Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture –3
Two-level System- I .

Hello, welcome to the second lecture of the course in this lecture we will start discussing 2-stage system in quantum mechanics. And these systems are extremely important because of its application in quantum information science and quantum technology. In fact, later on while discussing artificial atoms we will model those atoms as two-level systems.

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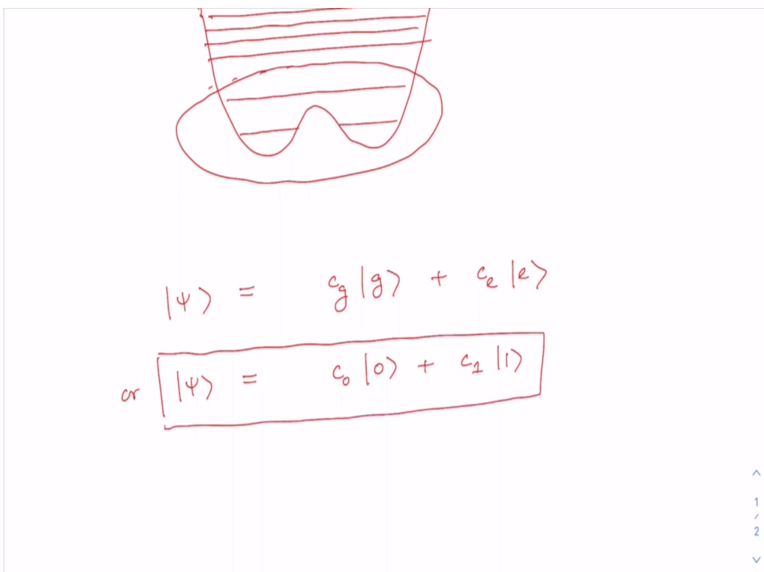

So, let us begin. Two-state systems or two-level atoms which are also referred to as qubits are backbone of quantum technology if one is interested in quantum information processing applications. By a two-state system we mean a quantum system where we have just 2 energy levels which are the most relevant and they are corresponding eigenstate. For example, we can have a two-level atom or two-level system having energy 0 and E with the corresponding eigenstate denoted by say ket g and ket e respectively.

In the real world we can consider a spin half particle or a spin half quantum system as a two-level atom or two-level system and we know that for a spin-half particle we have only 2 state one is the up-spin state and another one is the down-spin state and they can be represented by these kets. And even in the so-called double well potential, if we have a double well potential like this and if the low-lying energy levels are well separated from the higher energy levels, we can ignore all these higher energy levels and we can focus only on the 2 low-lying ground states and we can create this system as a two-level atom or two-level system. Now an arbitrary two-state system can be represented by a state vector say ket psi which would be the superposition of the ground state and the excited state and with their corresponding coefficient c g and c e. c g mod square gives the probability of finding the system in the ground state and c mod square gives the probability of finding the system in the excited state.

And sometimes this two-state system can also be represented in this notation where we will write it as a c 0 ket 0 + c 1 ket 1. In fact, this is the most you used notation when one talks about qubit.

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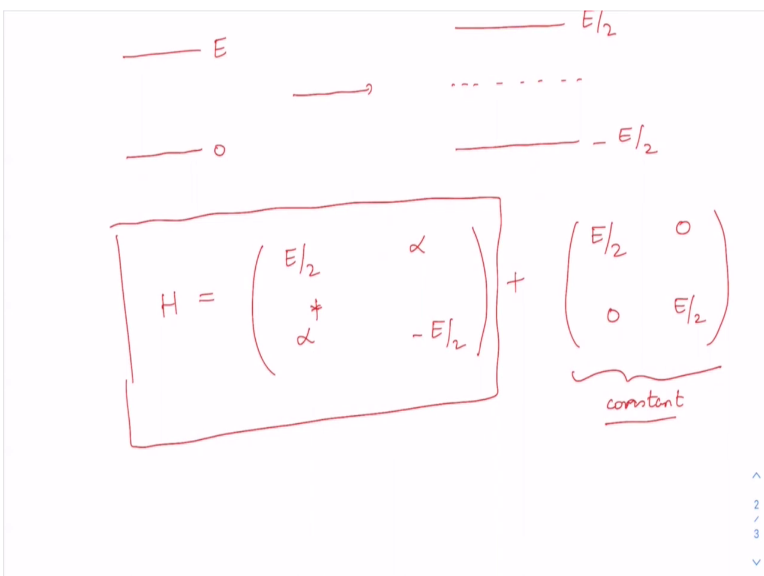

The Hamiltonian of a two-state atom can be represented by a 2 by 2 matrix. So, we have say, energy E and energy 0 this diagonal element refers to the energies and the off diagonal elements has to be complex and they have to be complex conjugate to each other because this Hamiltonian has to be Hermitian and this off diagonal elements refers to any coupling existing between the energy levels.

