind working in the regime of resolved side band regime and this is experimentally speaking or practically it is a very good regime, only issue is that because your kappa has to be much smaller. So therefore, you really need a very good quality factor, a very high quality factor cavity.

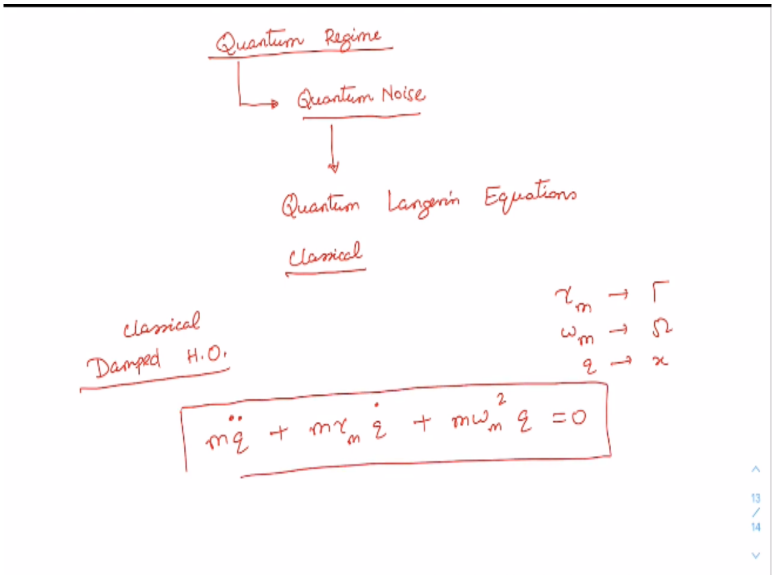
(Refer Slide Time: 28:34)



All these effects have been actually experimentally validated. This is some experiment that was done in the year 2008 by a group by Tufail group. And here you see, by the way, in this plot here, this omega m refers to our capital omega, the resonance frequency and these omegas here refer to delta omega. And this gamma is refers to our symbol gamma, and this is the cooling effect as we see this is for the cooling and heating part cooling and heating effect, this is your heating effect and this is the cooling effect.

And red dots are the experimental data and this solid curve is from the theory. As you see the; experiment and the theory matches pretty well. On the other hand, in this figure here is this is the frequency shift versus the detuning parameter. And here also you see the red one refers to the experimental data and the solid curve refers to the theoretical these things, theoretical plot and both of them matches pretty well. And whatever we discussed, this feature is, as you can see, this is actually done in the resolved side band regime and experiment and theory matches pretty well.

(Refer Slide Time: 30:09)

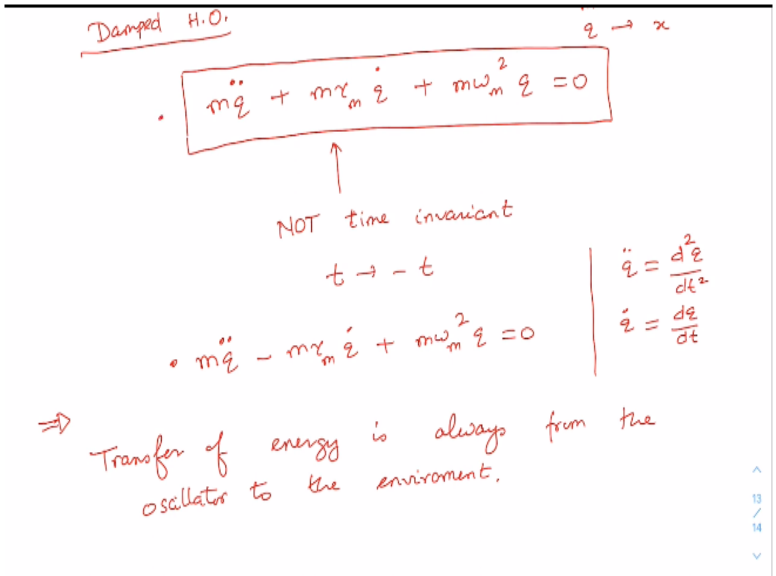


Here now, we are almost ready to discuss quantum regime. The quantum regime is particularly different from classical regime due to the so-called quantum noise. The quantum noise is generally discussed by the so-called quantum Langevin equations and knowing this quantum Langevin equation is going to be very useful for us. In fact, to appreciate quantum Langevin equation, let us first discuss the classical Langevin equation.

