Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture - 33
Cavity Optomechanics: Basic Physics II.

Hello, welcome to lecture 24 of the courses, this is lecture number 3 of module 3. In this lecture, we will continue our discussion on basic physics qualitatively then we will start discussing the so-called fluctuation dissipation theorem, which will be very important for our later classes. **(Video Starts: 00:52)** So, let us begin in the last class we discussed about the quality factor Q of a Fabry Perot cavity.

And we show that quality factor and finesse of the cavity are related very closely. In fact, they are actually the same only quality factor is expressed in a different frequency unit than that of the fineness and quality factor and fineness both refers to the fact that for example say Q = 10 to the power say 6 that means, that the photon will oscillate inside the cavity around 1 million times or 10 to the power 6 number of times.

Then we started discussing cavity optomechanical system and firstly based on logic we try to guess the Hamiltonian of the cavity optomechanical system. The total Hamiltonian of the cavity mechanical system will consist of primarily 5 parts one is due to the optical photon inside the cavity and optical photon as we know that it is basically a harmonic oscillator, then mechanical oscillator which is basically due to the movable mirror that is vibrating in the range of gigahertz or kilohertz.

So, they are basically quanta of vibration they are components. So that is also again another harmonic oscillator and then there is interaction between optical photon and phonon and that is because of the fact that when the photon hits the movable mirror, it displaces the mirror by an amount x and thereby, both photon and phonon get coupled by the parameter x, and this laser cavity optomechanical system is driven by a laser.

Apart from that term there are other terms due to the dissipation because both the photon as well as the phonon may decay. And then, because we know that the cavity photon is basically a harmonic oscillator and now, this resonance frequency of the oscillator is dependent on the displacement of the movable mirror. So, based on this if the displacement of the mirror is very small then we got 2 different terms one is due to the cavity photon that is circulating inside the cavity and another one is the optomechanical interaction term.

And we also saw how the radiation pressure force can be very easily worked out and this is related to the number of force, number of photon inside the cavity as well as the optical cavity resonance frequency and the optomechanical interaction Hamiltonian term can be expressed in a very convenient form like this. So that is what we did in the last class and we were able to write down the full Hamiltonian of the cavity optomechanical system.

However, we are not discussing how to write the expression for the dissipation term under laser drive term, this is we will do it in a later class. After that we started discussing the basic physics of a cavity optomechanical system in a qualitative way assuming that the photon that is getting incident on this movable mirror reacts instantaneously to the position of the mechanical mirror, this we can assume provided this mechanical mirror moves very slowly.

So, this is basically the static case and under this static case, we are discussing the so-called optical spring effect what happened is that because this mechanical oscillator is basically a spring, it is a harmonic oscillator. So, it has an intrinsic spring constant, but due to the radiation pressure force, the spring constant of the mechanical oscillator can be modified and that comes due to the radiation pressure force, this is also we discussed in the last class.

And we have taken another perspective where this whole thing could be understood using the radiation potential because the radiation potential is a forces related because force is equal to the negative of the derivative of the potential with respect to the position as you can see from here and this will result in a staircase kind of a potential and this potential has to be added to the harmonic oscillator potential which is the basically the intrinsic potential of the mechanical oscillator then, the resultant potential will look like this blue curve here.

And based on this we saw that and we discussed that we obtained what is the phenomena called multistability because, we can have a number of local minimum also we saw that the restoring force and the radiation pressure force in a typical cavity optomechanical system, they may balances at some points for example, here in this figure at point A, B and C the radiation pressure force is exactly equal to the restoring force.

However, only the point A and C are stable and B is unstable, because here as you see in the case of the point B the radiation pressure force is exceeding the restoring it is in the increasing direction at the point B on the other hand at point A and C the restoring force is in the increasing direction. So, therefore, point A and C that means, when the movable mirror is at the location A or at the location C.

Then it will have a stable position and in fact, the whole effect all these things have been experimentally observed demonstrated way back in 1980s by a Herbert Wather group for the first time. So, now because so far whatever we have discussed we assume that the light reacts instantaneously to the position of the mechanical object that means, the force is always an instantaneous function of position, but it is not actually the really.

Now, let us consider a cavity optomechanical system driven by laser light. If the laser light satisfies the resonance condition of the cavity, then light will be able to enter into this cavity and there will be circulating light inside the cavity, and this movable mirror say it displaces by let me denote a variable by x as usual. Now, let us switch off the laser light then what happens? At first instance the circulating light intensity is still the same as before.

However, as time goes on the light will leak away out of the cavity and slowly the circulating power will decrease or diminish. So, circulating power would diminish with time because the photon is leaking away from the cavity. And the rate at which the energy inside the cavity decreases is called the cavity decay rate and it is denoted by this symbol kappa and this is called the cavity decay rate kappa. Unless the mechanical motion is really slow, we have to take this cavity decay rate into account.

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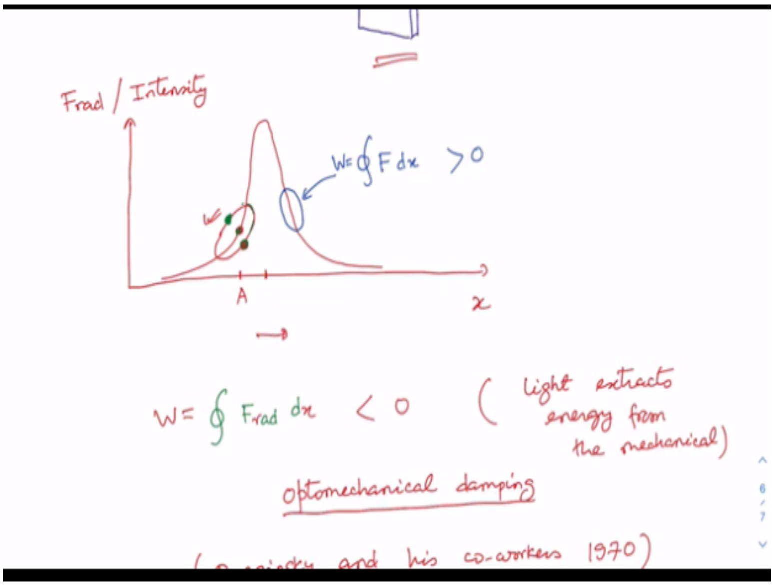

Let us consider the radiation pressure force against position plot again to understand this a little bit more clearly. So, we have this usual plot radiation pressure versus position and we already know that we are radiation pressure will be like this, we will just consider only one of the peak only. And by the way, the same kind of plot we are going to get for the circulating intensity inside the cavity as well.

Now, if we consider that the mirror is starting at some location say this location will be denoted by A and this mirror here is moving very, very slowly then as the position of the mirror is moved towards right then you can see that the force will increase as the mirror moves towards right. On the other hand, if it moves towards left, the force will decrease. This is easy to see. If the motion of the mirror is very slow, then there is no appreciable effect of the finite cavity decay rate we did not have to bother about the cavity decay rate that this stage.

Now let us consider the case where the motion is not that slow, that means this mirror is moving with some finite velocity, it is very small velocity, it is not a static case, this is the case for the dynamics. And in that case, this diagram would be a little bit modified. And this is what I am going to explain what we are going to get is this, let me just plot it. So, we will get this kind of plot if the mirror is moving with some finite velocity the movable mirror is moving with finite velocity.

Let me explain what is going on here as the mirror is moving towards right the light intensity for some time still remains low. So, suppose we are now here we are at this stage this is the static case. So, let me draw it a little bit different so, this is the dynamic case now, the mirror is moving towards right with some finite velocity. And therefore, the light intensity for some time still remains low compared to the static case, because the cavity still has to fill up the light corresponding to what would be expected for the new position.

