Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture – 27
Quantum Master Equation.

Hello, welcome to the lecture 20 of the course and lecture 10 of module 2. In this lecture we will discuss the so-called quantum Lindblad master equation and we will see how quantum dissipation can be taken into account using this approach.

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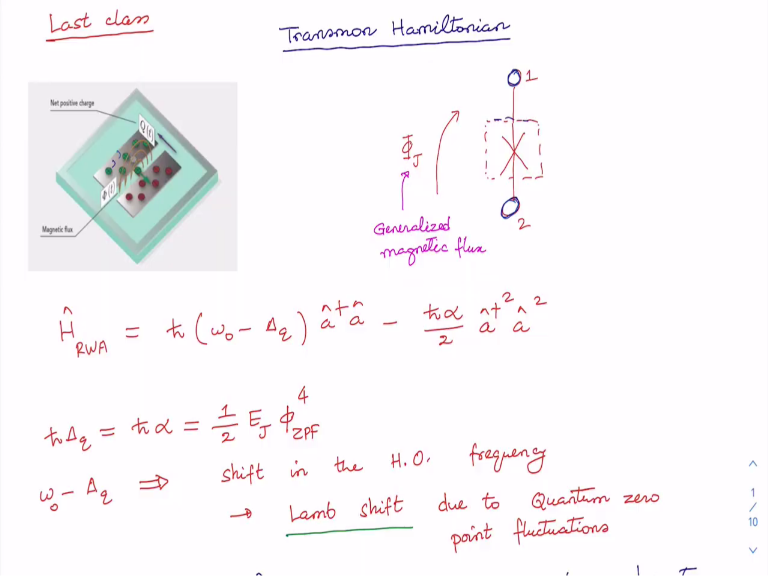

Last class

Transmon Hamiltonian

$$\hat{H}_{RWA} = \hbar(\omega_0 - \Delta_q)\hat{a}^{\dagger}\hat{a} - \frac{\hbar\alpha}{2}\hat{a}^{\dagger 2}\hat{a}^{2}$$

$$\hbar\Delta_q = \hbar\alpha = \frac{1}{2}E_J\phi_{ZPF}^{4}$$

$\omega_0 - \Delta_q \Rightarrow$ shift in the H.O. frequency

→ Lamb shift due to Quantum zero point fluctuations

So, let us begin. In the last class we continued our discussion on transmon Hamiltonian from the previous one. So, under the rotating wave approximation the Hamiltonian for the transmon was written in this particular form where h delta q is equal to h cross alpha and it is related to the fluctuation, zero-point fluctuation and the Josephson energy in fact this first term, it basically gives you the shift of the linear harmonic oscillator frequency and this is known as the Lamb shift.

And on the other hand, this particular term is coming due to the non-linearity of the Josephson junction. So, this equation can be, this Hamiltonian can be recast in terms of the number operator and using the number operator we can rewrite the Hamiltonian in this particular form and as you can see from this Hamiltonian that there are 2 terms one is linear term.

Linear in the number operator N and another one is nonlinear term and it is quadratic in N and here omega q is basically the shifted frequency of the harmonic oscillator nonlinear harmonic oscillator. And the energy of the Hamiltonian in the Fock state energy basis state is simply it can be very easily worked out using this eigenvalue equation. Because we know that this is what the Fock state represents. This is the eigenvalue equation for the number operator.

So, using this what we can do we saw that this energy level of the harmonic oscillator is no longer equally spaced because it is now no longer linear harmonic oscillator rather it is a nonlinear harmonic oscillator. As you can see from this expression very easily we saw that difference between the first excited state and the ground state is h cross omega q. On the other hand, the second excited state and the first excited state has an energy difference and it is not equally spaced it is h cross omega q minus h cross alpha.

And in fact, this particular Hamiltonian is experimentally realized and for that these certain parameters for example this capacitance was used as in lab it was taken to be 65 femto Ferat while the inductance was taken to be around 14 nano Henry, these are very small parameter. And the corresponding Josephson energy was 12 gigahertz and the charging energy was only 0.3 gigahertz.

And this omega q turns out to be 5 gigahertz while this parameter alpha turns out to be 0.3 gigahertz. And as you can see for these particular experimental parameters the difference between the ground state and the first excited state is 5 gigahertz. On the other hand, the first sorry, this would be 2, the second excited state and the first excited energy difference is 4.7 gigahertz.

So, energy levels are no longer equally spaced. So, we are able to have anharmonicity. Anharmonicity is introduced now and this is as we discussed in the last class that this is very critical because in that case we can just consider, we can utilize it as a qubit and it is very useful for quantum information processing kind of exploitation. And also, it turns out that there are quantum fluctuations, the fluctuation in the flux and the fluctuation in the charges.