And to do that, we will begin with our usual classical harmonic oscillator. And as you recall that we represented a classical harmonic oscillator by the damped harmonic oscillator model and we damped harmonic oscillator. Here I am going to discuss once again classical damped harmonic oscillator and we will see the shortcomings of the usual model. The equation of motion for the damped harmonic oscillator is this m q double dot + m gamma m q dot this is the dissipation term and we have m omega m square q = 0.

Just a caution here gamma m is the same is gamma that just we have sort well back we took and omega m is the same as this resonance frequency omega for the mechanical oscillator and this q is basically the same as x the coordinate of the mechanical oscillator. So, usually this is the damped harmonic oscillator equation and in most situations it is alright.

(Refer Time Slide: 32:16)

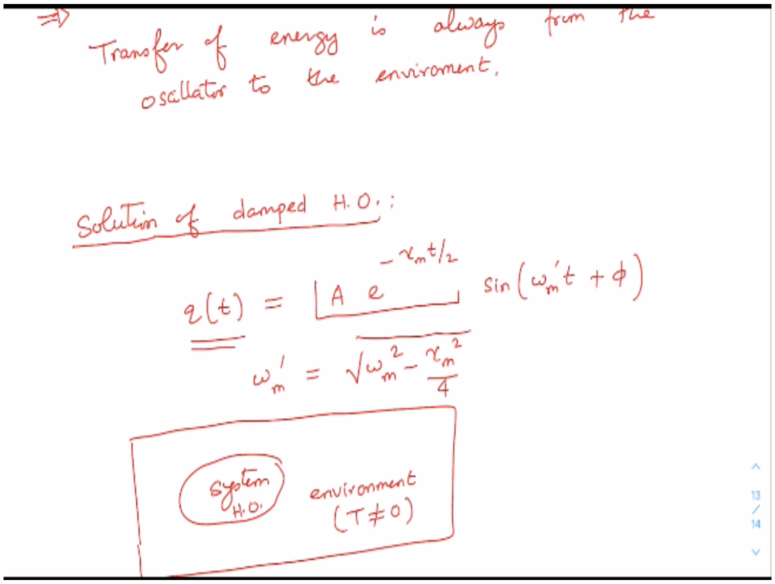


But there is a fundamental problem with this equation because this equation is it is not time invariant. So, not time invariant by this I mean that when you go from t to -t, then this equation could become mq double dot - m gamma m q dot + m omega m square q = 0. You see, this is because q double dot is d 2 q / dt 2. So, under time reversal it will remain the same. But q dot will change its signs because q dot = dq / dt. So, if t goes to -t q dot would become -q dot and that is why this is what we are having.

So, physically what it means is this that transfers of energy this implies because it is not time invariant. So, transfer of energy is taking place, transfer of energy is always from the oscillator to the environment and the reverse process never occurs. And also you see here in this equation of motion for the damped harmonic oscillator, all the variables are in terms of the harmonic oscillator only, but it is generally it interacts with the surrounding and there is not a single variable apart from this gamma m which connects the dissipation of the oscillator.

But there is no connection to the surrounding variables and so on. So, therefore, this equation is actually it is not correct in strict sense.