So, we are slightly below the usual curve that is the static one of the force versus position then, as we complete the cycle, that means if we go like this and go suppose now we reach this position now, that means we are going moving back to the left, the light intensity for some time here will remain higher compared to the usual static case this red one you see red dot, the light intensity would be higher here, because the light has to leak out of the cavity, and it takes some time so this is the reason we get such kind of an ellipse.

If we draw the force versus position plot, for such a situation we must take the finite cavity decade rate into account. So, this has actually important consequences. For example, the walk done by the radiation pressure force in the mechanical object would be given by this expression. So, there would be this close integral, this is the radiation F dx force into displacement. Now, if we look at the plot carefully here, then we will see that when we are here at the lower point, both the radiation pressure force and dx displacement has the same sign while moving towards the right. Of course, in that case the contribution is lower than the usual one. On the other hand, this force and dx would have opposite sign when the mirror moves towards left and has higher contribution than the usual one. So, overall what is obtained is this, overall this integration would be less than 0, because the negative contribution will be higher when we are at this location of the resonance curve that means the resonance peak is at this location.

So, we are at the left of the resonance peak, then this integration basically work done would be less than 0. And what it physically means is that the radiation pressure in such a cycle extracts energy from the mechanical motion. So, the radiation pressure force extract or rather let me say light because the radiation pressure force is due to light. So, I can say that light extract energy from the mechanical motion and if energy is extracted from the mechanical motion.

So that will result in the so-called damping of the mechanical motion and this is known as optomechanical damping. By the way this kind of calculations were done by the Russian physicist Brazinsky and his co-workers, way back in 1970, this kind of analysis were done. And this phenomenon is also called as I said optomechanical damping and such a mechanical object is you know this kind of cantilever is usually coupled to fluctuating thermal environment.

And because of this damping force, which is usually associated with little noise, it also cools mechanical mirror. So, very simply as we have seen, because the light is extracting energy from the mechanical mirror, so, as a result of mechanical mirror is getting cooled down. And this phenomenon is extremely important and we will discuss it in great details later on also quantitatively and this is called optomechanical cooling.

And by this process, we can actually, it is kind of a laser cooling in fact it is laser cooling. On the other hand, if we go to the other side here and if we do the analysis here in this side, what turns out that this work done work turn out to be greater than 0 and what does it mean? It means that the radiation pressure force, work done due to the radiation pressure forces is greater than 0 means the light is going to dump energy into the mechanical oscillator or the mechanical mirror and this will result in anti-damping or heating.

And if it heats basically the more because already as I said, this mechanical oscillator is coupled to thermal environment and further heating takes place this thermal excitation will increase further and also if this effect is pronounced and it will also result in instability. Now, we should be able to read out what is going on within the cavity in order to analyze the phenomenon associated with the static and dynamic cases which we have already discussed. This is easy because the cavity is simply an interferometer.

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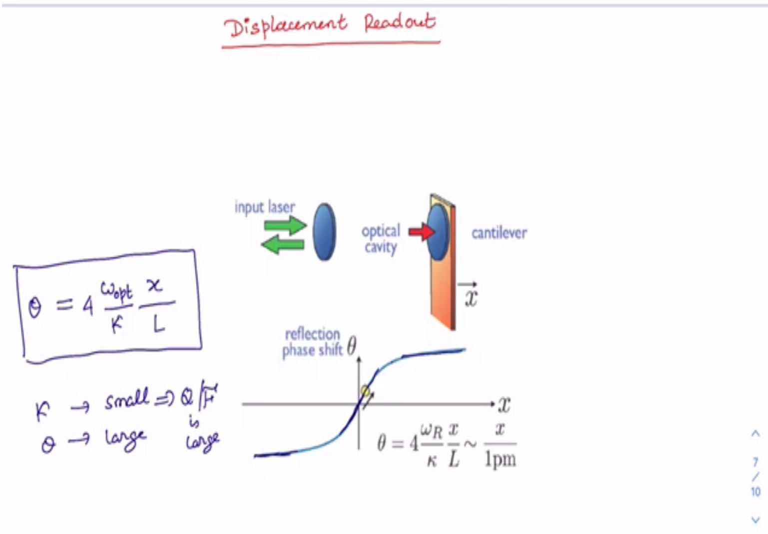

So, this is going to lead us to a discussion on what is called displacement readout. If the back mirror of this optomechanical system, so say this mirror is perfectly reflecting, then whatever the light is getting incident on this cantilever, because this is perfectly reflecting it will be completely reflected back and the reflected light will suffer a phase shift. Now, people or experimentalists look at this phase shift as a function of the position a typical plot is plotted here in the phase shift versus position.

As you can see from this diagram, initially when you are away from the resonance as you see here, there is no phase shift. The phase shift is there is no changing in phase shift basically it is constant, because whatever light is getting incident that is getting reflected back from the prom mirror itself, because light would be able to enter into the cavity only if the resonance condition is satisfied and when the resonance condition is satisfied light enters and it can be reflected from the back mirror.

And then there will be phase change and what is reflected in this slope here? So, there is a phase change and again as we move the mirror further than resonance condition would not be satisfied and the light would get reflected back from the front mirror itself and the phase change will again remain the same and there will be no phase shift actually. So, phase shift would be given by actually can be worked out and if it is worked out.

Then it turns out that the phase shift is directly proportional to the displacement of the mirror and inversely proportional to the cavity decay rate. So, if the cavity decay rate is very small cavity decay rate kappa has to be a small k, means theta large so we will be able to have a significant phase shift that we should be easy to measure and if kappa or cavity decay rate is small that means that quality factor or the finesse of the cavity has to be large and what does that mean?

That means that the number of oscillations the photon will make inside the cavity will be large or in other words, it means that the photon will survive inside the cavity for longer amount of time and this is the reason why in cavity optomechanical system, we need to have a we require high Q cavity or high fineness cavity. So, you see from phase information as you can see from this expression from phase information, we can have the information about the displacement of the mirror and one can actually plot the displacement versus time curve for the mechanical oscillator.

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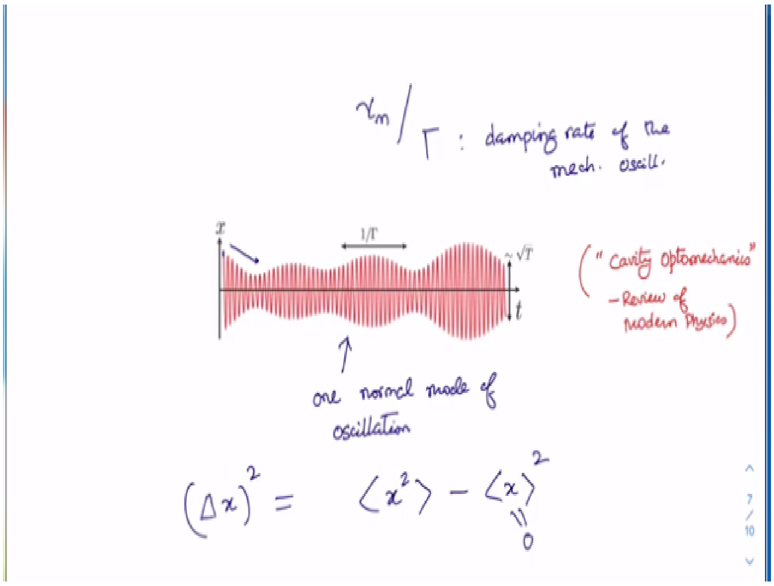

And a typical plot will look like this here this I have taken from the review paper cavity optomechanics. Now, you see the mirror is oscillating with I am just considering here only 1 eigen or normal mode of vibration is considered one normal mode of say oscillation of the mechanical oscillator is considered and here this gamma refers to the damping rate of the mechanical oscillator damping rate of the mechanical oscillator somewhere we have taken gamma m like this, but it is the same thing.