We can always shift the energy levels. Say here, we have 0 E. We can always shift the energy level such that we we can shift it to this is 0 here and here we can write E by 2 and this one has to be minus E by 2. So, that the difference between the energy levels remains the same that is E here and this can be done by adding a constant term to the Hamiltonian. So, our Hamiltonian now here we have E by 2 and minus E by 2 and here the off diagonal elements are alpha star and alpha.

And then we just have to add a constant term E by 2 0 0 E by 2.0 This constant term is of no relevance as regards physics is concerned and we can just keep this part of the Hamiltonian.

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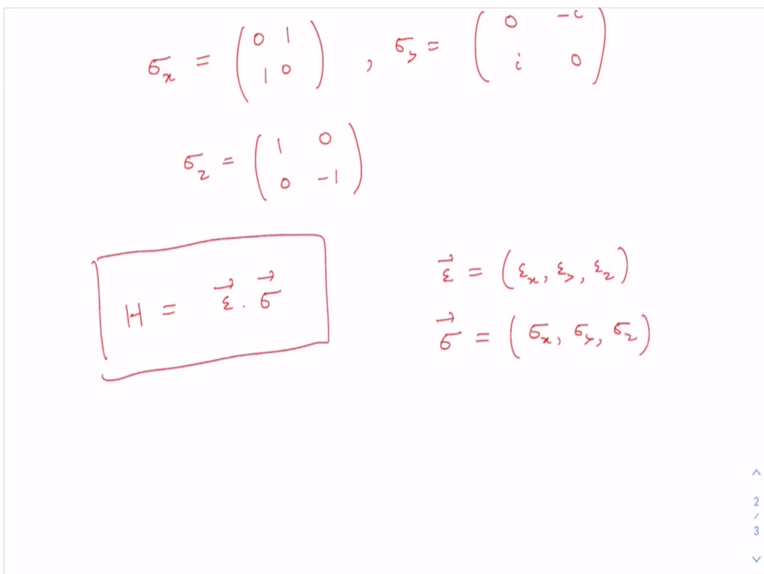

Writing the Hamiltonian in this form has a huge advantage as we will see now. Let me write a general Hamiltonian for a two-state system as follows; H is equal to epsilon z minus epsilon z and here we had earlier alpha and alpha star. So, let me consider alpha as epsilon x + i epsilon y and then alpha star here would be epsilon x - i epsilon y and this will ensure that this Hamiltonian is Hermitian.

These we can express in terms of the so-called Pauli matrices. So, you may know about these Pauli matrices. There are three Pauli matrices sigma x which is 0, 1, 1, 0, sigma y which is 0, -i, i, 0 and sigma z that is 1, 0, 0, minus 1. So, in terms of these Pauli matrices it is very easy to see that this Hamiltonian here I can write it in a compact notation like this I can write it as epsilon x sigma x + epsilon y sigma y + epsilon z sigma z.

Which I can further write in a very short form and I can write it as x is equal to epsilon vector dot sigma and epsilon vector has the components epsilon x, epsilon y, and epsilon z and this sigma vector which is the Pauli vector has components sigma x, sigma y, and sigma z. Let me, before I go further, quickly remind you about some useful properties of the Pauli matrices.

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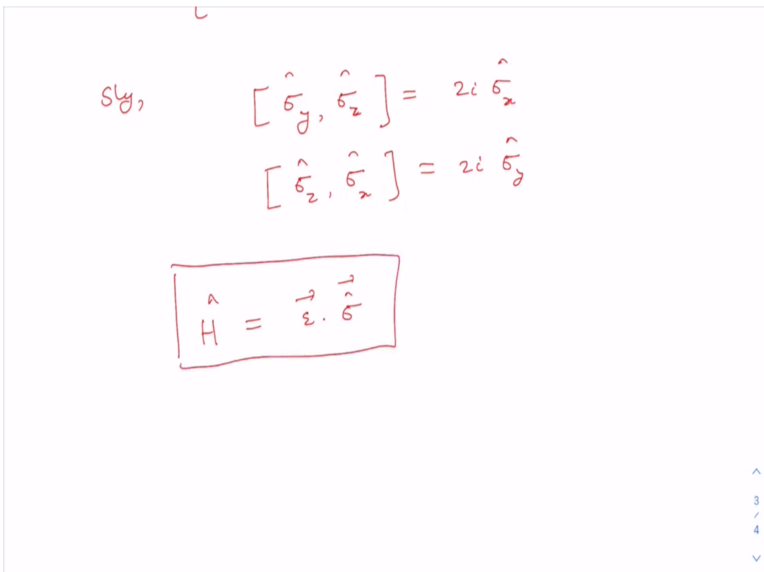

For example, you know that if you take sigma x square or again sigma y square all of them would be equal to identity matrix. You can verify it very easily. So, this would be identity and the product of the matrices sigma x into sigma y, but these are actually operators, so, let me write it like this, so, if you work it out you will get it as i sigma z. let me quickly show you. So, sigma x is 0, 1, 1, 0, sigma y is 0, - i, i, 0 and if you take the product here you will get it as i, 0, 0, - i which you can write it as i, 1, 0, 0, -1 and this is nothing but your sigma z. So, this is i sigma z.