(Refer Time Slide: 34:26)

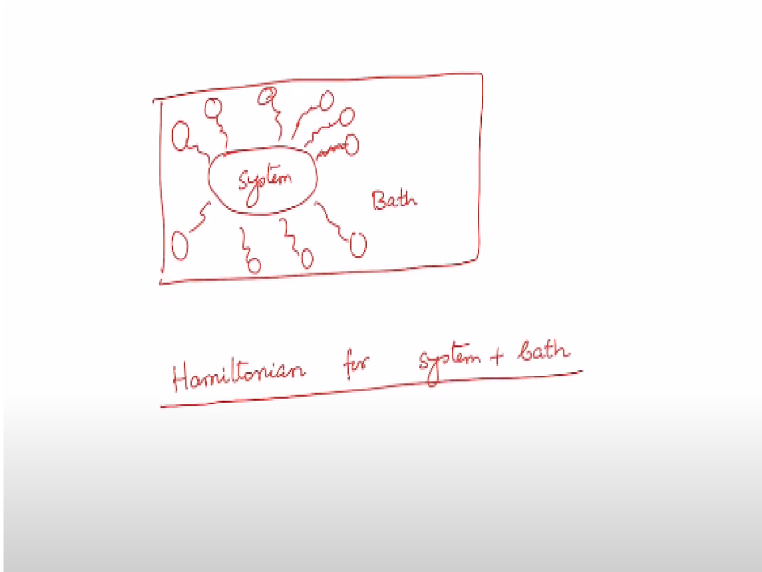


Because, also you can see from the solution of the equation that solution of this, let me write the solution there maybe you have already you must have studied in your undergraduate solution of damped harmonic oscillator, it is very easy to verify, if you can put this solution you can check it that this solution satisfy that damped harmonic oscillator equation. So, solution is like this say q of t is equal to some constant amplitude A e to the power -gamma m t / 2 and you have sin omega m dash t plus this constant this is phase of the oscillator.

And omega m dash is the shifted frequency or omega m dash is equal to because of dissipation you have omega m square - gamma m square / 4. So, by this you see the amplitude is of the oscillator is getting decreased as time goes on. So, this solution says that for any initial conditions the displacement of the oscillator eventually decays to 0. However, such behaviour does not represent realistically the situation where the oscillator is in thermal equilibrium with surrounding at nonzero temperature.

Say our system is there and the system is surrounded by the thermal environment, thermal environment means it has some temperature T not equal to 0 and here by this system in our case is the harmonic oscillator. Now, if it is interacting with the environment at thermal equilibrium the system is also going to have some temperature. so, there is definitely going to be the displacement. So, as time goes on this qt cannot go to 0. So, that is what the solution says. But in reality, that is not the case if it is in thermal equilibrium there be always some kind of a displacement.

(Refer Slide Time: 36:58)



So, how to address this realistic case and to model the realistic situation that needs to take into the thermal environment into account, what is considered is this that the system is interacting with the environment or the surrounding, let me say this call environment or it is also termed as bath, it is called bath environment. This bath is model as a collection of infinite number of independent harmonic oscillators.

So, this bath is actually considered to be that, as if the system is interacting with a lot of harmonic oscillators like this. These are independent harmonic oscillator. So, it is going to be model like this. So, I will show you how this is done. This is what is we are going to now consider that this bath is a collection of N independent harmonic oscillator and the Hamiltonian for the system plus bath can be written as follows. So, let me write out a Hamiltonian for system plus bath, so, why we are doing it? We are actually doing it so that we can have a realistic model of the interaction of the system with the environment.

(Refer Slide Time: 38:24)

$$H = \underbrace{\frac{p^2}{2m} + \frac{1}{2} m\omega_m^2 q^2}_{H_{system}} + \underbrace{\sum_{i=1}^{N} \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 q_i^2 \right)}_{H_{Bath}}$$

$$- q \sum_{i=1}^{N} c_i q_i + q^2 \sum_{i=1}^{N} \frac{c_i^2}{2 m_i \omega_i^2}$$

$$\left[\begin{array}{l} \dot{q}_j = \dfrac{\partial H}{\partial p_j} \\ \dot{p}_j = -\dfrac{\partial H}{\partial q_j} \end{array} \right.$$

And the Hamiltonian would be like this, say first of all the Hamiltonian with this part I am going to write it for the system. So, this is the kinetic energy let me consider the harmonic oscillator to be a single harmonic oscillator. So, its momentum is p, mass is m and you have half m omega m square q square. So, this is the potential energy part and the environment is now or the bath is considered to be a collection of N harmonic oscillator let us say.