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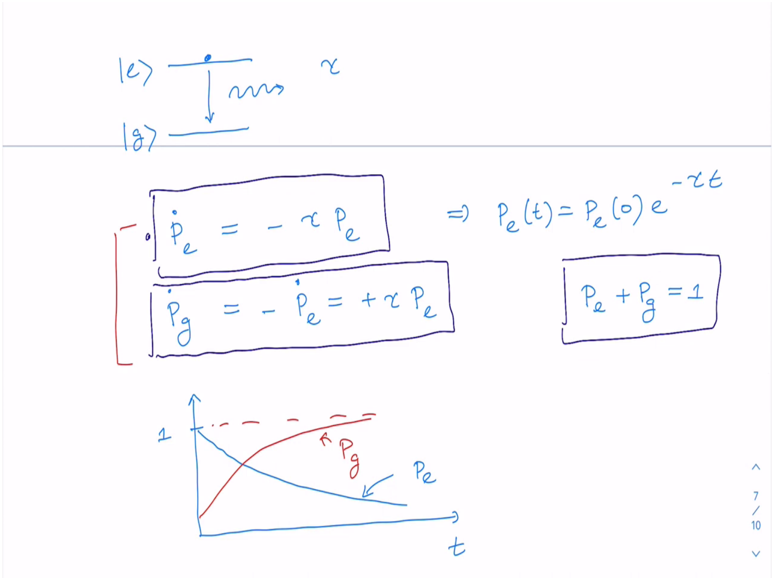

$$\boxed{\dot{P}_e = -\tau P_e} \Rightarrow P_e(t) = P_e(0)\, e^{-\tau t}$$

$$\boxed{\dot{P}_g = -\dot{P}_e = +\tau P_e}$$

$$\boxed{P_e + P_g = 1}$$

Fluctuation in the flux is half quantum, actually flux quantum that is 0.5. On the other hand, charge fluctuation is nearly one Cooper pair and what it physically means that in the ground state there is about one Copper pair worth of charge fluctuating back and forth between the 2 metal plates. On the other hand, we know that there are around 10 to the power 12 electrons per metallic plate or superconducting island are there.

So, therefore this fluctuation of one Cooper pair box says that we have to keep the transport at very low temperature and actually this leads to the next topic that we discussed in the last class that is the so-called quantum dissipation or quantum noise. And this is a very important problem in quantum physics or even quantum technological purposes.

For example, we find that the so-called spontaneous decay of atom is one example of quantum dissipation, then photon leaking away from a cavity and then the decay of the superposition state. These are examples, some examples of quantum noise or quantum dissipation. Then we went on to show that the so-called classical treatment the way we take dissipation into account for example in damped harmonic oscillator the way we model it, it cannot be utilized for the quantum case.

Because we land up into the issue where the uncertainty itself the uncertainty itself appears to be decaying. So, which is not the case or in other words you can also say that the commutation relation is no longer preserved if we extend the classical approach to the quantum approach. So, therefore we have to add up different approaches. And we discuss as an example of a decay of an atom and how to model it.

We took the example of spontaneous decay of an atom which we have just taken a 2 level atomic system and it is interacting with the surrounding or the vacuum. The vacuum can be considered to be a collection of independent harmonic oscillator or it is basically termed as bath. And the interaction between the atom and the bath is given by this operator sigma x F where sigma x introduces transition between the ground state and the excited state.

While this operator F represents basically the electric field which can be or any perturbation. In this case we can say that it is the electric field and we know the electric field is quantized and it can be represented in terms of this a and a dagger operator in normal modes we can write it this way where g k is the coupling interaction coupling between the bath and the field, bath and the atom actually right.

And using RWA this Hamiltonian can be exactly solved and it was done by Weisscoff and Wigner and this is known as the Weisscoff-Wigner solution and the idea was to find out the probability of getting the atom in the excited state at an arbitrary time t considering initially the atom to be in the excited state and the field to be in the vacuum. Then this probability of finding the atom in the excited versus time was plotted and what it turns out that except for very very short time and very long time, mostly it behaves exponential decay behaviour is there.

And this exponential decay behaviour can be very easily taken into account by very simple model where we write these 2 equations the rate of change of the probability of getting the atom in the excited state it decays like this by this expression and because the total probability has to be equal to unity. So, corresponding equation rate of change of probability for the ground state can be very easily written.

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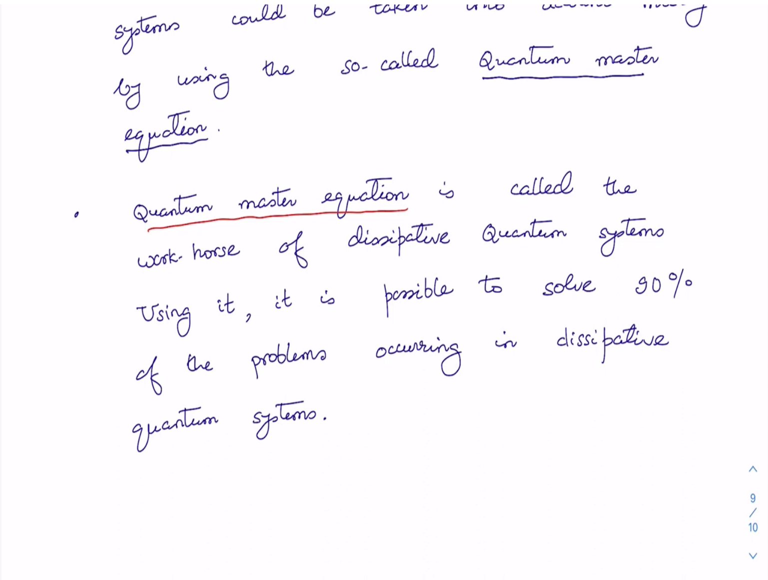
systems could be taken

by using the so-called Quantum master equation.