Similarly, you can show that sigma y dot sigma x is equal to - i sigma z. So, this clearly shows that sigma x and sigma y does not commute. So, we have this computation relation sigma x sigma y is equal to twice i sigma z. In fact sigma x and sigma y actually anti-compute. So, you will have sigma x sigma y. So, anti-computation means you have to add it up sigma x sigma y + sigma y sigma x. If you add it up you will get 0. In the similar way you can show these relations sigma y sigma z is equal to twice i sigma x. So, you note the cyclicity here y z then you have here x. Similarly, you can write sigma z sigma x. This combination relation would be twice i sigma y. Let me now discuss about the dynamics of a two-level system. when it is not driven by any external drive the system is described by this Hamiltonian H is equal to epsilon dot sigma this Pauli vector.

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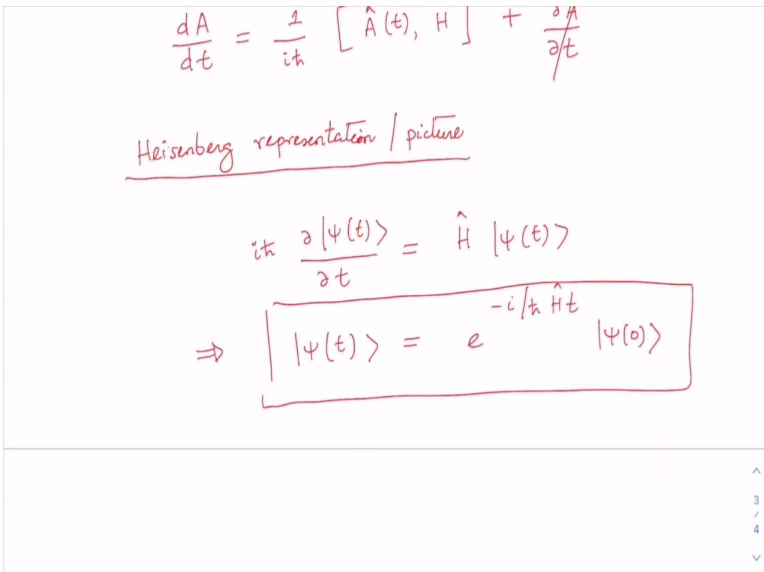

To know the dynamics of the system we have to find out how the Pauli vector sigma evolves in time. We can work out how Pauline vector evolves in time by using the so-called Heisenberg equation of motion. According to Heisenberg equation of motion the time rate of change of and Hermitian operator A is given by this equation dA dt is equal to 1 by i h cross and the commutation between the operator A and the Hamiltonian. If the operator is explicitly dependent on time then we have to take that also into account.

So, we have to add this term. However, for simplicity let us assume that there is no explicit time dependence and therefore we are going to ignore this particular term here. The Heisenberg equation is the result of the so-called Heisenberg representation which is also known as the Heisenberg picture. It is called Heisenberg representation or picture. Let me actually digress a bit and briefly explain this Heisenberg representation to you. All you are taught about is the Schrodinger equation in your basic quantum mechanics course.

You know that the time evolution of a quantum state is described by the Schrodinger equation that is i h cross del psi of del t is equal to H psi of t. This is the Schrodinger equation and from here we know that we can find out how this state vector evolves in time. And we already talked about it. This is the so-called evolution operator and it tells us how the wave function evolves in time.

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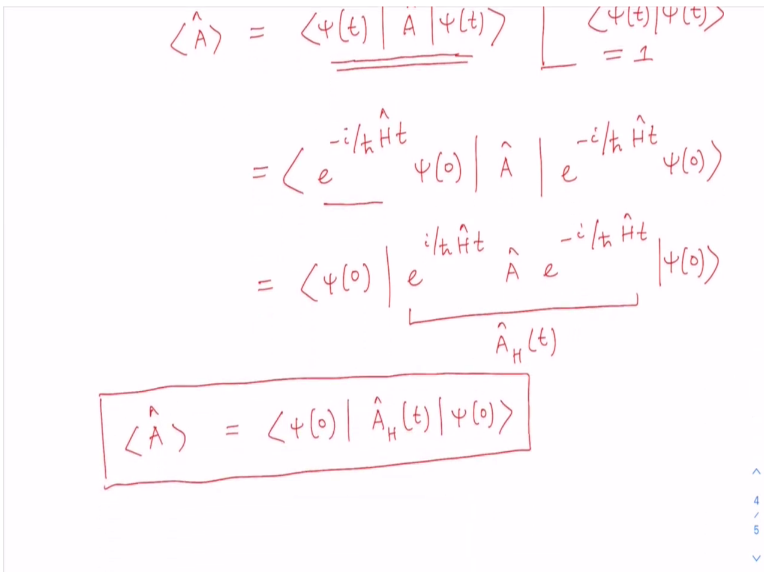

