Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture - 20
The Jaynes Cumming Model

Hello, welcome to lecture 15 of the Course, this is lecture number 5 of the module 2. In the last lecture, we learned how to quantize a transmission line and transmission line resonator. In this lecture, we will see how this quantization leads to the so, called Jaynes Cummings model and we are going to discuss this model in somewhat great details in this class. So, let us begin.

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Let's have a quick recap of what we have done in the last class, in the last class we quantize the infinite transmission line taking the magnetic flux as the coordinate and it turns out that charge would be momentum and Lagrangian of this discrete transmission line we wrote as the difference between the electrical energy and a magnetic energy and we got from this Lagrangian that indeed the momentum is nothing but the charge and you should recall that this magnetic flux is related to the voltage in the transmission line by this expression and from here you see that the charge at a particular branch and at a particular node, we can get it just by taking the time derivative of the flux and then multiply it by the capacitance. Again, the current can be found just by taking the difference in the magnetic flux below 2 neighbouring nodes and that is going to give you the current that is driven in that particular branch.

Then in terms of this magnetic flux, we can write down this Lagrangian and from here we immediately get our Hamiltonian, this is the classical Hamiltonian and it turns out that this momentum and coordinate they are canonically conjugate variables. So, therefore, we can invoke the canonical quantization method and we do that by writing the variable magnetic flux in terms of operator and the momentum in terms of the momentum operator and invoking these Heisenberg uncertainty relations we can quantize it.

So, the quantized Hamiltonian for the transmission line would be of this form and that is you can see that is quadratic in the coordinate and quadratic in the momentum, but our goal is to express this Hamiltonian in a more useful form we want it to be in the form of a simple harmonic oscillator Hamiltonian. So, therefore, what we do next is we just take we write the magnetic flux as in terms of plane waves, plane waves are the normal modes of the system and then we apply an appropriate boundary condition.

The periodic boundary condition is what we have taken and the reason being is that infinite transmission line is translationally invariant, whether you go towards the right or left in the transmission line it does not matter. So, we can imagine the transmission line to be some kind of a ring having circumference W and we divide this ring into a number of cells each cell having cell size a and we at a particular point x in the transmission line, we can find out all the physical parameters.

So, say x if we choose then x = n into a, n is an integer and the boundary condition wise as we discussed this is actually result in the wave number gets discretize and k n = 2 pi / W into an integer n is an integer and then answers we take it in this form where it is superposition or a lot of plane waves and A k is the amplitude and k a is between minus pi / a to + pi / a and because of the translational invariance we find that it is easy to see that A k = A - k and omega k = omega - k.

Then we put this expression of this magnetic flux as well as this momentum in this Hamiltonian then and then we can there will be tedious calculations are there, but straightforward calculation we did not do that, but it can be done as it is done in the problem-solving session. So, you see that when you square P n square, then you get this kind of a term here, but then you have to sum over two k's, k and k dash and because the Hamiltonian you see already it is summed over n.

So, there are effectively 3 sums are there, but this can be simplified further because of this orthogonality condition. So, this orthogonality condition we discussed and because of this orthogonality condition, the Hamiltonian is over only 1 sum that is sum over k. So, there will be various terms one of the terms in the Hamiltonian which is of this type, but, this term as we discussed is not wanted, because it is a bilinear combination of these annihilation operators.

We do not want that our Hamiltonian should have this term because ultimately, we want our Hamiltonian to have a k dagger a k plus half kind of form is what is we want. So, therefore, this term we cancel out if we can make this term inside this curly bracket to be 0 and that can be done if we take omega k like this and omega k is basically the frequency of the kth mode and this dispersion relation and in fact, this is the dispersion relation for a plane wave propagating on the transmission line.

Then we discussed this dispersion relation and we found that when k is near equal to 0 then this dispersion plot or curve is nearly linear. And as we go to the continuum domain, you will see as a tends to 0 this pi / a it is go in the outward direction in this side and similarly that goes this sides. So, that actually means that the cut-off frequency for the continuum case this actually the wave number would a tends to 0.

This velocity on the transmission line would be simply given by this expression where v would be 1 / square root of LC, L is the inductance per unit length and C is the capacitance per unit length. Then we had another term in the Hamiltonian, this term is the required one we have the desire form here and putting the dispersion relation that we derived. So, based on this equation, this Hamiltonian can further be simplified so, this can be written in this form.

Now, choosing A k in this form we can write our Hamiltonian in a simple harmonic oscillator Hamiltonian form which physically means that now, our transmission line is nothing but a collection of independent simple harmonic oscillators. We then have this magnetic flux as well as this potential also or the voltage we can work out very easily as we discussed in the last class. Then we went on to discuss the quantization of transmission line resonator.

A transmission line resonator can be obtained from the infinite transmission line simply by cutting the infinite transmission line at 2 points. So, then we will have a transmission line resonator, but now in the case of transmission line resonator obviously we cannot use plane propagating waves as our modes. So, we have to take standing waves as we discussed and these modes would be combination of either it will be sin function or cosine function depending on what we are taking n.

For n = 1, 3, 5 we have a sin, the shape of the mode would be sine function. For n = 2, 4, 6, 8 like that, you will have a shape of the function would be cosine. Then we also have this orthogonality condition, using all these things and taking our coordinate like this and the momentum like this. Now, we are actually taking the continuum model and if we put everything in this Hamiltonian, then we finally obtain again our Hamiltonian can be for the transmission line resonator can be written as a collection of independent harmonic oscillators. While doing this we have taken our constant A n to be like this and the dispersion relation is this. Now, from this thing we also discussed and how we can get the voltage in the transmission line resonator, which is a very, very important quantity.