So, initially as you see from the plot the amplitude rises here because the thermal fluctuation from outside heats up the motion and with time this fluctuation gets weakened, so, it becomes weak and the amplitude also depletes, now, the question is how large the variance of this fluctuation quantified by the variance of this harmonic oscillator or the mechanical oscillator. So, you recall that variance is given by this expression.

So, delta x square = x square average - average of x square but for harmonic oscillator we know already that the expectation of x is 0. So, therefore variance would be given by only this quantity.

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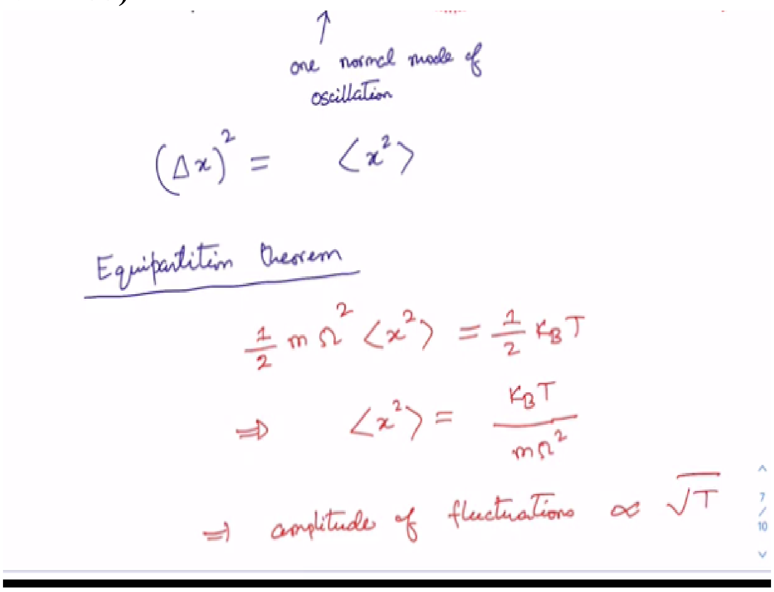

Now, how we can estimate it, this is easy, we can do that by using the so called equipartition theorem we can estimate the variance. Let me show you it is very straightforward. So, we know that the energy of the harmonic oscillator is half x square or m omega square or omega is the oscillation frequency of the oscillator and then this is the variance and this has to be equal to half K B T because we are consuming 1 mode of oscillation, so it implies that this variance is x square = K B T / m omega square.

So, it implies the amplitude of the fluctuations varies like square root of temperature this is easy to see. However, in fact, this is what is shown here that this amplitude is directly proportional to the square root of temperature. So, thereby, we can extract information about the temperature just by measuring the amplitude of the fluctuation. However, it is important to note that the displacement time plot that we have shown here is far more complicated in reality.

Because here as I said, we are considering only a single normal mode of oscillation, but in reality, there is an infinity of normal modes in the mechanical structure and the displacement of the mechanical oscillator at the position of the laser spot here this displacement consists of a superposition of all such normal modes of vibration and all these different normal modes of vibration has different frequencies. So, it is better to Fourier decompose it and look at the Fourier spectrum.

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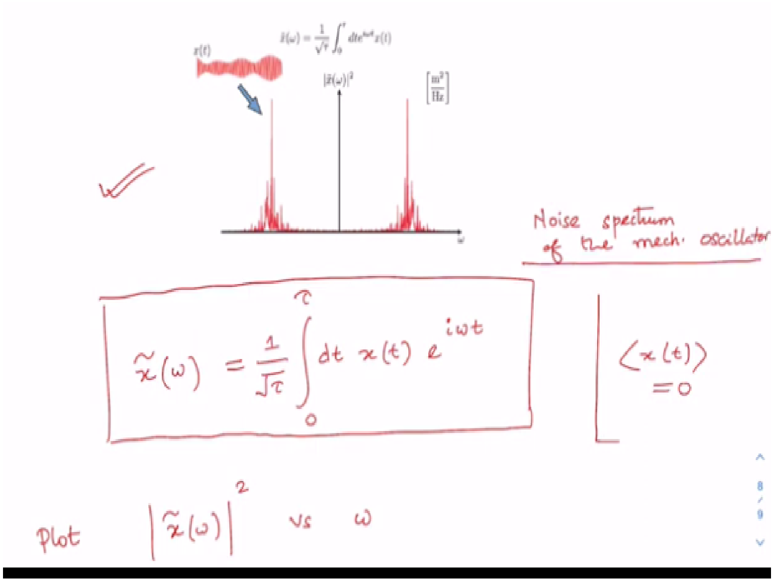

The Fourier transform of the displacement x of the harmonic oscillator in a time interval from say 0 to t. So, let us say the Fourier transformation of the displacement is denoted by x tilde omega and it is Fourier transformation of the displacement x of t. So, e to the power i omega t is the Fourier kernel then we are going to integrate in the time interval say 0 to tau and this is the normalization factor here 1 / square root of tau. So, this is the Fourier transformation of the displacement.

Now, this is obviously a complex quantity and average of this quantity is 0 because we know that if we take the average of this x of t for harmonic oscillator this is equal to 0. So, better take the mod of this quantity and square it and then plot x tilde, what I am saying is that you better take the mod of this particular quantity x tilde of omega. Let me write it properly so you have x tilde omega you take the mod and then you square it.

Now, let us plot x tilde omega mod square this quantity versus frequency omega and actually a typical plot is already shown here. As you can see from this plot, this is actually called noise spectrum, this plot is the so-called noise spectrum of the mechanical oscillator. As you can see, we are obtaining a pretty fluctuating spectrum here. However, in laboratory will make a series of measurements not just one measurement.

So, we need to take average of many such spectrum and it means that it is better to plot you better take this x tilde of omega mod square and then you take the average and plot this average quantity versus the frequency and when we plot it this plot is called average noise spectrum of the mechanical oscillator a typical plot is now shown here.

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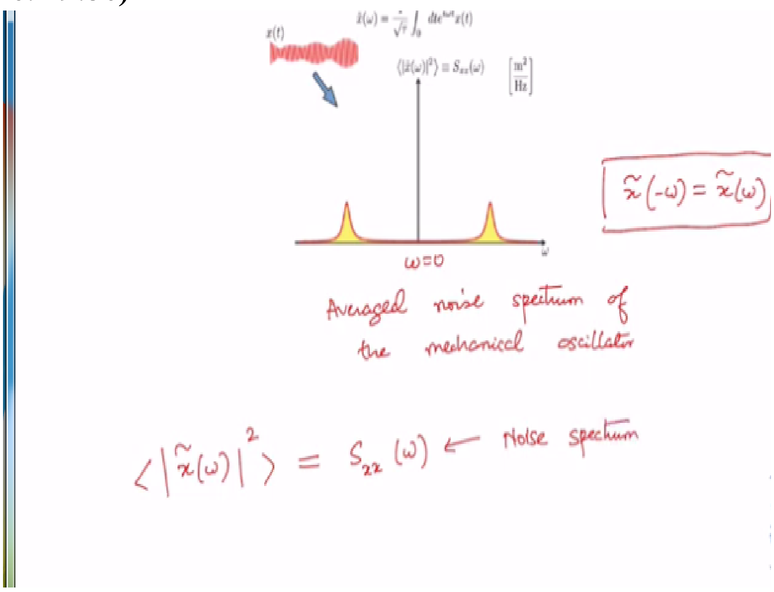

So, this is the averaged noise spectrum of the mechanical oscillator. One thing you can quickly notice is that the spectrum is symmetric around omega = 0 and what here it means is that x tilde of minus omega = x tilde of omega. So, this is what I mean by symmetry and this is a typical characteristic of noise spectrum in the classical regime. Now, in fact, it can be shown that this particular quantity x tilde of omega mod square.