In quantum mechanics the expectation value of an operator with respect to a state vector which gives us the average value of a physically observable quantity is found by working out this relation. So, expectation value of the operator A with respect to the state vector psi of t which is let us say it is normalized that means that the scalar product is equal to 1 here. Here you see that the operator A is time independent while the time dependency is in the state vector psi of t.

We can write using the expression for the time evolution of the state vector this relation we can write down the expectation value of this operator slightly differently. What we can do we can write it as - i by h cross, we are just applying this here, we have H t psi of 0 a and here I have again e to the power - i by h cross H of t psi of 0. If we now do it this way if I take this part to the other side then I will have here e to the power i by h cross H of t a e to the power - i by h cross H t psi of 0 and now if I say that this is my new operator, if I define the new operator A H which is as you can see the time dependency is now put into the operator.

While the wave function is now becoming time independent then the expectation value of the operator A can be worked out by using this relation. And in fact, nothing is getting changed only the interpretation is changing. Whether you work it in the Heisenberg picture or in the Schrodinger picture this way of representation where the wave function is dependent on time and the operator is time independent is known as the Schrodinger picture.

And it actually depends which particular situation you are facing or the problem at your disposal depending on that you can either use the Schrodinger picture or the Heisenberg picture.

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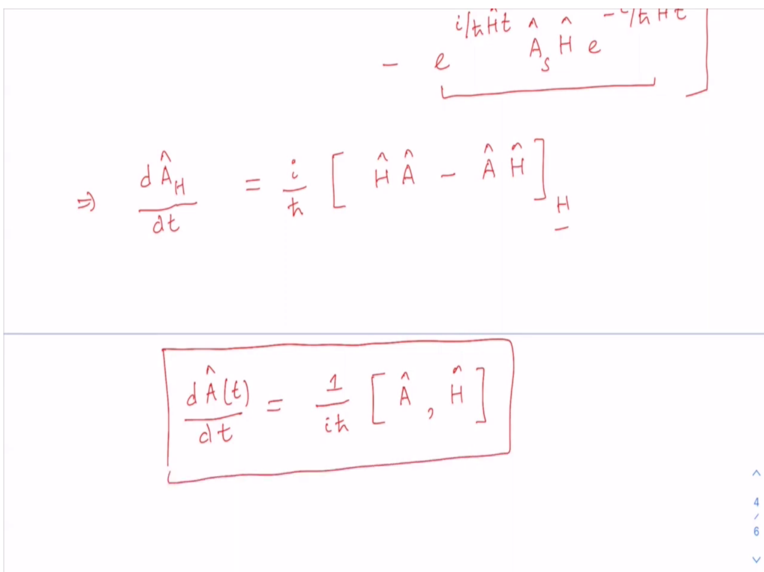

Now let me use this Heisenberg operator or the type dependent operator and let me show you how to derive the Heisenberg equation. So, let me first write the operator. So, this is my Schrodinger operator now and which is time independent. As time independent and we have this time dependencies included here. Now if I take the type derivative on both sides it is very simple and straightforward but let me show you how to do it. So, if I take the time derivative on both sides, I will have it i by h cross H e to the power i by h cross H t A. So, here I will have e to the power - i h cross H t and then I will have another term e to the power i by h cross H t As. This is time independent. So, now derivative if I take here I will have - i by h cross h e to the power - i by h cross H t. This I can write as i by h cross e to the power i by h cross H t. This I can now write H A e to the power - i by h cross H t and I have e to the power i by h cross H t. This is As okay.

I have here H e to the power - i by h cross H t. So, what you see that the product of this two operator is now in the Heisenberg representation because of this factor here. As you can see in the Heisenberg representation the operator is written in this form. So, similarly is the case here. So, therefore I can write it as i by h cross here H of A minus A of H and let me put H here just to emphasize that here I am writing these things in the Heisenberg representation and I have this equation dA H dt is equal to this one.

In general, we do not write the suffix x. So, we can simply write it as dA of t dt is equal to 1 by i h cross A H, commutation between A and H. So, this is the Heisenberg equation.