So, this is what now we are going to discuss, because now the question is how to couple this fluctuating voltage is V of x to the cooper pair box. We know the Hamiltonian for the cooper pair box and we know how an external electric field could be applied to tune the cooper pair box. This external electric field is proportional to the fluctuating voltage V of x. This is in fact analogous to the fact that when you put a real atom in an external electric field,

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$$\boxed{H_{int} = -\vec{\hat{\mu}} \cdot \vec{E}_{ext}}$$

$$H_{ch} = \left[\hat{Q} - (Q_G + \delta\hat{Q}_G)\right]^2$$

where the interaction is say given by this is I am talking about real atomic system where you put the atom in the external electric field and this interaction term is given by this one this expression where mu is the this is the atomic dipole moment and E ext this is the external electric field. So, analogous thing is also here is I am now going to discuss if you remember, our cooper pair box has this charging energy term we had Q square / C but then we have this Q and then to take the external field into account then we put this gate charge there.

Now, because of quantum fluctuation this gate charge is has some extra contributions say delta Q G this is the quantum fluctuation and then this is the charging energy term Q square by twice C.

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$$\delta\hat{Q}_G = \underline{\underline{C_G}}\,\hat{V}(x)$$

$$\hat{H}_{int} = -\frac{1}{C}\left(\hat{Q}-Q_G\right)\cdot\delta\hat{Q}_G$$

$$= -\frac{C_G}{C}\left(\hat{Q}-Q_G\right)\underline{\underline{\hat{V}(x)}}$$

$$\hat{H}_{int} = -\frac{C_G}{C}\,\omega_n A_n \phi_n(x)\left(\hat{a}_n+\hat{a}_n^{+}\right)\left(\hat{Q}-Q_G\right)$$

In fact, we can write this extra fluctuating charge as C G into the fluctuating voltage. C G is the gate capacitance which depends on the geometry and it depends particularly, where we put the cooper pair box in the transmission line resonator. So, this is basically a geometrical quantity and it takes actually detail so many things, but in particular it is important where we are putting the cooper pair box in the transmission line.

Now expanding this Hamiltonian just taking only the cross terms, because cross terms are only responsible for interaction. So, if we take only the cross term it is easy to see that I can write my interaction Hamiltonian in this form, if you expand it, you will see you will get a term like this 1 / C Q - Q G into delta Q G. And this I can then use these expressions then I will have minus C G / C Q - Q G into the voltage that is how I can couple my cooper pair box with the voltage.

Now, if we focus only on the single field mode only one single mode rather than many modes as you see we have this. So, if we just focus on one single mode than we need not have to deal with the summation just one mode, then I can write this interaction Hamiltonian in this form minus C G / C omega n A n phi n, that is a shape of the mode and then I have a n + a n dagger and I have Q - Q G I can simply put the expression of V x in here and that is what I get.

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$$N_G = \frac{1}{2}$$

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$$|g\rangle = \frac{1}{\sqrt{2}}\left[\,|N=0\rangle + |N=1\rangle\,\right]$$

$$|e\rangle = \frac{1}{\sqrt{2}}\left[\,|N=0\rangle - |N=1\rangle\,\right]$$

Now I can exploit the fact that I can write my charge operator in terms of the number of operators we discussed in I think lecture 12 earlier. So, I can write it as twice q e into N - N G, N is the number operator. Now, let us concentrate only on the ground state and the first excited state. So, take the charge degeneracy point at N G = half and our cooper pair box would then be a 2-state system and this will enable us to express the cooper pair box or this particular term in terms of Pauli matrices because now it is a 2-state system.

As discussed in lecture 12 we saw that in lecture 12 we discussed that N G = half, the ground state of the cooper pair box is a symmetric combination of we discuss it there, N = 0 and N = 1 and the excited state is a anti symmetric combination of N = 0 and N = 1. So, this if we take this ground state and the excited state as our basis state as our basis what we can do

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$$|e\rangle = \frac{1}{\sqrt{2}}\left[\,|N=0\rangle - |N=1\rangle\,\right]$$

$$\begin{aligned}(\hat{N} - N_G) = {} & \langle g|\hat{N} - N_G|g\rangle\, |g\rangle\langle g| \\ & + \langle e|\hat{N} - N_G|e\rangle\, |e\rangle\langle e| \\ & + \langle g|\hat{N} - N_G|e\rangle\, |g\rangle\langle e| \\ & + \langle e|\hat{N} - N_G|g\rangle\, |e\rangle\langle g|\end{aligned}$$

now we can express this operator N - N G we can write it in this form. So let me just quickly give you we can write it as g N - N G you just have to expand it, g ket g bra g plus so all these there will be 4 terms. So, very straightforward you can do the calculation here and then we have other 2 terms would be g N - N G e here to be g e and we have e N - N G g e g this you can do.

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$$\hat{A} = \sum_{i,j} |i\rangle\langle i|\hat{A}|j\rangle\langle j|$$

- with $N_G = \frac{1}{2}$

$$\hat{N} - N_G = -\frac{1}{2}\left[\,|e\rangle\langle g| + |g\rangle\langle e|\,\right]$$

$$= -\frac{1}{2}\sigma_x$$

If you have forgotten, let me just quickly remind you. Suppose you have any operator A I can always write it I can sandwich it in this form I can always sandwich A between the basis state i j and then I just sum it. So, this is what I have used while writing this expression here. Now, with N G is equal to half it is straightforward to show that N - N G is equal to minus half e g plus g e so, this actually tells you this operator tells you that you can go from ground state to the excited state or from the excited state to ground state. So, obviously, this is our Pauli operator sigma x.