- Quantum master equation is called the work-horse of dissipative Quantum systems. Using it, it is possible to solve 90% of the problems occurring in dissipative quantum systems.

And if we plot it we see that exponential decay from the excited state is of course it is it should be and we recover that dominant result in the Weisscoff-Wigner solution. But now the issue was that what about you know if we start rather than the atom to be in the excited state if say the atom is initially in a superposition state like this, superposition of the ground state in the excited state while the bath is in the vacuum state.

Then this approach fails and because of the fact that there are phase relationships between the complex amplitude C g and the C e which cannot be taken into account by the wave function approach. So, we need to take the density matrix formulism and in this context I just quickly reminded about the density matrix for the pure state which is this and for the mixed state the density operator is defined like this and we have already discussed the density matrix formulism in module one.

Then the total system sometime rather than the total system we may be interested in a subsystem. For example, if we focus only on the atom then this can also be taken into consideration by using the so called reduce density matrix, then we saw how just we actually revised how again starting with the atom to be in the superposition state initially and the bath is in vacuum, how we can get the only the density operator for the atom.

Then we see that we just get this density matrix for the atomic state and off diagonal elements contain the information about the phases while the diagonal elements contains the information about the probabilities. Now it turns out that the dissipation in quantum system could be taken into account by using the so-called quantum master equation approach. This is the topic for today's lecture.

And in fact, this quantum master equation is known as the work horse for dissipative quantum systems and it is possible to take into account around 90% of the dissipative problem probably related to dissipation using this quantum master equation. So, we are now going to discuss quantum master equation. A very simple discussion rather than going to the rigoristic. I will discuss it with certain examples.

And this would be very useful for taking many issues that we occur in quantum technological technology particularly circuit quantum electrodynamics and the topic that we are going to discuss in the next module module 3 that is quantum optomechanics.

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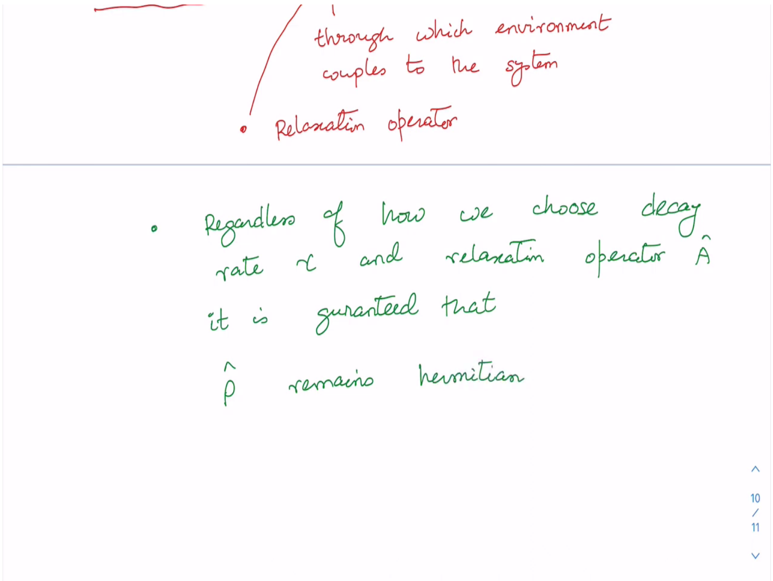

through which environment couples to the system

- Relaxation operator

- Regardless of how we choose decay rate τ and relaxation operator $\hat{A}$ it is guranteed that $\hat{\rho}$ remains hermitian

Let me introduce the quantum master equation quantum master equation which is also known as Linblad quantum master equation and this equation give us the time rate of change of the density operator. So, dot refers to the derivative with respect to time. So, basically this is what I mean time rate of change density operator it is equal to 1 by i h cross the Hamiltonian of the system commutation relation between the Hamiltonian and the density operator.

This is one part of the master equation then the other part is this which I am going to explain. So, there is an operator called Linblad operator L, which is function of some arbitrary operator say A j and it operates on the density operator and then it these gamma are the so called the decay rates. Anyway I will explain it. So, this part actually it refers to the coherent evolution of the system and this can be very easily worked out using the Schrodinger equation.

This refers to the coherent evolution of the system. It is also called Von Neumann equation this part along with this one and this second part takes dissipation into account because gamma are the dissipation rate gamma or the decay rate which I am going to discuss very soon. I will explain this particular part dissipation is this refers to the dissipation dissipating part of the equation.

Now please note that in this equation each part in time, the time derivative is given by the present state and there is no memory and this is why this master equation is more aptly termed as Linblad Markovian quantum Master Equation. So, let me put here Linblad Markovian quantum master equation or generally it refers to as the Linblad quantum master equation but appropriately it is a basically refers to a Markovian process where we do not bother about the history.