And this would be all the oscillators has different momentum at different masses all of them are independent. So, we have this sum of these here, m i omega i square q i square and this is the bath part of the Hamiltonian. So, let us say H bath and this is we have the system which is our harmonic system harmonic oscillator and the system at the bath is interacting with each other and the interaction is taken into account by this bath here q is the system parameter and the bath parameter is there are N oscillators and corresponding to these N oscillators there are q i number of coordinates.

So, various coordinates are there. So, system coordinate is getting coupled to the bath coordinates and c i is the corresponding coupling between one particular bath variable and q i and system variable q and there is an additional term that is added to these things as to take into account or the cancel the shift in frequency of the system oscillator you see frequency is associated with this q square.

So, this particular term is just added for convenient purpose. So, that it can cancel the shift in frequency of the system oscillator where it is interacting with the environment. So, this is what is done. So, we have a term like this. Now, from this Hamiltonian, we can write down the first order differential equations for the bath and the system variable. So, you will recall that the equation of motion. We can write in terms of the Hamiltonian like this a q dot j = del H del p j and we have p j dot that is equal to minus del H del q j.

(Refer Slide Time: 41:16)

$$\boxed{\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}}$$

System

$$\dot{q} = \frac{p}{m} \rightarrow (2)$$

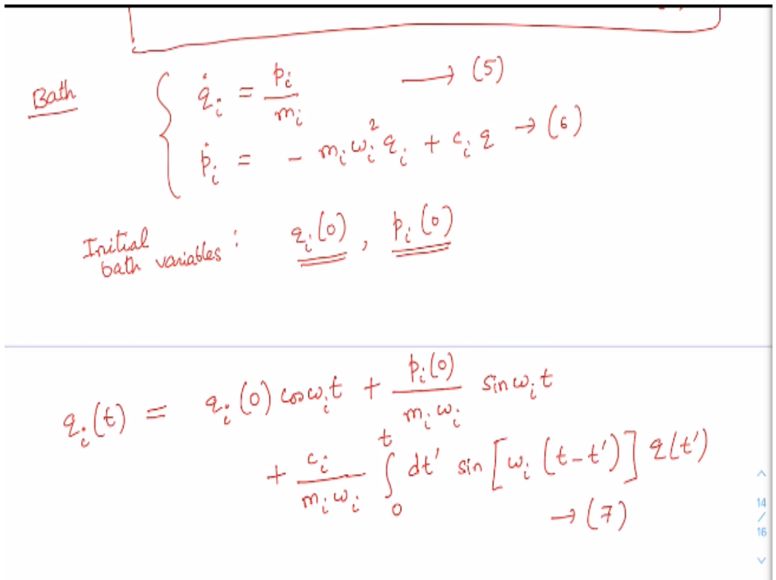
$$\dot{p} = -m\omega_m^2 q + \sum_{i=1}^{N} c_i q_i - q\sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i^2} \rightarrow (3)$$

$$\Rightarrow \boxed{m\ddot{q} + m\omega_m^2 q + q\sum_{i=1}^{N}\frac{c_i^2}{m_i\omega_i^2} - \sum_{i=1}^{N} c_i q_i = 0 \rightarrow (4)}$$

So, using this we can write the equations for the system let me first write it for a system variable that will be q dot = p / m and p dot is equal to you can easily verify p dot = -m omega m square q + summation i = 1 to N c i q i and we will have - q summation i = 1 to N let me take this here. Let me write this part separately to avoid confusion. So, I have -q summation i = 1 to N. you can I urge you to verify it whether we are writing it correctly or not you can check it.