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$$\hat{H}_{int} = \frac{C_G}{C}\,\omega_n A_n \phi_n(x)\left(\hat{a}_n + \hat{a}_n^{\dagger}\right) 2q_e \frac{1}{2}\sigma_x$$

$$= \hbar g\, \sigma_x \left(\hat{a}_n + \hat{a}_n^{\dagger}\right)$$

$$\hbar g = \frac{C_G}{C}\,\omega_n A_n \phi_n(x)\, q_e$$

$$\boxed{\hat{H}_{int} = \hbar g\, \sigma_x \left(\hat{a}_n + \hat{a}_n^{\dagger}\right)}$$

So, using this form we can now write our interaction part of the Hamiltonian simply by this. So let me write it, it would be C G / C omega n A n phi n of x a n plus a n dagger twice q e then in place of this one I have this minus, minus get cancelled out because I had here this is minus so therefore, I will have a half sigma x. So, finally what I can do I can write this whole expression I can write it as h cross g sigma x a n plus a n dagger, where I have put this constant h cross g includes this particular term which is a constant C G / C omega n A n phi n of x n q e. So, finally this is an important expression that we obtain that is h cross g sigma x a n plus a n dagger. So, this term Hamiltonian basically tells us how the 2-level system or artificial atom is now getting coupled to the transmission line resonator. You should know that h cross g has the dimension of energy and g is in frequency units.

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g : coupling parameter

$g \gtrsim$ MHz

~ 100 MHz

Jaynes - Cumming Model

$$\underbrace{\text{field}}_{\text{SHO}} + \underbrace{\text{atom}}_{\text{TLS}} + \text{atom - field}$$

g is known as the coupling parameter or coupling coefficient and it gives us the coupling strength between the transmission line resonator and the artificial atom. In a typical experiment, g is on the order of it is in frequency unit and so it will be in the order of megahertz. One can actually have g up to or more than 100 megahertz, so, we can have a very, very strong coupling between the artificial atom and the microwave field depending on how we design the transmission line or the whole system.

By the way, this model has a name and this model is known as the so-called Jaynes Cummings model and it is a popular model and it appears in many areas in physics, and we are specifically discussing it in the context of transmission resonator and 2 level artificial atom. Now, our objective would be to solve this model and understand its physics. So, Jaynes Cummings model basically describe the interaction between a single field mode with an atom. for our case the field mode is a single harmonic oscillator.

So, we have field and we have atom and then we have this atom field interaction is there and already I have written down the atom field interaction term. So, in our case the field is a one single mode we are considering and that is a simple harmonic oscillator and atom we are considering is a 2-level system.

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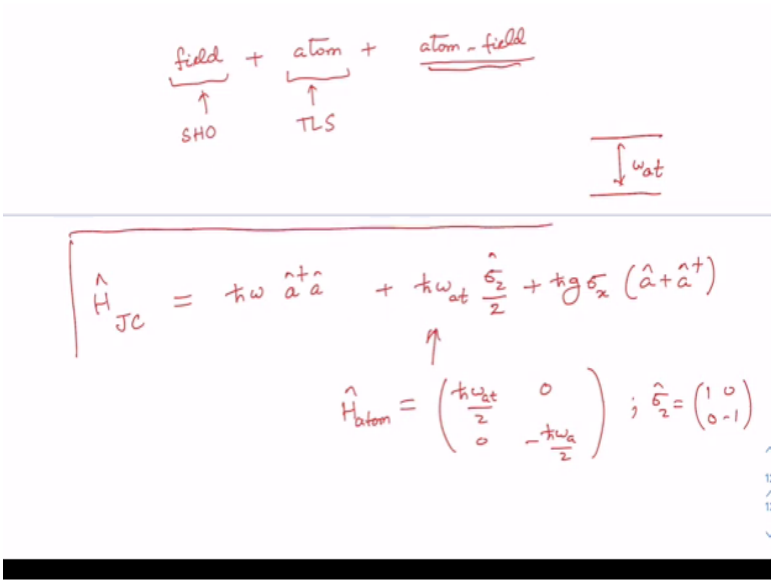

So let us now write down the full Jaynes Cummings Hamiltonian, so first of all we have a field mode so that we can write as a simple harmonic oscillator a dagger a then h cross omega that is the frequency of the mode, I am not writing plus half here, because that is just a constant and then the atom, we are going to model it is a 2-level system. So, we have this atomic transition frequency omega atom and let me just write it as a sigma z / 2, thereby mean that my atomic this transition frequency is omega atom.

In fact, if I just specifically talk about the atomic part of the Hamiltonian, then it would look like this h cross omega atom here by 2 and minus h cross omega atom by 2 because you know that sigma z is equal to 1 0 0 -1 and finally, we have this atom field interaction term is there and that we have already known as h cross g sigma x a + a dagger. So, this is, what is the full Hamiltonian for the Jaynes Cumming model. The underlying physics described by the Jaynes Cumming model is easy to understand.

The first term in the JC as Hamiltonian represents the field mode energy. The second term is about the energy of the 2-level atom while the third term represents the interaction between the atom and field mode.