So, what it says is that at each point in time the time derivative as I said is given by the present state and there is no memory. The Linblad operator this operator L which is the Linblad operator and it also because it operates on the density operator this is also called super operator. This is also sometimes referred to as super operator or simply the Linblad operator and it has the following structure let me explain.

So, Linblad operator it is a function of some arbitrary operator A and it operates on the density operator rho and this is it has this structure the whole thing is the structure. So, the density operator it is getting operated by this arbitrary operator A and again it is operated. So, from right hand side this way from right a dagger and so basically this is the structure A dagger a rho and minus half rho a dagger A.

So, this is the structure of the Linblad here Linblad or this whole thing has this particular structure this operator arbitrary operator A is actually it is the operator through which through which the system actually gets coupled to the environment or through which environment we will see some examples soon environment couples to the system couples to the system.

In fact, sometimes this is also referred to as the so-called relaxation operator. So, this is also referred to as relaxation operator relaxation operator and this specific structure of the Linblad operator this particular one of the master equation is extremely important because it is the only form. So, this is the only form because of which you know it ensures actually it makes sure that regardless of how we choose the decay rate.

Let me write here regardless of how we choose decay rate gamma and relaxation operator and relaxation operator A. In fact, this can be proved which will not do here it is guaranteed this is guaranteed that the density operator rho remains a Hermitian. We know the density operator has to be Hermitian.

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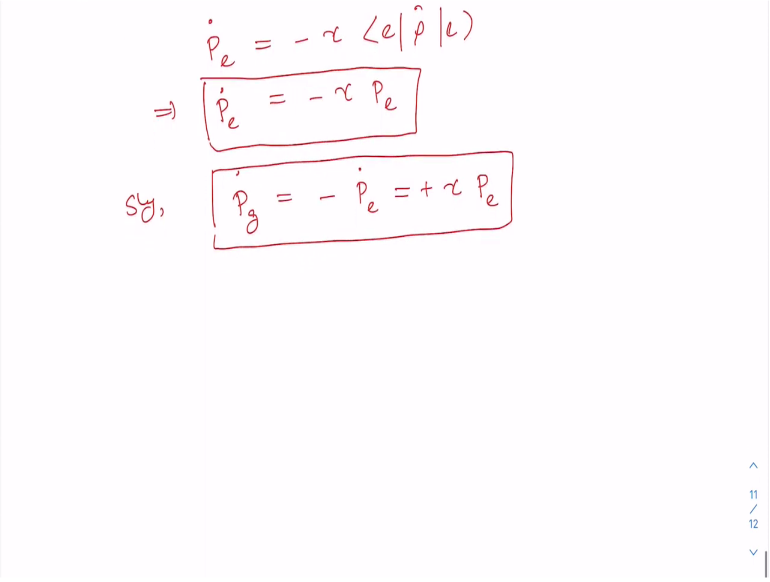

$$\dot{P}_e = -\gamma \langle e|\hat{\rho}|e\rangle$$

$$\Rightarrow \boxed{\dot{P}_e = -\gamma P_e}$$

Sly, $$\boxed{\dot{P}_g = -\dot{P}_e = +\gamma P_e}$$

So, this is the only structure. If you pick up any other structure then this is not going to be the case and also what it ensures is that the trace of rho is maintained, it is equal to one and also what it ensures is that rho remains semi-positive definite that means the diagonal elements of the density operator is always positive it cannot be negative.

Now let me give some examples to illustrate the power of this quantum master equation. Firstly, let us consider 2 level system relaxing to the ground state via spontaneous emission this is already familiar to us, 2 level system relaxing via spontaneous emission. So, we have this excited state and then we have this ground state here and the atom is relaxing from the excited state to the ground state with the rate say gamma by emitting a photon.

So, the decay rate is given to us. So, in order to apply the master equation, we first need to find out the relaxation operator A and in fact this can be rigorously calculated but it is in this case it is very easy to guess. So, it would be like this, so we are going from the excited state to the ground state. So, this is our relaxation operator or it is basically our sigma minus, atomic lowering operator. The atom is going from the excited state to the ground state.

Let us see whether our guess is correct or not then the master equation. Let me write the master equation once again. So, we have rho dot is equal to 1 by i h cross, H rho, okay. Because here only we have only one decay process is involved. So, in the summation sign only one term is there. So, decay rate is here gamma and the form of the master equation as we have already seen that this is rho.

And then this is we have A, this is A A dagger and I have minus half rho A dagger A then minus A dagger A rho okay. So, this is the master equation but here we know what is rho okay. So, let me open it up just by putting it A operator which we have already guessed. So, let us utilize it. So, I have 1 by i h cross h rho + gamma A is I have g e rho and A dagger would be I would have here e and g this is the A dagger part and this is my A this is A this is A dagger.

And then I have the other terms minus half rho A dagger is e g here and A is ket g bra e - half A dagger is e g and A is ket g e and then it is we have rho. So, this is what the master equation we write. Now let us look at the diagonal entries of the density matrix first and see how it gets evolved in time. So, basically what it gives the diagonal element for example this particular term gives the probability of getting the atom in the excited state and the time rate of change of this is going to give the time rate of change of the probability.