So, this is what equation let me name this equation as or let us first let us say this is equation number 1, let me this equation number 2, this is equation number 3. In fact, combining these 2 equations rather I can write it this is mq double dot p dot would be equal to mq double dot right from here I can write p dot as mq double dot and this would be mq double dot + m omega m square q + q summation c i square / m i omega i square i = 1 to N, and you have minus summation i = 1 to N c i q i = 0. So, let us say this is my equation number 4, this is going to be a very important equation, we are going to come back to this equation a little bit later.

(Refer Slide Time: 43:33)



And these are for the system and for bath, for bath I have the equation as q i dot = p i / m i I think yes and then we have p i dot = - m i omega i square q i + c i q. let us say this is my equation number 5, this is my equation number 6. So, these are for the bath variables. Now, solution to the bath equation can be easily written in terms of the initial bath variable say q i of 0 and p i of 0 are the initial bath variables, position and the momentum, let us say a time t = 0, we have q i 0 and time t = 0.

We have momentum p i 0, suppose we know all these initial bath variables then the solution can be written like this. So, this is also you can verify just by directly putting it in the equations. So, let me write down the solution here. So, we will have q i of 0 and we can have cos omega i t + p i 0 / m i omega i. You may find it very difficult, but actually it is not difficult are a straightforward algebra only people do. But I think my motivation here or intention here is to do the things in details.

So, that you can see the whole picture, let me do it and you have c i m i omega i integration 0 to t dt dash sin omega i t - t dash and then you have q t dash, let me say this is equation number 7. Now, I want to do one thing I want to put this equation number 7 in equation number 4. Because, I am interested in knowing what is going on with the system. I have the system equation here.

(Refer Slide Time: 46:08)

$$m\ddot{q} + m\omega_m^2 q - \sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i}\int_0^t dt' \sin\left[\omega_i(t-t')\right] q(t') + q\sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i^2} = \sum_{i=1}^{N} c_i\left[q_i(0)\cos\omega_i t + \frac{p_i(0)}{m_i\omega_i}\sin\omega_i t\right] \rightarrow (8)$$

$$\Downarrow$$

$$m\ddot{q} + m\omega_m^2 q + \sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i^2}\int_0^t dt'\, \dot{q}(t')\cos\omega_i(t-t') = \sum_{i=1}^{N} c_i\left\{\left[q_i(0) - \frac{c_i}{m_i\omega_i^2}q(0)\right]\cos\omega_i t\right.$$

So, putting equation 7 in equation 4, let me write it down you can actually take a pen and paper and do it yourself and you can verify it and let me just write down the whole thing here. This would be m q double dot + m omega m square q - i = 1 to N c i square m i omega i integration 0 to t and I will have dt dash sin omega i t - t dash q of t dash + q do not worry it looks very difficult but intimidating, but actually it is we are going to simplify it.

And we will write a very simple expression at the end and you will see that m i omega i square and that is equal to on the right hand side i will have i is equal to 1 to N c i and you will have q i 0 cos omega i t + p i 0 divided by m i omega i and it have sin omega i t. So, let us say this is my equation number 8. In fact, here do you see this particular term this third term in equation 8 on the left-hand side, this can be simplified by using integration by parts.

And if we do that, you will see simplify it and put it in the equation 8 after simplification, this is very straightforward and then if we do that, I can write this equation number 8 as follows that would be m q double dot + m omega m square q. Now the third term is simplified. So, I will have i = 1 to N c i square m i omega i square integration 0 to t dt dash q dot t dash cos omega i t - t dash and here I have on the right-hand side I have i = N to c i, I think the same expression putting here I will have q i 0 - c i / m i omega i squire q of 0. And this term actually coming after simplification of the third term, and we have here cos omega i t.