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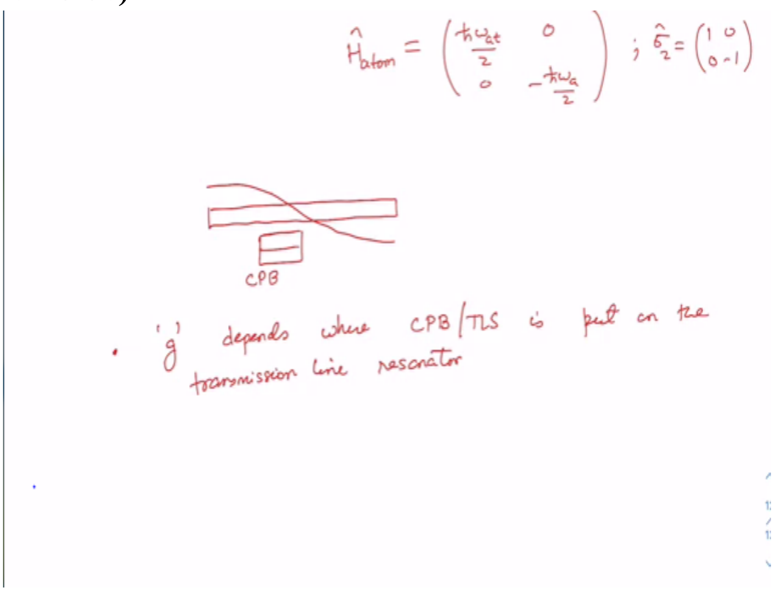

In our case, we have a situation like this where we have our transmission line resonator and in the transmission line resonators say we have a field mode like this and this field mode is interacting with the cooper pair box or any artificial 2-level system and the coupling between this cooper pair box and the field mode microwave field mode depends where we are putting our cooper pair box on the transmission line.

The strength of this coupling parameter or its frequency, g depends where cooper pair box or 2-level system is put on the transmission line. So, it completely g is dependent on the geometry. Now we will discuss the solution of this Jaynes Cumming Model.

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$$\hat{H} = \hbar\omega \hat{a}^{\dagger}\hat{a} + \hbar\omega_{at}\frac{\hat{\sigma}_z + 1}{2} + \hbar g \hat{\sigma}_x (\hat{a} + \hat{a}^{\dagger})$$

$\hat{\sigma}_z \rightarrow +1 \Rightarrow$ atom is in the excited state

$\hat{\sigma}_z \rightarrow -1 \Rightarrow$ atom is in the ground state

For our convenience, let me take the ground state of our 2-levels system to be at 0 and excited energy state would have an energy h cross omega atom where omega atom is the transition frequency between the ground state and the excited state. Earlier we had a situation like this where the ground state energy was taken to be the - h cross omega atom / 2 and the excited state have an energy, h cross omega atom / 2 and the transition frequency of course still it is omega atom.

So, nothing is actually getting change only, I am just shifting my energies and that has actually is would turn out to be very convenient, then I have to make a little bit of changes in the Jaynes Cumming Hamiltonian, let me do that. So, first term would remain the same in the Jaynes Cumming Hamiltonian, that is the h cross omega a dagger a that is the field mode energy and the atom energy would be of this form now omega atom sigma g, I will just put a plus 1 there divided by 2 and this interaction term is remains the same h cross g sigma x a + a dagger.

Now you see that this guy sigma g, when this is equal to plus 1, this refers to the fact that the atom is in the excited state. And you can immediately see that if I put sigma z is equal to plus1 here then my energy of the excited state of the 2-level atom would be simply as h cross omega atom. And on the other hand, when I have sigma z is equal to minus 1 then the atom is in the ground state.

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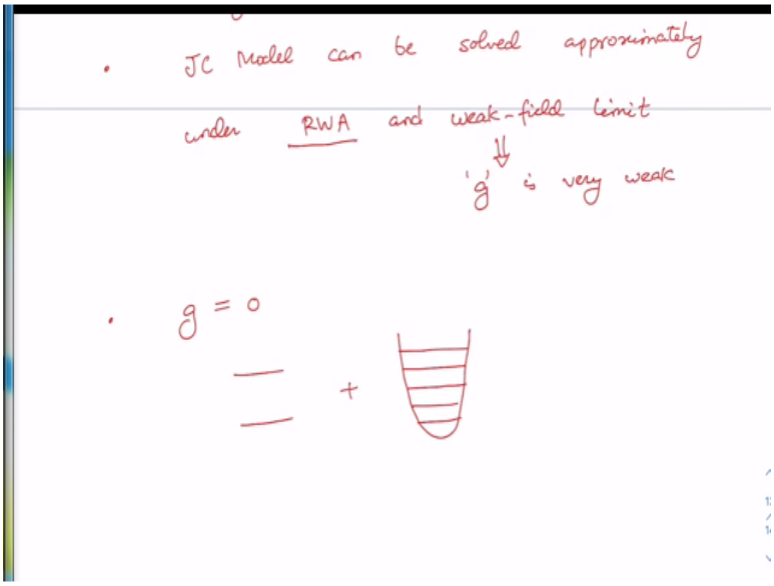

Now, surprisingly, this Jaynes Cumming model has no analytical solution. So, JC model does not have I mean exact analytical solution, though it is such a simple model, but we can still solve approximately under the so-called rotating wave approximation RWA and weak field limit weak field condition by weak field limit I mean that it implies that g this coupling parameter g is very weak in fact, that means, that this last term this coupling term is now going to be considered as a kind of a perturbation.

And also, please note that when these parameters g is equal to 0 that is there is no coupling between the atoms and the field, we cannot solve the Jaynes Cumming model because in that case, what we are having simply the atom which is a 2-state system and we are having this field mode, which is a harmonic oscillator as we know and this harmonic oscillator has infinite energy levels and all of them are equally spaced and nothing happens. So, we are not going to have any solution in the model nothing useful we can get out of this.