So, we can immediately very easily work it out. So, what we will have we will have 1 by i h cross actually this term okay. I will illustrate this term first term will go to 0 because this Hamiltonian is diagonal in the e g basis. I will explain it first let me write it down. So, I am taking the multiplying this from one side, let me do it. So, I have here e g e rho e g and here it would be e.

Of course, this term is going to be zero because e and g are orthogonal and then what about the other term that is minus gamma by 2 e rho e and this is identity, normalized. So, I am left out with e and multiply e okay this is also identity this is also 1 this is 1 and then I have here half I have e e ok this is again identity and this is also identity. So, therefore I will have e rho e.

So, what I will be left out this implies that I have time rate of change of probability for the excited state it would be 1 by i h cross e H rho e and I will be left out with minus I will have okay here it is gamma. So, this term and this term combining these 2 terms I will get minus gamma e rho e. Now this term I claim that this is equal to 0 because you can easily see that e h rho okay this is equal to e h rho e minus e rho h e.

Now this Hamiltonian you know, this is the eigen state. So, let us say our this 2 level system this difference is say h cross omega 0 and this has say 0 ground state has zero energy and the excited state has h cross omega. So, therefore what we will have we'll have h cross omega 0 that is the energy eigenvalue. So, this is what we will have and because of this I can see that H this I think already you can make it out that the first term is going to give me h cross omega 0 e rho e.

And then similarly I will have h cross omega 0 e rho e. So, this is equal to 0. So, therefore we will be left out with only that P e dot is equal to minus gamma gamma e rho e which is nothing but the probability of getting the atom in the excited state. So, this is what I will have. Similarly, you can get P g dot you can get it very easily it would be minus P e dot in fact what you will get is plus gamma P e.

This is what you will get. In fact, if you recall that these are the same exactly the same set of equations that we got earlier by using our simple model.

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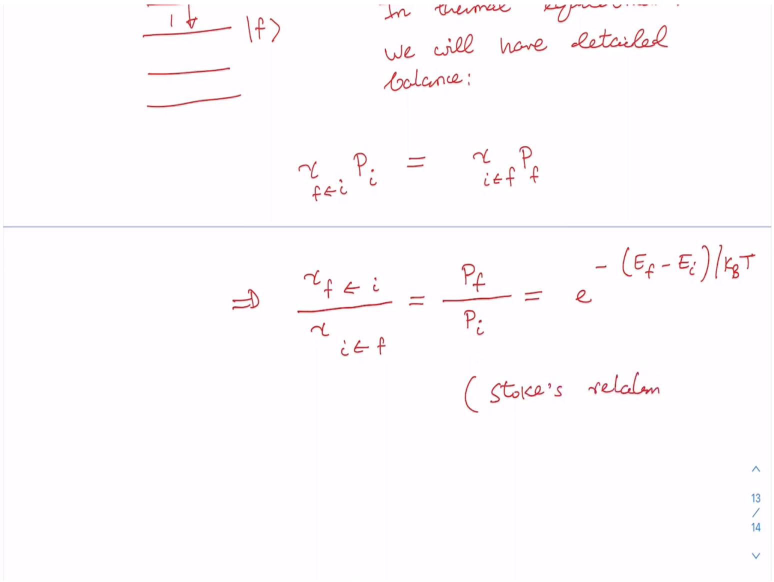

$|f\rangle$

In thermal equilibrium

we will have detailed balance:

$$r_{f\leftarrow i} P_i = r_{i\leftarrow f} P_f$$

$$\Rightarrow \frac{r_{f\leftarrow i}}{r_{i\leftarrow f}} = \frac{P_f}{P_i} = e^{-(E_f - E_i)/k_BT}$$

(Stoke's relation

But this master equation gives us something extra for example what about the coherence? What about what about coherences? So, let us work it out for example coherence term is say rho e g this is one of the coherence terms. So, let us see how it time evolution of this coherence term. So, again if you put it in the master equation so you will have 1 by i h cross here you have to multiply it by g H rho e.

This term is again you can show it to be zero and then you will have just let me okay you have to write it just let me write down the terms. So, the second term you will have gamma and you are multiplying this g here g e rho e and you will have g e of course this term is going to be 0 then you will have minus gamma by 2 g rho e. Then you will have g g e e and you will have gamma by 2 g e g g e. I think you will get e rho e this is what you will get and again this term is going to be zero this is identity one this is one.

So, what will be left out. This means that this rate of change of this particular coherence term would be equal to minus gamma by 2 g rho e. Only this particular term will survive. So, what it says is this that the off diagonal elements which is, this is actually the decoherence this is the decoherence, off diagonal elements off diagonal elements decay.

And decay at the rate gamma by 2. So, you see this model the quantum master equation gives us the information about the evolution of the coherences also which our simple model was unable to give. Now let us consider the case where suppose there is some thermal excitation is there earlier this the example that I have given was for spontaneous decay what happens if there are some thermal excitation. Suppose the atom is excited from the ground state to the excited state.

And the excitation rate is say gamma plus okay then how we can take this into account in our master equation. So, here again this rate is given that is gamma plus we have to find out. So, here I am doing thermal excitation, thermal excitation of the 2-level system. So, here this decay or the relaxation operator I can again guess that because I am now going from the ground state to the excited state okay.