And if we take the average and this is also denoted by this symbol S xx of omega this is actually known as noise spectrum, this quantity is referred to as noise spectrum, it can be shown, because we know the what is x tilde of omega.

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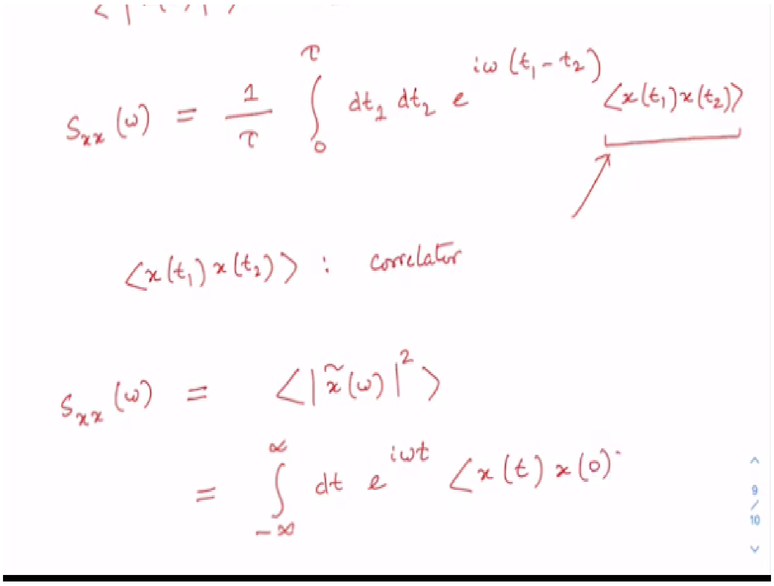

So, it is very easy to show that this is equal to or let me just write here that I can write x omega, this noise spectrum = 1 / tau 0 to tau dt 1 dt 2 e to the power i omega t 1 - t 2 and we have x of t 1, x of t 2, it is very easy. If you find it difficult, do not worry, we will actually address it in our problem-solving session. This is what I have and this particular quantity you see average of x t 1 and x t 2 this is called correlator.

So, let me write here this is an important quantity displacement at 2 different times, this is known as the correlator or the correlation function and it basically gives us the information of the displacement of the oscillator at 2 different times. Now, in the steady state no point in time is special because it is the steady state and the correlator will depend only on the time difference. So, it can be shown just by changing variables that I can express in the steady state, S xx of omega which is the noise spectrum and this is x tilde omega mod square and the average. So, it can be shown that it is minus infinity to plus infinity because all times are equivalent. So, therefore this correlator I can just say the time variable and I can write it like this.

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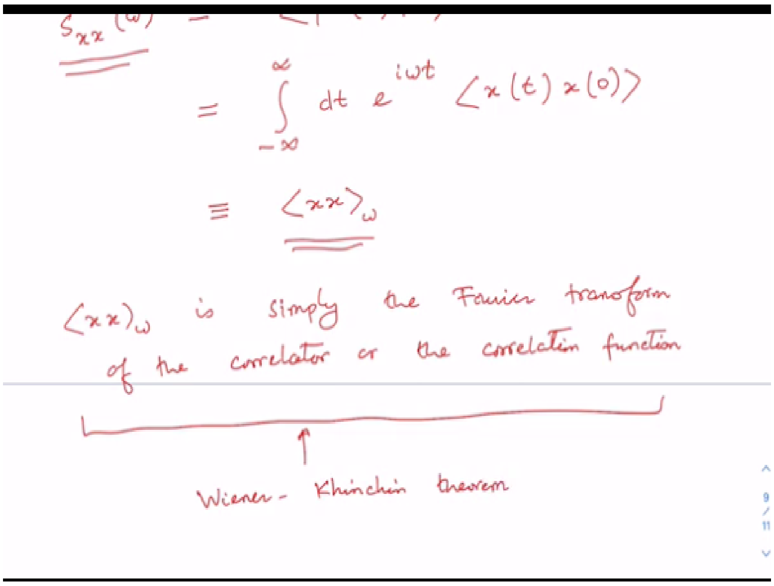

This is also sometimes it is symbolically represented by this xx of omega and clearly either this quantity or this or this is the same quantity this means that or rather let us say this quantity is simply the Fourier transform of the correlator or the correlation function. So, what you see basically is that the noise spectrum is nothing but the Fourier transform in the correlation function. And this is known as, and it is referred to as Wiener Khinchin theorem, so Wiener Khinchin theorem states that the noise spectrum is nothing but it is simply the Fourier transform of the correlation function or the correlator.

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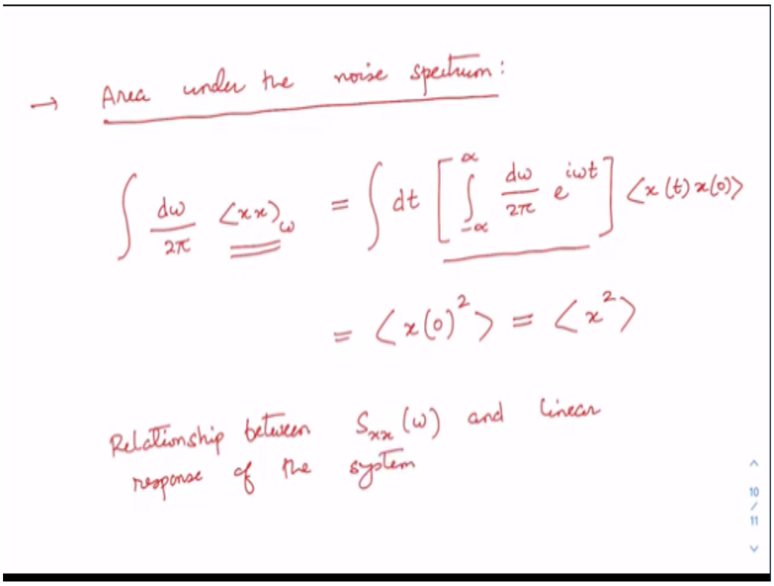

Now, one can very easily calculate the area under the noise spectrum. So, what I mean by that is you just have to by conventionally, we will put 2 omega this will be divided by 2 pi and this is the quantity we can calculate. And you know that this would be d of dt because we know what is this noise spectrum is then I have here this is from minus infinity to plus infinity d omega / 2 pi e to the power i omega t and then I have x of t x of 0 and this is very easy to calculate because you know this is nothing but the delta function.

So, this will lead us to x 0 square here and because all times are equivalent in steady state, so, I can simply write it at x square. So, what you see that the area under the spectrum give us the variance of x. So, area under the spectrum gives us the variance of x and we know that the variance of x is directly related to the temperature of the harmonic oscillator in thermal equilibrium. Now, let us find out the relationship between the noise spectrum and linear response of the mechanical system.