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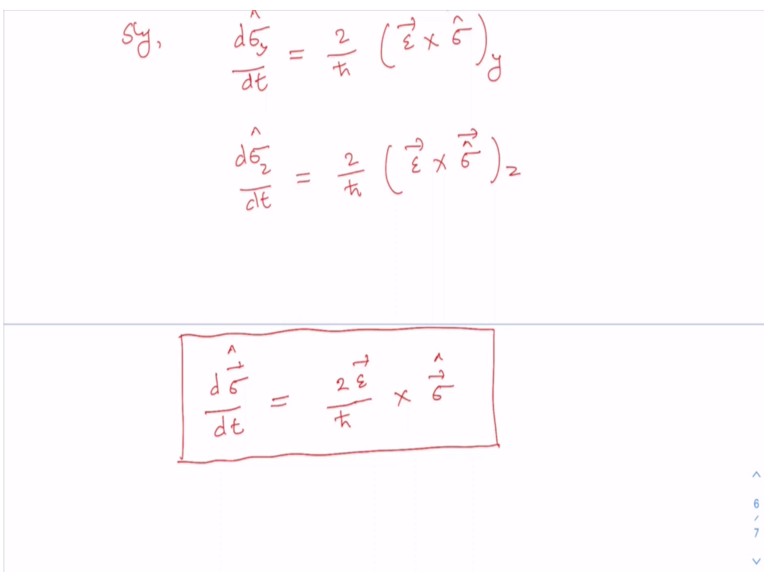

Let us now apply this equation to the Pauli matrices. For example, let me first do it for the x component of the Pauli matrix say for sigma x, then this Heisenberg equation says we will have d sigma x of dt is equal to 1 by i h cross sigma x and the Hamiltonian. Now here this

Hamiltonian we know for the two-level system, it is equal to, let me write here, this is epsilon dot sigma, let me write open it up, epsilon x sigma x + epsilon y sigma y + epsilon z sigma z. Okay let us work it out.

So, we will have 1 by i h cross or let me take it this side i h cross to the other side then I have this one and I will have relations sigma x sigma x epsilon x epsilon y sigma x sigma y + epsilon z sigma x sigma z. Now this is obviously 0. So, we will have epsilon y the commutation between sigma x and sigma y already we know that was 2 i of sigma z and sigma x sigma z we have minus 2i epsilon z sigma y. I can write d sigma x of dt is equal to I have 2 by h cross.

Let me write here epsilon y sigma z minus epsilon z sigma y. This I can write little bit cleverly as 2 by h cross. It is basically the cross product of the vector epsilon and sigma, the x component, if you now you recall that a cross b if I take the x component of it then I have here a y b z minus a z b y. So similarly, you can see that this is nothing but the x component of this cross product epsilon cross sigma okay.

Similarly, we can get d sigma y dt is equal to 2 by h cross epsilon cross sigma the y component and d sigma z of dt is equal to 2 by h cross epsilon cross sigma the z component. So, now we can combine all these results and we can immediately write an equation for the time evolution of the Pauli vector sigma and that would be d sigma dt is equal to 2 by h cross the epsilon vector into the Pauli vector okay.

So, this gives us the time evolution of the Pauli matrix and this equation should remind you about the gyroscope. You may have studied in your classical mechanics course.

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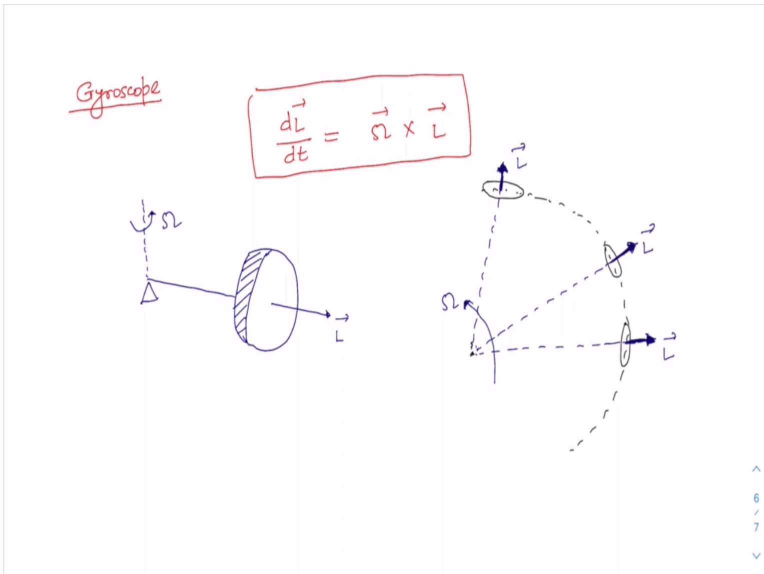

The case of gyroscope, you may have encountered this equation dL dt is equal to omega cross L where L is the angular momentum directed along the axial and omega is the angular velocity of the axial about the vertical. Actually, L processes around omega with angular velocity modulus of omega. So, as L precesses around this vector omega with angular frequency modulus of omega vector. In fact, in the similar line we can interpret this d sigma dt is equal to 2 E by 2 epsilon by h cross into this Pauli vector.