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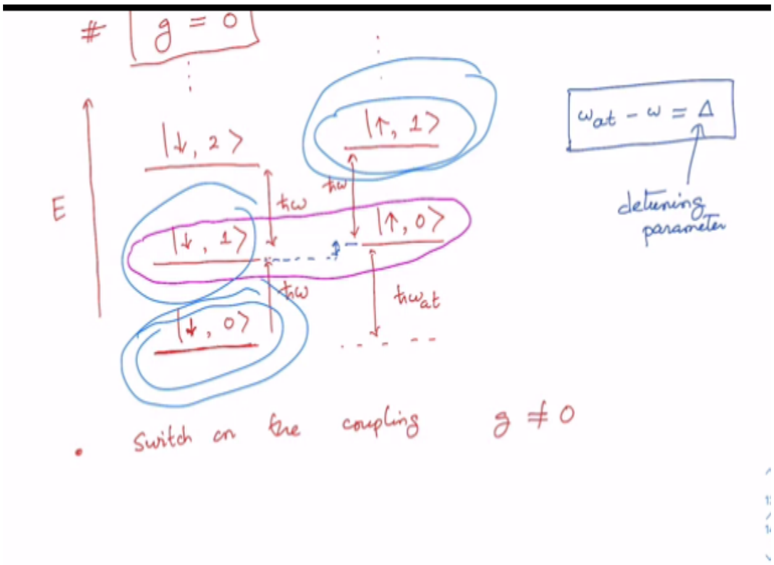

So now let us draw the energy level diagram for the combined atom field system and in this process, we are going to take the number of photons. So, what we are going to do, energy level diagram, we are going to draw energy level diagram for atom field system and while doing that, we are going to take the number of photons inside the field mode with the atom to be either in the excited state or in the ground state.

Firstly, for simplicity let me consider the easiest case that is, when the coupling is 0 between the atoms and the field. We can view the atom as a spin half system as it has only 2 energy level, let us say we have the atom is in the ground state. So, let be represented by the down here down spin it is in the ground state and there is no photon in the field mode 0 photon in the field mode. So, clearly this is the ground state of the combined system combined atom and field so this represents the ground state energy of the system.

Now we can put 1 photon into the field the atom still being in the ground state and then we have this 1 photon in the field mode. So, the energy would be then energy of the system would enhance by h cross omega because h cross omega is the energy of the field and then if we put another photon then energy would be enhanced further with respect to the ground state the energy would be twice as h cross omega.

Now atom is in the ground state and there are 2 photons in the field mode and so on we can go only one the upper direction. So, this is what we have is the energy only the other hand, we can have a situation where the atom is in the say excited state and there is no photon in the field mode then, because the atom has energy h cross omega atom. So, this energy level is shifted from the ground state energy of the combined system by h cross omega atom.

And as you see, this particular level that let me put this colour here, this particular level and this level, it has some energy difference, that energy difference or frequency difference is, as you can see from the diagram, that would be omega atom - omega and this difference in frequency has a name it is called detuning parameter and it is very important. So, delta we are going to denote it as the so-called detuning parameter. So, this is detuning parameter and when the system is in resonance so in that case omega atom is equal to omega.

Now, if we put another proton in the field mode then energy state would be this combined system would have energy this. So, this would be because we have put field mode so, it will be h cross omega and so, this way you can go on draw the other energy levels. Now, let us switch on the here we discuss the case when the coupling between the field and the atom is 0. Now, let us switch on the coupling so that means when g is not equal to 0 and of course, as I say that we are going to consider the coupling to be very, very weak.

And then we can exploit the fact that the energy levels which are in resonance or they are nearly resonance. For example, if you just look at these 2 energy levels, they are nearly resonant lying nearby these energy levels are far more important than the off resonant transitions or energy levels for example, we can neglect the transition between say this energy level and this energy level or say this energy level and say this energy levels hope you are getting the idea.

So, we can just neglect interaction between these energy levels and in that case basically we are neglecting all the off-resonance transitions and this is the so called rotating with approximations.

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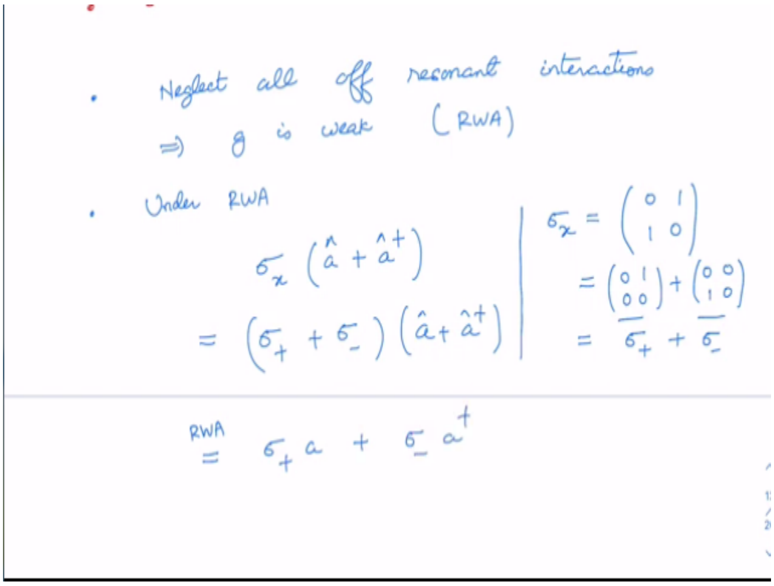

So, we are going to neglect all off resonant interactions as we will be clear that is possible only if we have this coupling parameter g is very weak then we can neglect all the other off resonant interactions now under the so called RWA under RWA we can write this last term in the Jaynes Cummins Hamiltonian sigma x a a dagger we can write in this form sigma x can be written as sigma plus into sigma minus let me explain that, a + a dagger you see sigma x is equal to 0 1 1 0 this we can write as 0 1 0 0 + 0 0 1 0 now if you can recall from your earlier class that this part is nothing but the atomic raising operator sigma plus and this one is atomic lowering operators sigma minus.