(Refer Slide Time: 49:48)

$$m\ddot{q} + m\omega_m^2 q + \sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i^2} \int_0 dt' \, \dot{q}(t) \cos\omega_i (t-t')$$

$$= \underbrace{\sum_{i=1}^{N} c_i \left\{ \left[q_i(0) - \frac{c_i}{m_i\omega_i^2} q(0) \right] \cos\omega_i t + \frac{p_i(0)}{m_i\omega_i} \sin\omega_i t \right\}}_{\xi(t)}$$

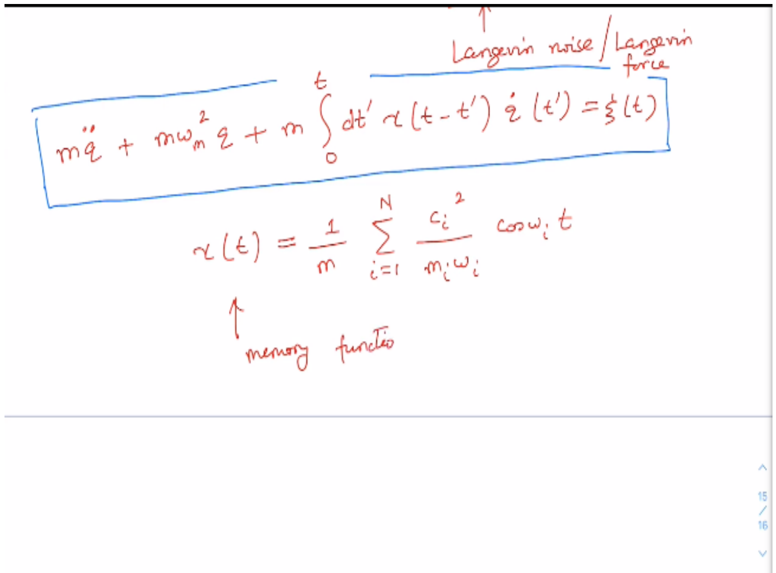
Langevin noise / Langevin force

$$\boxed{m\ddot{q} + m\omega_m^2 q + m \int_0^t dt' \, \gamma(t-t') \, \dot{q}(t') = \xi(t)}$$

And we will also have this term is what I will have plus p i of 0 / m i omega i sin omega i t. I am doing the things in pretty details, but eventually we will see we will simplify it further just by using some notation. Now, the right-hand side of this equation is this equation is important, this right hand side of this equation is due to the bath variables primarily you see and you have the dimension of force because mq double dot right that is force and this term on the right hand side of this equation has dimension of force.

And it is associated with a bath variable. And this is known as the Langevin force and it is denoted by the symbol xi of t and it is called Langevin noise or it is called Langevin force also Langevin force and it arise completely due to the surrounding and therefore, the simplification of this notation, we can further simplify this equation or we can write it as mq double dot + m omega m square q + m integration 0 to t dt dash gamma I will define what this guy is t - t dash q dot t dash = xi of t. And this is what we have the so, called classical Langevin equation and this is really important, we will see a lot of application of it very soon.

(Refer Slide Time: 51:57)



And here this guy gamma t it is a time dependent function, it is called a memory function it is termed as memory function and it is basically shorthand notation of this particular term here. Let me write here gamma of t = 1 / m summation i = 1 to N we have c i square divided by m i omega i and we have here cos omega i t and this is termed as the memory function. This is a say, memory function. I will talk about it on later. And it is termed as Langevin noise or Langevin force.

Let me stop here for today. In this lecture we have discussed optomechanical cooling and optical spring effect using classical picture. Then we started discussing classical noise because quantum noise distinguishes the classical regime and to appreciate quantum noise better we have to first understand classical noise. And to do that we have derived the so called classical Langevin equation. In the next lecture, we will discuss quantum Langevin equation. So, see you in the next lecture. Thank you.