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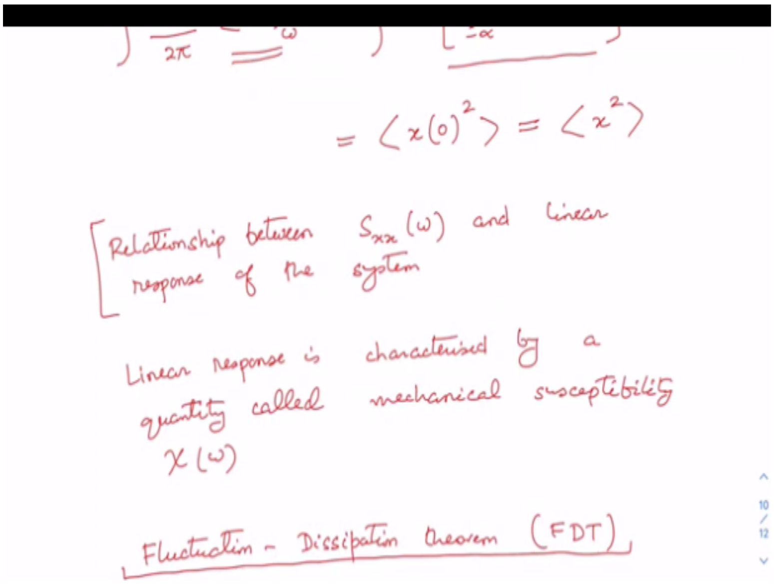

Now, this linear response of the system is characterized by quantity called susceptibility of the mechanical oscillator. So, linear response is characterized, I will explain it, by a quantity called mechanical susceptibility and it is denoted by chi of omega at a particular frequency and these are relation between the noise spectrum and the linear response of the system. In statistical physics, it is known as the fluctuation dissipation theorem or it is called FDT.

Now we will actually not discuss it in details, only the theorem is needed for us and that is what I am now going to state. The fluctuation dissipation theory or FDT in sort could be stated as follows.

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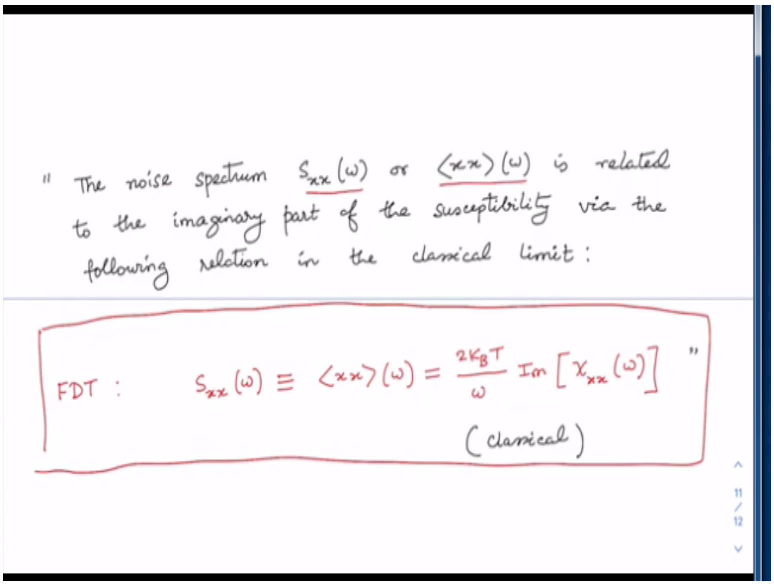

The noise spectrum S xx omega or also it is denoted by this quantity by this xx of omega is related to the imaginary part of the susceptibility via the following relation, this relation is given in the classical limit. Now, let me explain it and first let me explain about the susceptibility because you already know what is this noise spectrum term, you are already may be clear about it. So, regarding susceptibility you know that we have encountered this term called susceptibility in many places in physics and other areas of science.

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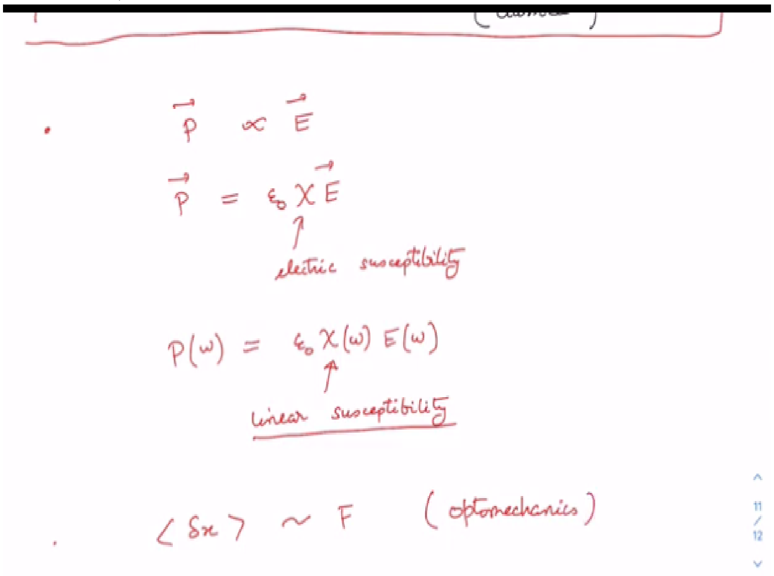

For example, if you recall so called the polarization say the polarization P that you encountered in electromagnetism, which is the electric dipole moment per unit volume in a dielectric arises due to the application of an electric field and this polarization P is directly proportional to the external electric field that is applied and it is actually expressed in this form. So, P = epsilon 0 chi of E and here chi is the electric susceptibility it is the proportionality constant and this is what we call electric susceptibility.

So, what happens is that this polarization sometimes it is very strong at a particular frequency of the electric field and hence, it is also written in this form that this polarization is a very strong at this particular frequency corresponding to the application of the electric field at that particular frequency and this proportionality constant would be this way chi of omega now chi as I said this is electric susceptibility and this is actually called linear electric susceptibility, it is linear because you see the polarization is the linearly dependent on the electric field.

However, there is if the electric field strength is very large, then only their terms are also has to be incorporated but we will not go into those domain, here we will assume that the electric field that is applied to the dielectric is not very strong. So, in that case the polarization in electric field they share a linear relationship between them and that is the bridge between these relations are actually given by the so-called linear susceptibility.

Now, in the similar way, the average displacement say delta x of the movable mirror in an optomechanical system is directly proportional to the radiation pressure force that is imparted on the mechanical mirror, so this is in optomechanics, we can have the similar thing. So, obviously, hopefully we are now guessing it where this linear susceptibility term will come. Now, again here also the response of the mirror to this radiation pressure force is strong at a particular frequency say omega and this is represented in this form.

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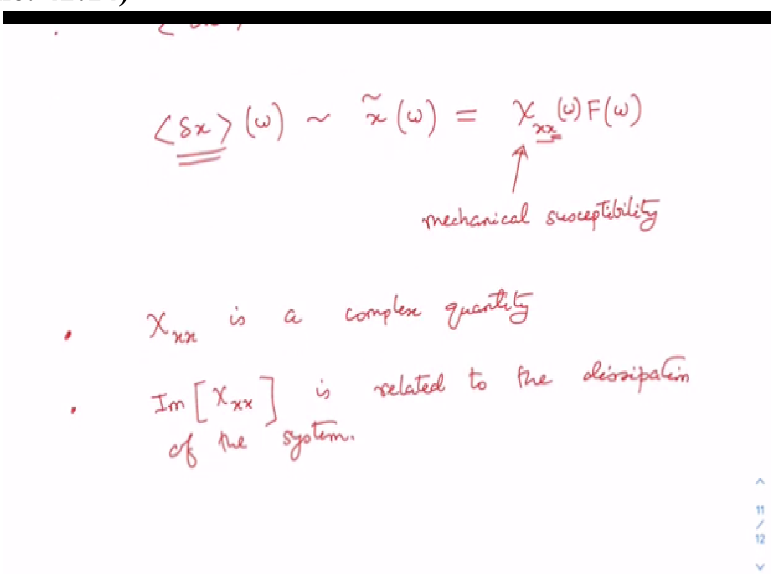

So, this displacement of the movable mirror it is a very strong responses at frequency omega or I can simply write it as the displacement and frequency omega. And this is equal to the radiation pressure force at the frequency omega and the proportionality constant here is given by this term and this is your susceptibility term here this is called mechanical susceptibility, because we are now dealing with mechanics. So, this is mechanical susceptibility of the movable mirror or the optomechanical system susceptibility.