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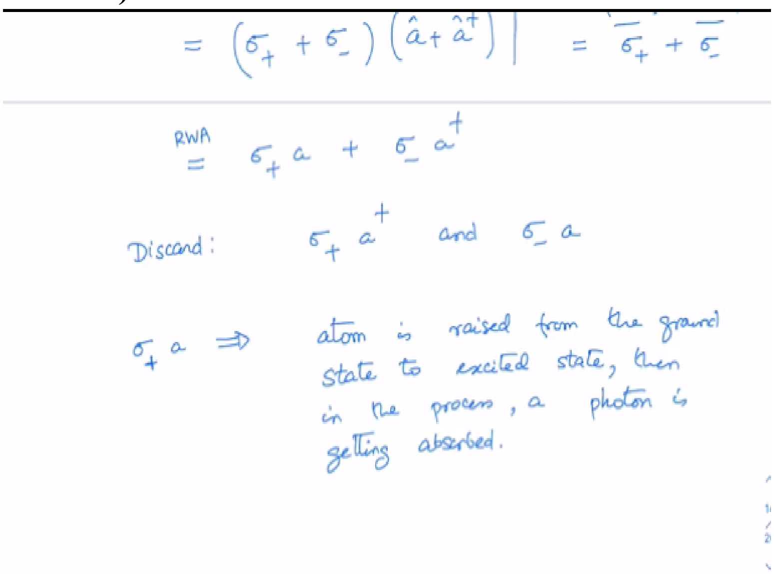

Now under RWA or the so-called rotating wave approximation, I can write if I expand this whole term, I will get 4 terms but out of these 4 term because of RWA I can just keep only 2 terms and those would be sigma plus a and sigma minus a dagger. Here we are discarding 2 terms those are sigma minus a dagger and sigma minus a this is physically reasonable, because for example we know that this sigma plus a, this whole thing implies that because this is sigma plus is atomic raising operator, so it means that atom is raised from or excited from the ground state to the excited state and then in the process a photon is getting absorbed and we know that this is indeed true and this is physically what happens.

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and in the process a photon is emitted.

- $\sigma_+ \hat{a}^+$, $\sigma_- a$ — ignore

- $$\boxed{H_{JC} \overset{RWA}{=} \hbar\omega\, \hat{a}^+\hat{a} + \hbar\omega_{at}\frac{\hat{\sigma}_z+1}{2} + \hbar g\left(\sigma_+\hat{a} + \sigma_-\hat{a}^+\right)}$$

Similarly, in this case you see sigma - a dagger it means atom is now getting lowered atom is transiting from excited state from let me just write like this from excited state to the ground state. And in the process, a photon is getting created a photon is emitted creation of photon is happening a microwave photon happens in the case of our system that we are considering a transmission line resonator interacting with a cooper pair box so this is very reasonable.

On the other hand you say these 2 processes say sigma plus and a plus this means that what is happening that when the atom is getting excited, a photon is also getting created these are very unlikely, very probability of occurrence of these 2 events, say this one and another one is this sigma - a, which means that when the atom is now getting de-excited from the excited state to the ground state a photon is getting absorbed that is not the case right that is these are 2 unlikely events.

So therefore, we can safely we can ignore these 2 terms, I hope it should be you know clear to all of you. So, in that case then we have this so called RWA Hamiltonian the Jaynes Cummings Hamiltonian under RWA we can now write it as h cross omega a dagger a plus h cross omega atom we have it as sigma z plus half and then we have h cross g I have sigma plus a plus sigma minus a dagger. So, this is what we have and we are going to discussed now.

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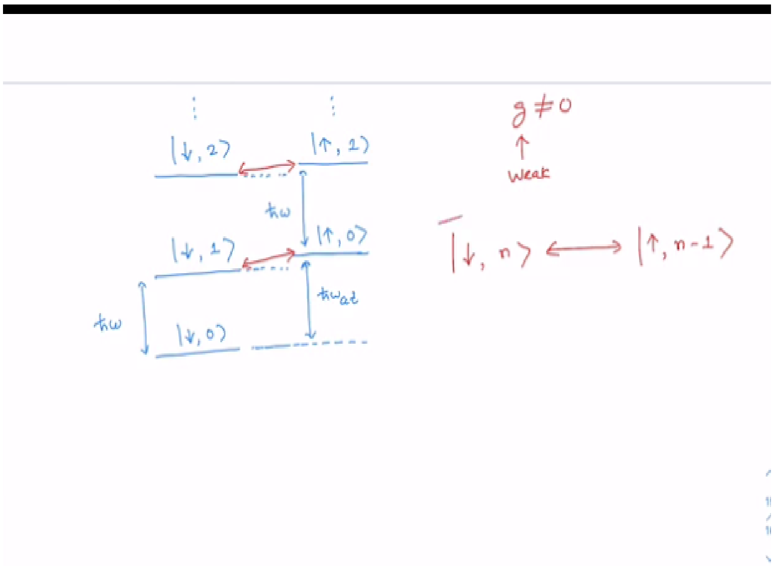

Let us switch on the coupling parameter that is when g is not equal to 0 and see how the level scheme looks like. We have g none 0 and this coupling parameter is weak and because of that as per the rotating wave approximation all the neighbouring energy levels for example, this energy level and this energy level are connected to each other similarly, this energy level and these energy levels are connected and similarly, for all higher energy levels in the ladder.