That this Pauli vector precesses around this epsilon vector with angular frequency angular frequency 2 E by h cross. Actually, to be rigorously correct as you can see this is an operator so we cannot plot an operator as such but what we can do we can take the expectation value of the operator and then we can plot it graphically. I can show you. let us say our epsilon vector is directed along this direction.

And this Pauli vector is directed along this direction and we have to take the expectation value of this Pauli vector and then it processes around this epsilon vector in this way. So, this is x-axis this is y-axis. The expectation value of the Pauli vector is also known as the block vector because it completely characterizes a two-level system. In fact, let me once again repeat that if we have a 2 by 2 matrix say a11, a12, a21, a22 this can always be expressed in terms of the 3 Pauli matrices and the identity matrix.

This is why knowing the Pauli vector gives us the complete knowledge about a two-level system. The block vector generally defined in the context of quantum information science is discussed in a supplementary lecture. Now let us work out the eigenvalues and eigenvector of our two-state system which is H is equal to epsilon dot sigma vector. let us work out the eigenvalues.

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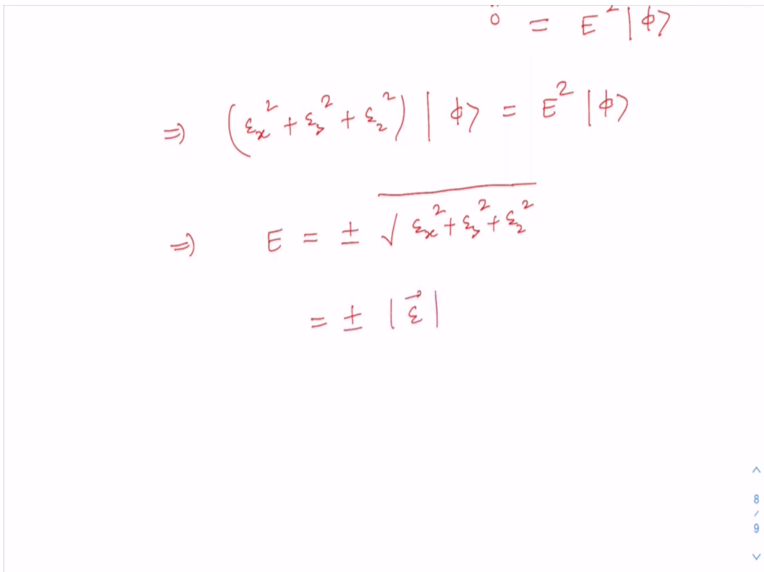

Eigenvalues of this Hamiltonian. Before that let me consider a simple case where say, epsilon vector has only z component only. In that case this Hamiltonian H would be simply become epsilon, 0, 0, minus epsilon. That means there is no coupling between the energy levels and clearly the eigenvalues of the Hamiltonian are here E is equal to + - epsilon. And the eigenvector corresponding to this value epsilon would be, as you can see, it would be simply 1, 0.

And eigenvector corresponding to the eigenvalue minus epsilon is 0, 1. In the general case, it can be shown that the eigenvalue would be given by E is equal to plus minus modulus of this epsilon vector. Let me show you the proof. We have this eigenvalue equation that x is equal to say this is the eigenstate and eigenvalue is E and we have this ket phi and let me first write it as epsilon dot sigma. It is applied on the eigenvector phi ket phi.

And we have this E phi. Applying the operator x again on both sides we will get it epsilon dot sigma square ket phi is equal to E square ket phi. Now if I open it open this up then I will get let me write the complete term you will get epsilon x square sigma x square + epsilon y square sigma y square epsilon z square sigma z square then we will have epsilon x epsilon y into sigma x sigma y + sigma y sigma x.

Then, I have say epsilon y epsilon z sigma y sigma z + sigma z sigma y. All these are operators matrices, Pauli matrices and I have finally epsilon z epsilon x sigma z sigma x + sigma x sigma z. Now then this is applied on ket phi and we have E square ket phi. Already we know that this is anti-commutation relation between sigma z and sigma x. Similarly, this and all these terms you know they anti commute.

So, this would be 0. On the other hand, sigma x square sigma y square sigma z square identity. So, we'll be left out with epsilon x square + epsilon y square + epsilon z square applied on this ket phi that would be equal to E square ket phi. So, you can see that E is equal to + minus square root of epsilon x square + epsilon y square + epsilon z square which I can write as plus minus modulus of the epsilon vector. So, that is how we can prove it.