So, let me take it as my relaxation operator and again by the same kind of prescription please do it yourself you can do it very easily by following the same prescription that the rate of change of the probability of the excited state you will get it to be plus gamma plus here okay that is the decay that is the rate relaxation rate. So, this is equal to P g which is the probability of the ground state that means probability of getting the atom in the ground state, by this I means rho g g.

So, this is what you will get and also you will get that the coherences, this coherence term will decay at the rate of gamma plus by 2. So, this is interesting because what it basically says is that off diagonal elements this implies that the off diagonal elements off diagonal elements still decay though there is excitation but whenever whether this excitation or de-excitation it does not matter off diagonal elements decay whenever there is some kind of relaxation processes, relaxation may be upward or in the downward direction.

Now in the most general case in fact that can also be taken into account in the most general case for example say a multi-level system, if I consider a multi-level system rather than a 2-level system like say we have various energy levels okay like this and say we want to grow from go from one particular state say ket i to some other state say get f initial step to the final state or from f to i.

Then this also can be taken into account and then in that case you can guess that the relaxation operator. So, we are going from say initial state to the final state. So, the idea would be to take the operator would be of this type. Now in thermal equilibrium what we are going to have is the so-called detailed balance. In thermal equilibrium in thermal equilibrium where thermal equilibrium will be reached when the upward transition rate of upward transition would be equal to the rate of downward transition this is what is called detailed balance.

We will have will have detailed balance what I mean by this is this say flow from the upper level suppose initially the probability that we are in the upper level order initial state is Pi and so flow from the upper level to the lower level. So, we are going from this initial state to the final state that has to be balanced by suppose now we are in the downward state f and then it has to be it would be the rate with which we are going from f to i.

So, this is how this has to be equal and this means that there would be this ratios, you are going from i to f this decay rate ratio of this decay from going from i to f or going from f to i is equal to the ratio of the probability of getting the state in the state f by P i right. And in thermal equilibrium this is equal to e to the power minus from statistical physics we know that is E f - E i by K B T where KB is the Boltzmann's constant T is the temperature and in fact this is a very useful expression this is also called Stoke's relation.

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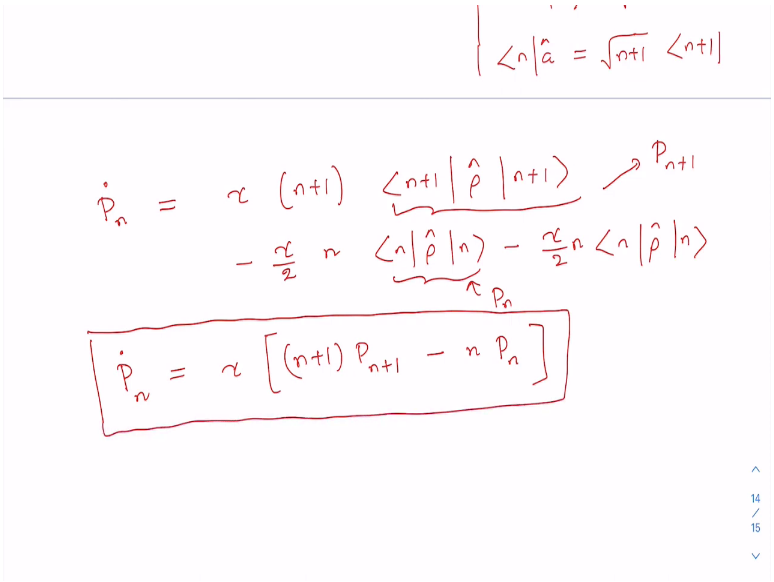

$$\langle n|\hat{a} = \sqrt{n+1}\,\langle n+1|$$

$$\dot{P}_n = \gamma\,(n+1)\,\underbrace{\langle n+1|\hat{\rho}|n+1\rangle}_{} \rightarrow P_{n+1}$$

$$-\frac{\gamma}{2}\,n\,\underbrace{\langle n|\hat{\rho}|n\rangle}_{P_n} - \frac{\gamma}{2}n\,\langle n|\hat{\rho}|n\rangle$$

$$\boxed{\dot{P}_n = \gamma\left[(n+1)P_{n+1} - n\,P_n\right]}$$

Now let us discuss the case of damped harmonic oscillator. Earlier we sorted the classical treatment cannot be extended to the case of linear harmonic oscillator with damping for the moment let us consider the damped harmonic oscillator to be at zero temperature at say T is equal to 0 because if we take the harmonic oscillator to be at zero temperature we just need to consider only one kind of relaxation process that would be the downward transition.

Suppose the photon is in the Fock stage say n + 1 then it will go in the downward direction only. So, they will try to relax to the ground state and in this case it is easy to guess the relaxation operator. So, that would be say A is equal to a that is the annihilation operator and the rate is happening at gamma. And the Linblad master equation in this case is would be we can write it as rate of change of the density operator would be equal to 1 by i h cross the Hamiltonian of the harmonic oscillator H rho plus gamma that is the rate decay rate a rho a dagger minus half a dagger a rho minus half rho a dagger a.