Or we can say simply that the state where the atom is in the ground state and there are n number of photons in the field mode and atom in the excited state and there is one less photon in the field mode, they are connected to each other and the matrix element in the Jaynes Cummings Hamiltonian which connect these 2 energy levels or these 2 states can be worked out very easily.

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$$\langle \uparrow, n-1 | \hbar g (\sigma_- a^\dagger + \sigma_+ a) | \downarrow, n \rangle = \hbar g \sqrt{n}$$

$$\sigma_- a^\dagger | \downarrow, n \rangle = \underbrace{\sigma_- | \downarrow \rangle}_{=0} \; \underbrace{a^\dagger | n \rangle}_{\sqrt{n+1}\, |n+1\rangle} = 0$$

$$\sigma_+ a | \downarrow, n \rangle = \sigma_+ | \downarrow \rangle \; a | n \rangle = | \uparrow \rangle \sqrt{n}\, | n-1 \rangle = \sqrt{n}\, | \uparrow, n-1 \rangle$$

So let me work that out. We have this state atom in the ground state, n photons in the field mode and the atom in the excited state and 1 less photon in the field mode and this is connected via coupled via this term in the JC Hamiltonian h cross g sigma a dagger plus sigma plus a. So, this is what we have to work out now, actually, it will turn out that this would be simply h cross g square root of n let me so how it can be worked out very easily.

Let us see the action of this operator on the state say sigma a dagger this one so, this atomic now this is the lowering operator sigma minus atomic lowering operator would act on this atomic state and a dagger is going to act on this photon state field state and we know that this is simply n plus 1 into actually it is square root of n plus 1 and n plus 1 ket here. Now, this is the atomic lowering operator it is acting on the ground state so, it cannot lower it further so therefore, this is going to give us 0.

So overall we are going to get 0 because of this. Let us look at the other term where we have sigma plus a this one acting on the state sigma plus atomic raising operator is acting on the ground state of the atom and then annihilation operator is acting on the field mode. You know that when it is acting on the field mode, we are going to get simply a square root of n, n - 1and on the other hand this because of this act on this ground state is going to be raised to the excited state. So therefore, we will simply have square root of n atom in the excited state and field is having n - 1 number of photons so let me put this term here.

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$$= 0 \cdot \sqrt{n+1}\,|n+1\rangle \qquad = |\uparrow\rangle \sqrt{n}\,|n-1\rangle$$

$$= 0 \qquad\qquad = \sqrt{n}\,|\uparrow, n-1\rangle$$

$$\langle \uparrow, n-1 | \hbar g \sqrt{n} | \uparrow, n-1 \rangle$$

$$= \hbar g \sqrt{n}$$

So, what you are going to get because of that you will have these up-state n - 1 and h cross g square root of n and I have this is my state. Now, this is just a number and these 2 states are the same state. So, they are normalized and because of normalisation we have simply h cross g square root of n. So, this is how these 2 energy states or these 2 are connected to each other when there is very weak coupling between the states.

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JC

Basis states : $\{ \underbrace{|\downarrow, n\rangle}_{|\alpha\rangle} , \underbrace{|\uparrow, n-1\rangle}_{|\beta\rangle} \}$

$$\hat{H} = \sum_{i,j} |i\rangle\langle i| H |j\rangle\langle j|$$

$$\hat{H}_{JC} = \langle\alpha|H|\alpha\rangle\, |\alpha\rangle\langle\alpha| + \langle\beta|H|\beta\rangle\, |\beta\rangle\langle\beta| + \langle\alpha|H|\beta\rangle\, |\alpha\rangle\langle\beta| + \langle\beta|H|\alpha\rangle\, |\beta\rangle\langle\alpha|$$

$$\langle\beta|H|\alpha\rangle = \langle \uparrow, n-1 | \hat{H} | \downarrow, n\rangle$$

Let us find the matrix from of the Jaynes Cummings Hamiltonian in the rotating wave approximation. So, our Jaynes Cummings Hamiltonian let me rewrite again, it is h cross omega a dagger a plus h cross omega atom sigma z plus one by two plus h cross g I have sigma a digger plus sigma plus a so, this is our Jaynes Cummings Hamiltonian in the RWA. My intention now here is to work out the matrix form of this Hamiltonian and for that.

Let me choose my basis state as this, these are my basis states are simple this is atom in the ground state, n number of photons in the field mode, atom in the excited state, one less photon in the field mode. Taking this as my basis states, in fact, let me write it in a short hand notation, let me denote it by this ket alpha and let me denote this state by ket beta. So, the Hamiltonian then just recall that any arbitrary Hamiltonian I can always write in this form I can sandwich this Hamiltonian between these kets.

Because of the completeness condition and I have to take sum over all these here. So, therefore, this JC Hamiltonian now I can write it as, because I have only these 2 basis states alpha beta so I can write it as alpha, H alpha, alpha, alpha and I have beta H beta, beta, beta. I have alpha H beta alpha beta plus beta H alpha beta alpha. So, let us work out all these elements.

Already, we have worked out these elements, beta H alpha, which I can actually denote it as H beta alpha, and beta we have taken as upstate of the atom in n minus 1 that is the field photons H downstate of the atom and this is n that's how I have taken my alpha and beta so these already have worked out.