And by the way here as you see there are 2 x I am putting the first x is related to the response that we are looking because we are looking at the response, this one, that is the displacement and the second x here this second x is related to the direction along which the force is applied. So, here this second x that means, the force is applied along the x direction and the response is also measured in the x direction that is what it means. So, susceptibility is a complex quantity and because, it is a complex quantity.

So, let me write here that this mechanical susceptibility is a complex quantity and that means that it has a real part and the imaginary part and the imaginary part of the susceptibility is related to the dissipation of the system. Let me explain this actually dissipation of the system to make you understand it.

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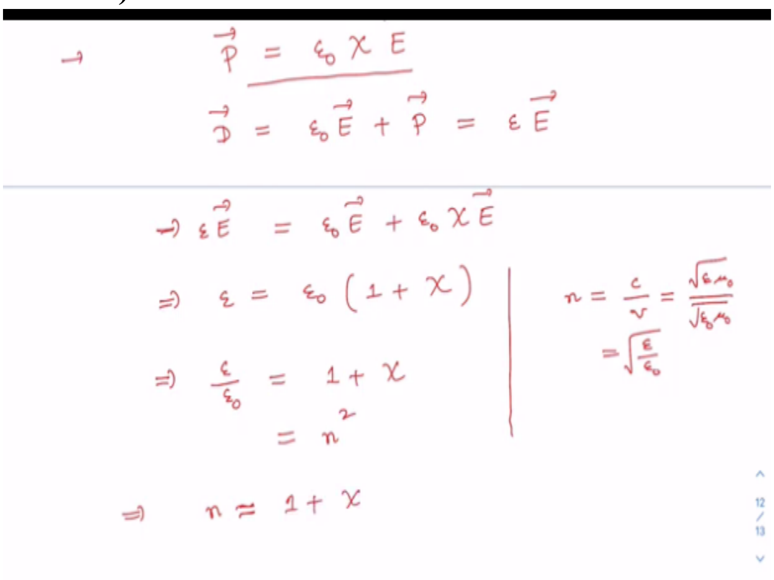

Let me go back to the case of electric susceptibility where we had this say polarization is directly proportional to the electric field. Now, all of you must have learned the electromagnetic theory, and in electromagnetic theory you know that the so called displacement vector you know that is related to the polarization by this expression you know epsilon 0 E plus the polarization which actually is related to this, epsilon 0 is the electric permittivity in free space and epsilon is the electric permittivity in the medium and from here as we have P is equal to this.

So, immediately we can write let me write here this means that I have epsilon E = epsilon 0 E + epsilon 0 chi into E and what it means that I have epsilon = epsilon 0 into 1 + chi that is the electric susceptibility and this quantity epsilon / epsilon 0 = 1 + chi. Now if you recall that refractive index just I am giving it a very you know, it is not exactly the way we should, but just to recall we should do but just tentatively let us understand it this way.

That speed of light is square root of epsilon 0 mu 0 and v is the velocity of or the speed of light in the medium that is again 1 / square root of epsilon and mu, if the medium is nonmagnetic, then mu = mu 0. So, let us we have that kind of a medium. So, you have n = epsilon / epsilon 0 square root. So, therefore, this quantity is simply equal to 1 + chi is nothing but refractive index n square.

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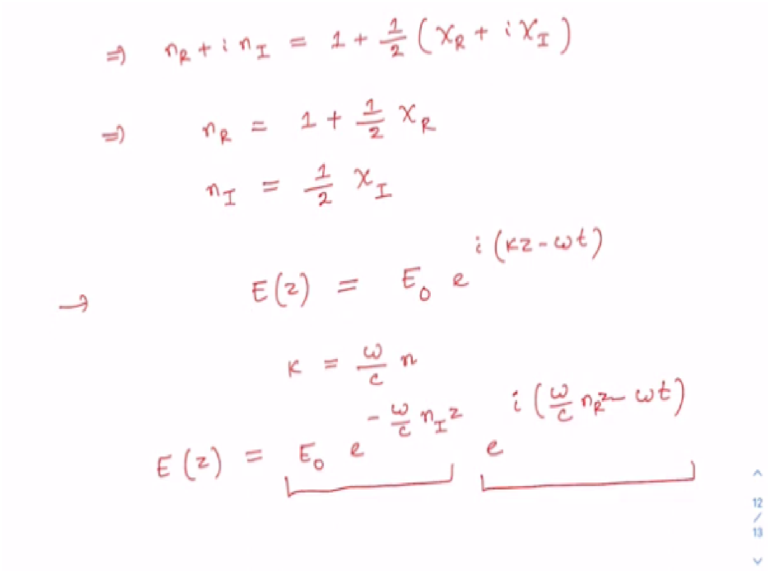

And this if I can also if say, chi is a small quantity, then I can write n = 1 + half of chi and refractive index is a complex quantity. So, it has a real part and an imaginary part, then I have this 1 + half susceptibility has also a real part and an imaginary part. So, let me write it this way i into chi of phi. So, the real part of the refractive index is equal to 1 + half of chi of R and the imaginary part of the refractive index is related to the imaginary part of the susceptibility like this.

Now, you know that if a plane electromagnetic wave passes through a medium, its electric field at a distance says z is given by this E of z, you may be aware of Lambert Beer law, so I am going to that here. So, say electric field initial is amplitude is E 0 and as it passes to a distance z so you have this phase vector here and also you know that this k is the propagation vector k = omega / c into the refractive index.

So, therefore I can write because refractive index is a complex quantity, so, I can write it as E 0 e to the power - omega / c the imaginary part of the refractive index then I have this propagation part of the electric field that is i into omega / c real part of the refractive index - omega t. So, this part represents the propagating part and this represents the amplitude. Now, what you see here also z is there, here also z is there. So, what you see?

That the amplitude is as the electric field is propagating through the medium amplitude is decreasing and dissipation of the amplitude is related to the imaginary part of the refractive index and who is in turn is related to the imaginary part of the susceptibility. Now, you see that this imaginary part of the susceptibility is generally related to the dissipation. So, hope you are getting now convinced. So in the similar way, we can say that, even in the mechanical case the imaginary part of the mechanical susceptibility is related to the dissipation of the mechanical system.

Now, you if we go back to the fluctuation dissipation theorem here as you see this particular quantity the susceptibility is independent of temperature, it only depends on the spring constant the damping rate mass etcetera, but you know the fluctuations do depend on temperature and hence in the fluctuation dissipation theorem, the temperature T is also appeared in the expression here.

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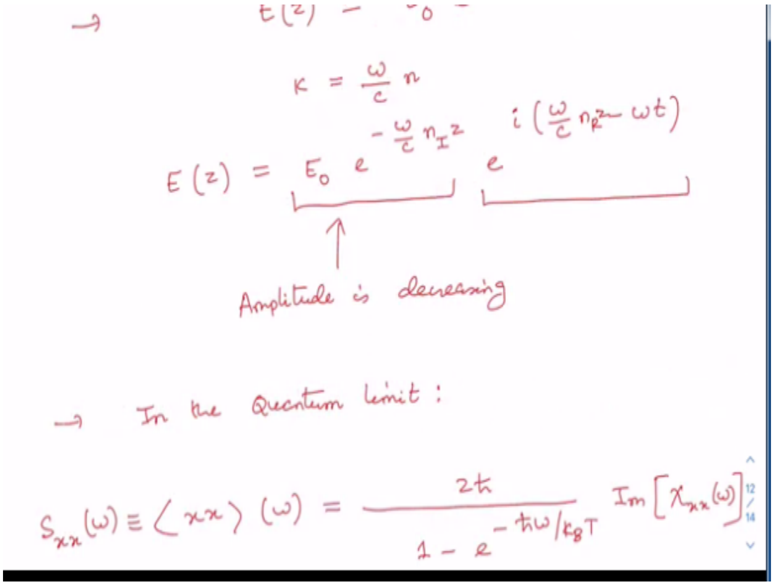

Now, if we go into the quantum limit, which we will discuss in a tutorial problem, in the quantum limit the FDT the fluctuation dissipation theorem is actually given by this expression that is noise spectrum is equal to twice of h cross divided by 1 - e to the power - h cross omega / K B T. And this is related to again imaginary part of the mechanical susceptibility. At high temperature the quantum version coincides with the classical version, and we will discuss about it in the problem-solving session. To appreciate fluctuation dissipation theorem. Let us consider a classical damped harmonic oscillator.