Now let us see how the probability of finding n photon changes with time. Probability of finding n photons. In fact, we are interested in rate of probability of finding n photons that would be given by the probability we know it would be given by n rho n and rate of change would be given by this expression. So, this we can easily work out.

So, I have to take 1 by i h cross here n here it will be H rho this is basically the coherent evolution part and because the Hamiltonian is diagonalized in the Fock state. So, therefore this term is actually going to be 0 and will be left out with the other terms like say gamma n let me write it down a rho a dagger n. And then we will have minus gamma by 2 and a dagger a rho n and we have gamma by 2 n rho a dagger a n.

And so you recall that this relations we know from our quantum mechanics that a when operates on the fork state n will get root n n minus 1 and a dagger and it will give us square root of n + 1 then it will give the Fock state n + 1. So, what will have because the first term vanishes and also you can utilize the fact that when an annihilation operator operates on the bra here the ket bra this bra of n.

So, you will get simply square root of n plus 1 and you will have n + 1. So, you can utilize this and then it is easy to see that we are going to get P n dot would be equal to gamma n + 1 n + 1 rho n + 1 and other terms would be gamma by 2 n rho n minus gamma by 2 n rho n. So, that is what we are going to get. So, combining all this what we can write is here I think yes here another term n will be there.

So, we will then have gamma n + 1 and you see this particular term is nothing but P n + 1 probability of getting this harmonic oscillator to be in the n + 1 state and this refers to the probability P n. So, we can then have it as P n + 1 - n P n. So, this is the equation we get, time rate of change of the probability for having the harmonic oscillator in the state n will be given by this one.

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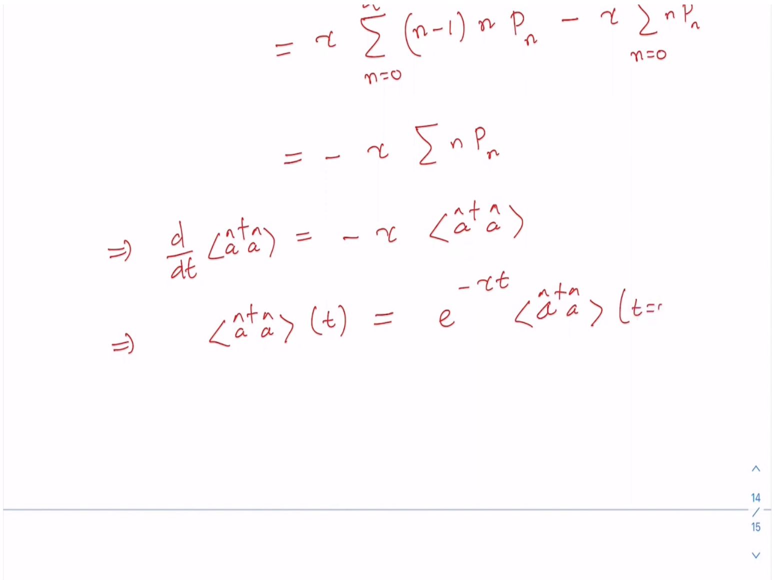

$$= \gamma \sum_{n=0}^{\infty} (n-1)\, n\, P_n - \gamma \sum_{n=0} n P_n$$

$$= - \gamma \sum n P_n$$

$$\Rightarrow \frac{d}{dt} \langle \hat{a}^{\dagger} \hat{a} \rangle = - \gamma \langle \hat{a}^{\dagger} \hat{a} \rangle$$

$$\Rightarrow \langle \hat{a}^{\dagger} \hat{a} \rangle (t) = e^{-\gamma t} \langle \hat{a}^{\dagger} \hat{a} \rangle (t=$$

Now what about the time evolution of the photon number what about time evolution of say photon number? That will be given by expectation value of this number operator. So, this we can work out easily because we know that d of dt expectation value of a dagger a is which you can write it as d of dt that would be nothing but n into P n this is the probability. So, this is the expectation value of the photon number which we can write as simply n is going from 0 to say infinity n P n dot.

And already we know the expression of P n dot. So, therefore we can write it as using that equation we can write n is equal to 0 to infinity and we will have n let me write down this P n dot is n + 1 p n + 1 - n p n. Which I can write as gamma you see what I have here n P n is there and here this suffix is n here it is n + 1 I can make it n just by changing.

Let us look at the first term. If I look at the first term then I can write it as if I replace n by n - 1 here and this would be then n and here it would be P n this is the first term and the second term I can keep as it is that would be gamma n P n, n is going from 0 to infinity I can very easily do that and if I do the math, so, you can easily see that you are going to get minus gamma n p n.

And what we see that n P n is nothing but this particular term this is expectation value of a dagger a. So, therefore I can write it as, so, therefore what I have p n dot is actually okay this is d of dt a dagger a and this is minus gamma a dagger a. So, we can easily write down this solution that the average number of say photon at in the arbitrary time t is equal to e to the power minus gamma t a degree a at time t is equal to zero. So, the average number of photons decays with time exponentially at the rate gamma.