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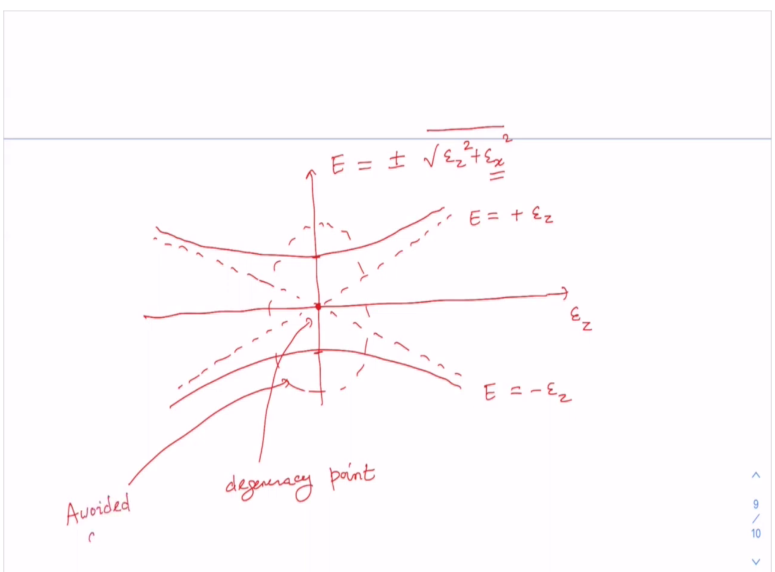

Now let us consider a two-level system with the Hamiltonian like this say we have now epsilon z minus epsilon z and now we have non-zero off diagonal elements epsilon x here. Let me consider epsilon y to be 0 and then eigenvalues of this Hamiltonian would be plus minus square root of epsilon x square + epsilon z square. Let us assume that the coupling field or the parameter epsilon x is fixed say epsilon x is fixed and the parameter epsilon z is controllable or variable.

We can control it or we can vary it or it is a variable parameter. Let me plot the eigenvalue E as a function of this controllable parameter epsilon z and we will get something interesting here. If I plot the eigenvalue E as a function of the controllable parameter epsilon z keeping epsilon x fixed. So, eigenvalue is epsilon z square + epsilon x square. Now when this epsilon z is equal to 0 you see E is equal to simply plus minus E x.

So, say this is + E x this is minus E x. Again, if epsilon x is 0, this particular point, then you will have E is equal to + minus epsilon z. So, for E is equal to + epsilon z I will get this positive slope. So, this is for E is equal to, in the asymptotic limit, you will have + epsilon z here and on the other hand when epsilon x is 0 then here we will have E is equal to minus epsilon z and this means that these two energy levels are crossing each other at this particular point at this particular point that is the reason it is called a degeneracy point.

This is called the degeneracy point. But when epsilon x is non-zero this is non zero then these two energy levels will no longer cross and we will have a curve like this and this will correspond to the plus square root of epsilon z square + epsilon x square and another curve will get like this and this two energy levels are no longer crossing and this particular area where this crossing is avoided is known as the avoided crossing. This is called avoided crossing.

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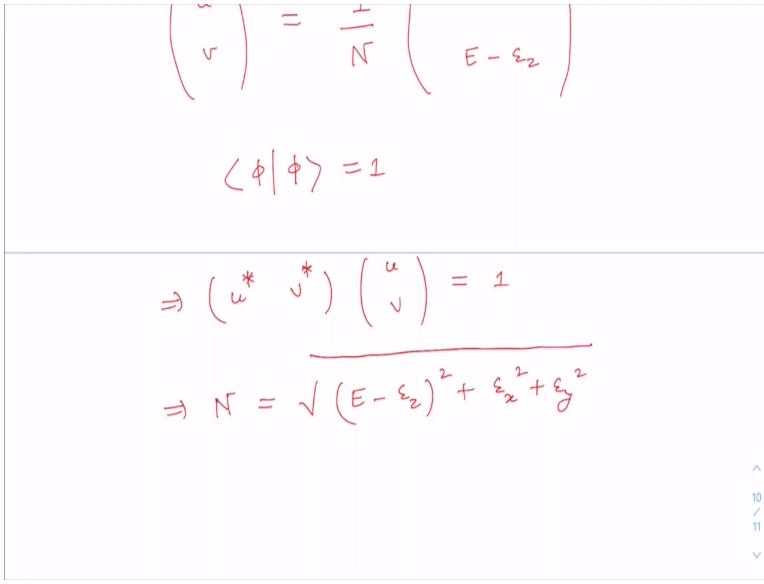

What about the eigenvectors? let us work it out. The eigenvalue equation we have is H ket phi is equal to E ket phi. As we are in 2-dimensional Hilbert space we can write ket phi as a column vector with two elements u and v. Our goal would be to find the elements u and v. Now from the eigenvalue equation, I have this eigenvalue equation; epsilon z minus epsilon z epsilon x + i epsilon y epsilon x - i epsilon y into u v is equal to E u, v.