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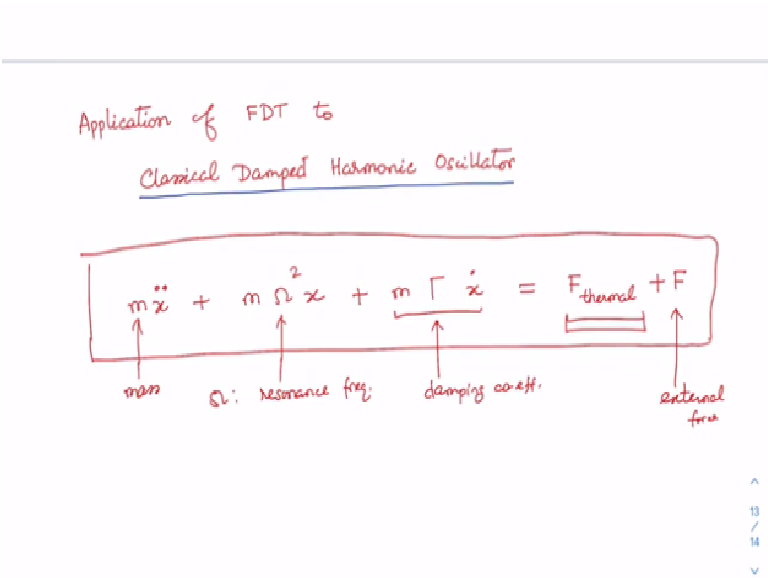

So, you know the equation of motion for a classical damped harmonic oscillator is given by this expression. Here this small m is the mass of the harmonic oscillator, this capital omega is the resonance frequency of the harmonic oscillator, this gamma is the damping coefficient and this F is the external force and F thermal you know even if there is no external force, in thermal equilibrium we are and also you can get the heat from this presence of this damping term that.

Even if there is no external force, there is always some kind of a thermal environment and this thermal environment is quantified by this term F thermal. Now, the idea in defining susceptibility is to look at the average response of the harmonic oscillator. And because this F thermal is a fluctuating term so, while we average it then we will be able to get rid of the thermal force and we are now going to average it to obtain this one.

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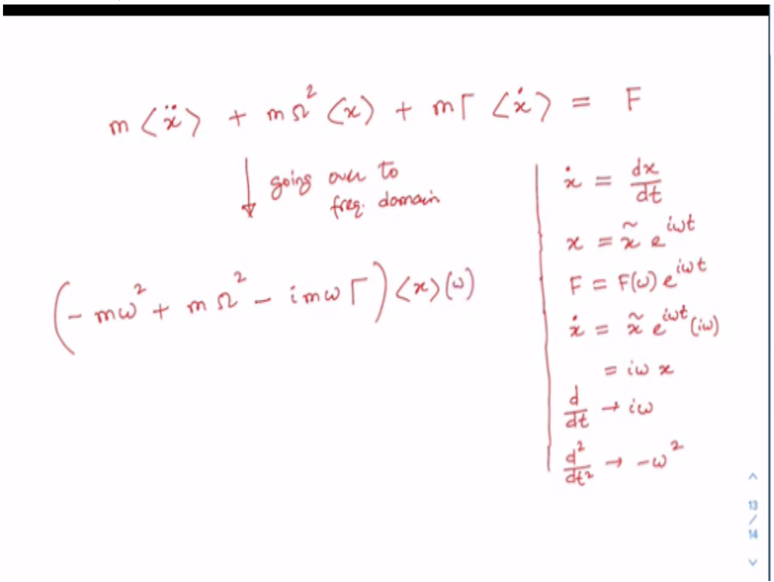

So, if we average it we will get m x double dot, this is average m omega square average of x + m gamma average of x dot and F thermal average would go to 0 so, we will just left out with this external force only. By the way, you let me just clarify that when I say x dot I mean to say the time derivative of x this is what I mean. Now, it is always better to work in the frequency domain.

So, if we work in the frequency domain by that I mean say if we take x is equal to say x tilde e to the power i omega t, and F is equal to say F of omega e to the power i omega t, then if I take x dot then I will get x tilde e to the power i omega t into i omega and which is simply i omega x so, as you can see, I can always replace delta t or dt here I can replace these dt / i omega so that is the prescription when we go over to the frequency domain so taking up this prescription.

So, d 2, dt 2 would become minus omega square. Taking this prescription going over the frequency domain let me write here and going over to frequency domain I can write it as minus m omega square + m capital omega square - i m omega gamma and here I have this x let me just write average of this but it is in the frequency domain and this is equal to F of omega.

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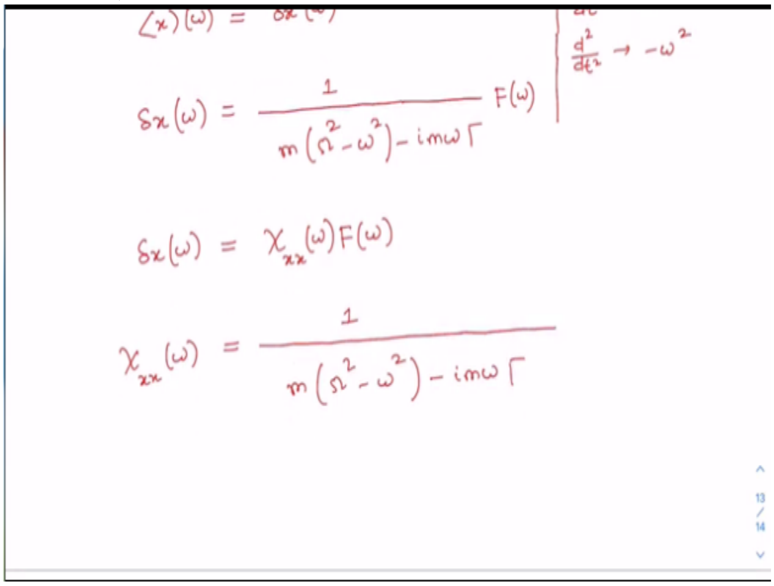

Because this displacement is very small so, I can simply to emphasize that I can write it as delta x of omega so, I have delta x of omega = 1 / m capital omega square - omega square so, here I have omega square - i m omega gamma into F of omega. Now, you recall that this is the response of the harmonic oscillator and it is directly proportional to the force that is applied external force and this proportionality constant as you know this is given by the so-called susceptibility of the mechanical oscillator.