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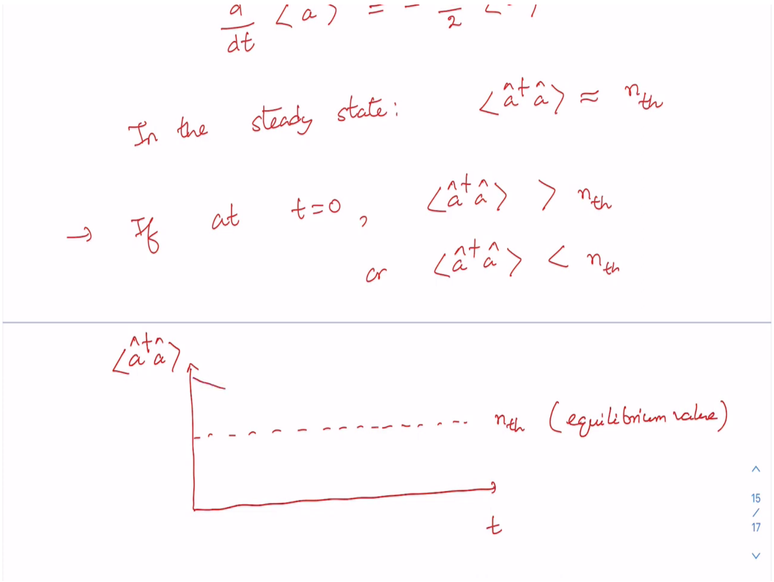

$\frac{d}{dt}\langle a \rangle = -\frac{\ }{2}\langle \ \rangle$

In the steady state: $\langle \hat{a}^{+}\hat{a} \rangle \approx n_{th}$

→ If at $t=0$, $\langle \hat{a}^{+}\hat{a} \rangle > n_{th}$

or $\langle \hat{a}^{+}\hat{a} \rangle < n_{th}$

Again, we can find out say what about the time rate of change of what about the time rate of change of expectation value of the annihilation operator? These also can be worked out I am leaving it as a exercise for you, you can do it and you can show that the time rate of change of this expectation value this annihilation operator would be you will get this equation.

So, gamma minus gamma by 2 a. So, this expectation value therefore decays at the rate gamma by 2. Now let us quickly check out the case when the harmonic oscillator harmonic oscillators at finite temperature, how we can discuss that finite temperature say finite temperature T. Now in this case there cannot be only one process because there is because at finite temperature there is thermal excitation as well.

So, we have this harmonic oscillator. So, there may be more than one process here one process may be in the downward direction say let us say that would be at the rate gamma minus and another process would be at a rate say gamma plus. So, both upward transition and the downward transition would be there. And here also we can have 2 processes say A 1 relaxation 2 relaxation processes A 1 and A 2 is there.

And here also you can guess it A 1, it refers to the downward relaxation process relaxation process in the downward direction A 2 is the upward transition. So, therefore we can guess that that would be represented by the creation operator. As regards the decade we have this gamma minus that is the downward decade that is the decade which would be actually we can write it in terms of the old decade gamma which is gamma is the decade at time t is equal to sorry temperature T is equal to 0 which is going to get enhanced by a factor of one plus say nth thermal.

nth thermal is the average number of thermal photons so this is this is a average number of thermal photon. In fact if you go to the t is equal to zero case the thermal average number of thermal photon would be zero then you will retrieve the earlier decay rate okay and actually this thermal number of photon would be given by this equation 1 by e to the power h cross omega by K B T - 1 where this is what we have as h cross omega spacing between the energy levels.

On the other hand, we have this another process that is the excitation process would be given by this and the decay rate or the relaxation rate would be gamma plus that would be actually because of the presence of the thermal proton it would be gamma n th. As you can see if we are having a zero temperature there would be no, because nth should be equal to zero at zero temperature and there would be no relaxation process in the upward direction.

Now using the Linblad quantum master equation we can actually show that the time rate of change of the average number of photons you should be able to establish this equation it will be gamma n th a dagger a or again here also just like in the case of the 2 level case or because relaxation process is there this is decaying at the rate minus gamma by 2 okay this is what even at finite temperature we will get.

Now in the steady state you can easily see that in the steady state or in thermal equilibrium this time rate of change of this is going to be zero. So, therefore the average number of photons that is a dagger a would be equal to nth. In fact depending on where you start say if say at t is equal to 0 we have this a dagger a that is greater than the average thermal photon or we have this average number of photons to be less than this equilibrium value.

This will get this behaviour as I can depict it in this diagram basically what I am doing is I am just depicting the solution of this equation you will see that in thermal equilibrium it will, so, here it is y axis is expectation value of a dagger a and this is the time and at thermal equilibrium it is going to reach this fellow it is the nth that is the equilibrium value equilibrium value.

So, if we initially start at a higher number of photons then it is going to decay exponentially and it will decay like this. On the other hand, if we start at say zero photon. So, then it will go and approach the thermal equilibrium at long time. So, this is what this solution basically represents. So, let me stop here for today in this class by using various examples we saw how quantum master equation or the so called Linblad quantum master equation can be used to study quantum dissipation processes.

In the next class we will continue our discussion on quantum dissipation and we will discuss the so-called pure dephasing and we will see how quantum master equation can be used to model it. So, see you in the next class thank you.