And from here one can easily get epsilon z u + epsilon x - i epsilon y into v is equal to E u and from this equation, this equation I can write as epsilon z minus E into u + epsilon x - i epsilon y v is equal to 0. So, we can write u and v. If I take u is equal to say epsilon x - i epsilon y, this if I take to satisfy this equation I must take v as E minus epsilon z and it has to be normalized. So, let us say N is the normalization parameter and because of the fact that this ket state is normalized which means that we have u star v star. This is the row matrix and complex conjugate you get and then you have u, v.

This has to be satisfied and from here you can immediately get that the normalization parameter would turn out to be square root of E minus epsilon z whole square + epsilon x square + epsilon y square.

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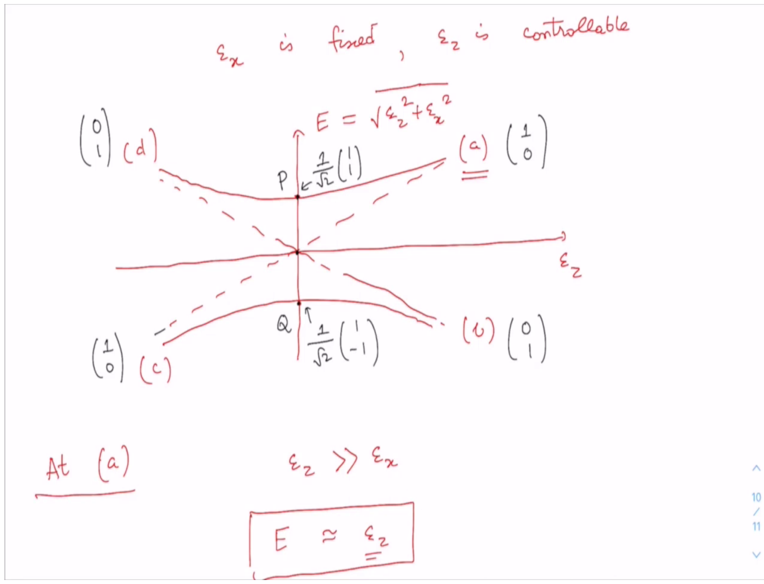

To illustrate let us again consider the Hamiltonian H is equal to epsilon z, minus epsilon z, epsilon x, epsilon x, and epsilon x let me consider it to be fixed say, epsilon x is fixed as I considered earlier and epsilon z is a controllable parameter. Then we obtain the E versus epsilon z plot like this as we discussed a short while ago. Epsilon E versus epsilon z. So, we had this plot here. So, when epsilon x is non- zero they do not cross. We get this avoided crossing.

Now let us find the eigenvectors at asymptotic limits that means say at the point a, at the point b here, at the point c, and at the point d. So, to do that let me first show you how to get it at the point a. So, at point a, at the asymptotic limit actually, at a in the asymptotic limit, when I say asymptotic limit it means that here I have epsilon z is much greater than epsilon x at the point a and therefore I can take E to be nearly equal to epsilon z. You can see from the eigenvalue equation; eigenvalue E is equal to square root of epsilon z square + epsilon x square. Now epsilon z is much larger than epsilon x.

So, I can consider in the asymptotic limit capital E to be equal to epsilon z and from this eigenvector equation we have u, v is equal to 1 by square root of, let me first write the normalization parameter here; E minus epsilon z square + epsilon x square, and here I have epsilon x E minus epsilon z. Now, this is what we have, but now at point a I have u, v, capital E is equal to epsilon z, and therefore I will get 1 by epsilon x here and here it would be epsilon x and it would be 0. So, therefore it would be simply 1, 0. So, this particular point simply refers to the eigenvector 1, 0 and because this point c lies on the same line this point c will also correspond to the eigenvector 1, 0. In fact you can do by rigorous calculation. These calculations are simple. You can so that the point c will also correspond to 1 0.

On the other hand, this point b will correspond to 0, 1 and by similar logic point d will correspond to 0, 1 okay. What about this point p and this point q when epsilon at epsilon z is equal to 0. You can do the calculation and you can show that the point p corresponds to the eigenvector 1 by root 2 1, 1 while this point q corresponds to eigenvector 1 by root 2 1, -1. Let me stop here for today.

In this lecture we discussed a generic 2-state system, we have written down the Hamiltonian and work out the energy eigenvalues and the eigenvectors. However, the system that we consider is not externally driven. This issue we are going to take up in the next class. So, see you in the next class, thank you..