So, we can easily read out the expression for the susceptibility for this classical damped harmonic oscillator and that is equal to 1 / m capital omega square - omega square - i m omega gamma. This expression can be simplified, but before that as you can see this susceptibility is a complex quantity and it has a real part and the imaginary part

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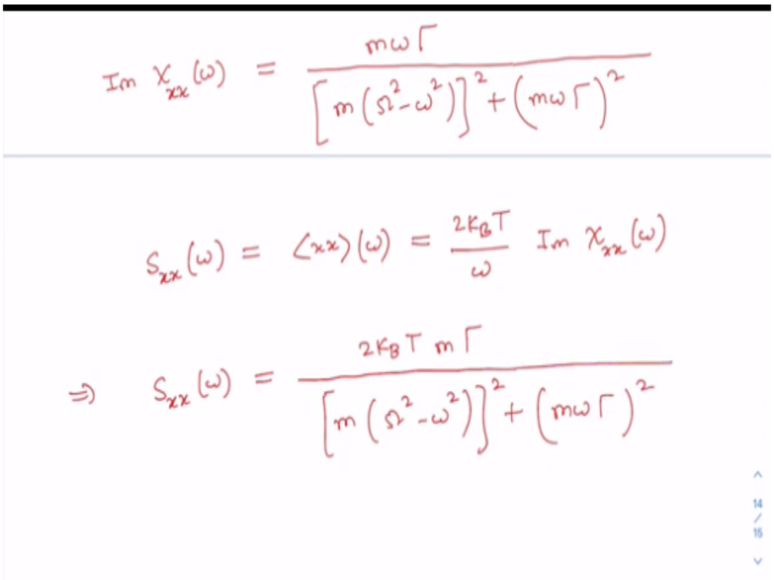

And imaginary part of the susceptibility is very straightforward to get and you can easily get it and if you do the algebra, then the imaginary part of the susceptibility will turn out to be m small omega gamma / m capital omega square omega square then whole thing whole square here plus m omega gamma whole square. So, this is what we have as the imaginary part. Now, you see, imaginary part of the susceptibility is associated with the noise spectrum as per our fluctuation dissipation theorem.

So, this noise spectrum which is actually the same as this term and this is equal to twice K B T by omega and imaginary part of the susceptibility. So, therefore, the expression for the noise spectrum this particular term is equal to if I put the terms from the susceptibility I have twice K B T m gamma / m capital omega square - omega square this whole square + m omega gamma whole square.

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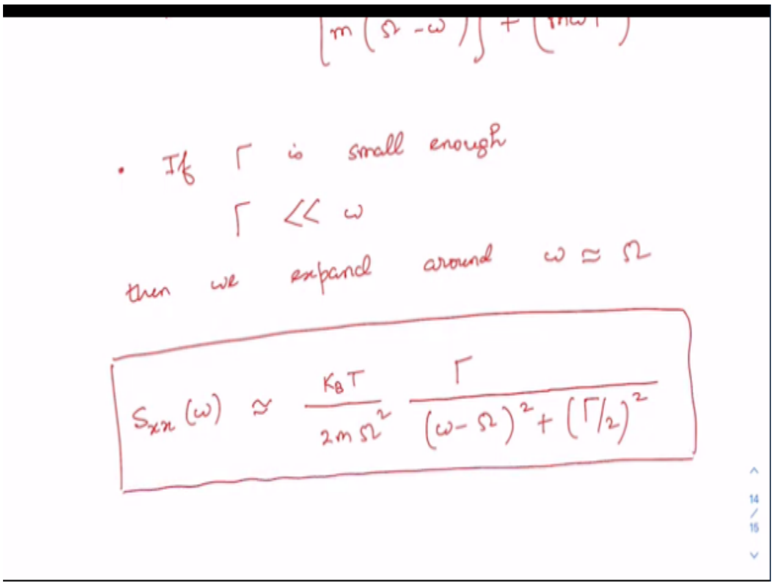

This expression can be further simplified if say if gamma is small, dissipation is small enough and you will have say gamma is far smaller than the frequencies, gamma is far smaller than frequency omega, then we can expand around omega = capital omega, then this expression for the noise spectrum, these expression can be further simplified. And you can do it very easily. So, I am leaving it to you to do it, otherwise, we will do it in a problem-solving session.

So, we will have K B T divided by twice m omega square divided into gamma / omega - capital omega whole square plus gamma / 2 whole square. So, this is the expression we will have now you see, this expression is has a Lagrangian form and the typical plot would look like this.

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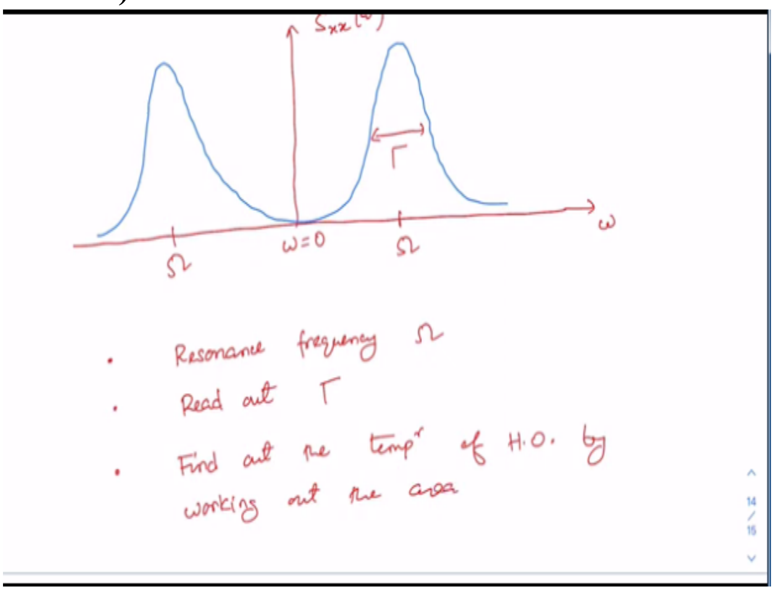

So, let me just plot it. So, we will have here the noise spectrum, let me put it here in the y axis and omega in the x axis then we will have a plot something like this. So, you will see that this is symmetric around omega = 0 and the full width at half maximum here you can easily find it out full width and half maximum would be gamma here and what are the other things that we can find out? We will find that a peak is situated at capital omega resonance frequency in both cases here or both maybe I am not drawing it properly.

But you take it as a completely symmetric around omega = 0 and what are the other things that you can find out from this simple diagram? A couple of things, one is you can see, you can find out the resonance frequency of the harmonic oscillator omega can be readout from this plot and also what you can do? You can find out the damping rate. We can read out gamma the damping rate just by measuring the full width at half maximum of the spectrum and also what we can find out? We can find out the temperature of the harmonic oscillator by working out the area and this is very straightforward because already I explained it I think earlier.

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- Resonance frequency Ω
- Read out Γ
- Find out the temp^r of H.O. by working out the area

$$\int \frac{d\omega}{2\pi} S_{xx}(\omega) = 2\,\frac{1}{2\pi}\,\frac{2K_BT}{2m\Omega^2}\int_{-\infty}^{\infty} d\omega\,\frac{\Gamma}{(\omega-\Omega)^2+\left(\frac{\Gamma}{2}\right)^2}$$

$$= \frac{2K_BT}{m\Omega^2} = \langle x^2\rangle$$

So, what I have to find out? I have to find out this integral d omega / 2 pi and this is the noise spectrum this area you can work it out it has to be twice because I have 2 peaks then I have 1 / 2 pi I have 2 K B T by twice m omega square from this expression here. So, I will have say minus infinity to plus infinity d omega gamma / omega - capital omega square plus gamma / 2 whole square. If you do the calculations and it is very straightforward please do that you will get twice K B T / m omega.

So, this would be capital omega this would be m omega square and this is nothing but the variance. So, that is how you will be able to find out the temperature also just by finding out the area. Let me stop here for today, in this lecture, we have completed our discussion on basic physics qualitatively and also, we discuss the so-called fluctuation dissipation theorem, which basically prepares us for discussion of cavity quantum optomechanics quantitatively from the next class onwards and in the next class we are going to discuss the classical regime of cavity quantum optomechanics. So, see you in the next class thank you..