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$$H_{\beta\alpha} = \langle \beta | H | \alpha \rangle = \langle \uparrow, n-1 | \hat{H} | \downarrow, n \rangle$$

$$= \hbar g \sqrt{n}$$

$$H_{\alpha\beta} = \langle \alpha | H | \beta \rangle = \hbar g \sqrt{n}$$

$$H_{\alpha\alpha} = \langle \alpha | H | \alpha \rangle = \hbar \omega n$$

$$H_{\beta\beta} = \langle \beta | H | \beta \rangle = \hbar \omega (n-1) + \hbar \omega_{at}$$

$$= \hbar \omega n + \hbar \underline{(\omega_{at} - \omega)}$$

$$= \hbar \omega n + \hbar \Delta$$

And this is simply h cross g square root of n. Similarly, you can work out H alpha beta, that means alpha H beta, this would be also equal to h cross g root n and try to find out what is H alpha, alpha, that is alpha H alpha, that will be equal to h cross omega n you can very easily work it out and H beta, beta that is beta H beta tha be h t would be h cross omega n minus 1 plus h cross omega atom this actually I can write it as h cross omega n plus h cross omega atom minus omega and these you know, this is the detuning parameter. So therefore, I can write it as h cross omega n plus h cross delta so I have all the elements in the Hamiltonian.

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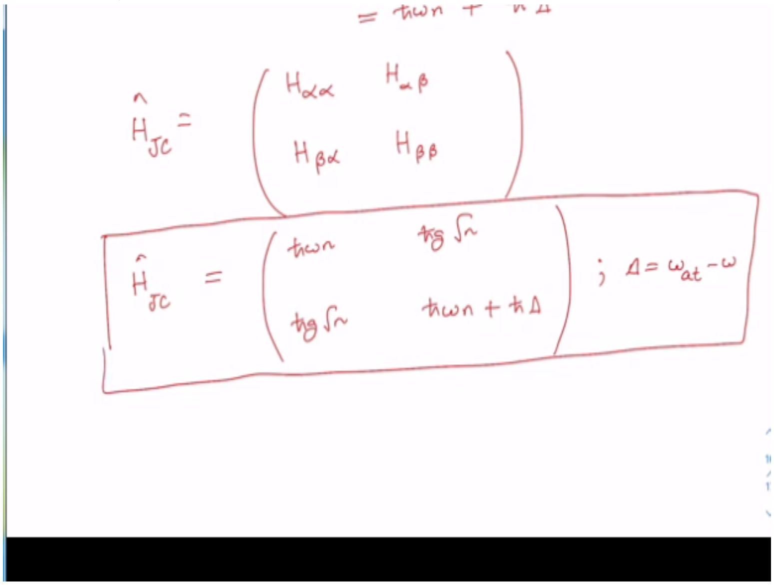

So, I can write my Jaynes Cummings Hamiltonian in the matrix from now, so we have H alpha, alpha, heads alpha beta, H beta alpha, H beta, beta. So, as I have worked out all the elements, so let me just write it down, that would be h cross omega n, and h cross g these are the coupling term h cross g root n and I have h cross omega n + h cross delta, where delta = omega atom - omega. Now, let us find out the Eigen values of this Jaynes Cumming Hamiltonian in the RWA that can be done very easily.

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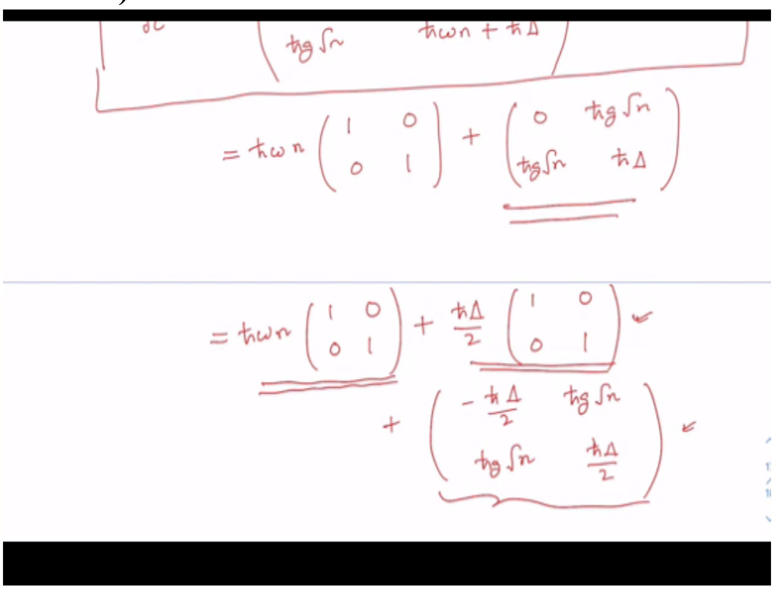

Let me show you this Hamiltonian I can write in a simpler, let me separate it out in a number of terms let me write, to find out the Eigen values, I need to diagonalize it and I can do it this way, I can write it as h cross omega n 1 0 0 1 then I would be left out with 0 h cross g square root n, h cross g square root and then I have here h cross delta, it is easy to see and then I can further write these is h cross omega n 1 0 0 1. And this I can write this further as h cross delta by 2 1 0 0 1 and I have minus h cross delta by 2 here and here, h cross delta by 2.

So, as you can see, I have if I take this term and take this term, I will have the diagonal elements here. And this one would be simple, there will be h cross g square root n and h cross g square root n the good thing is that this particular part in the Hamiltonian is already diagonalized this is also diagonalized. Now, I have to just, diagonalize this part of the Hamiltonian.

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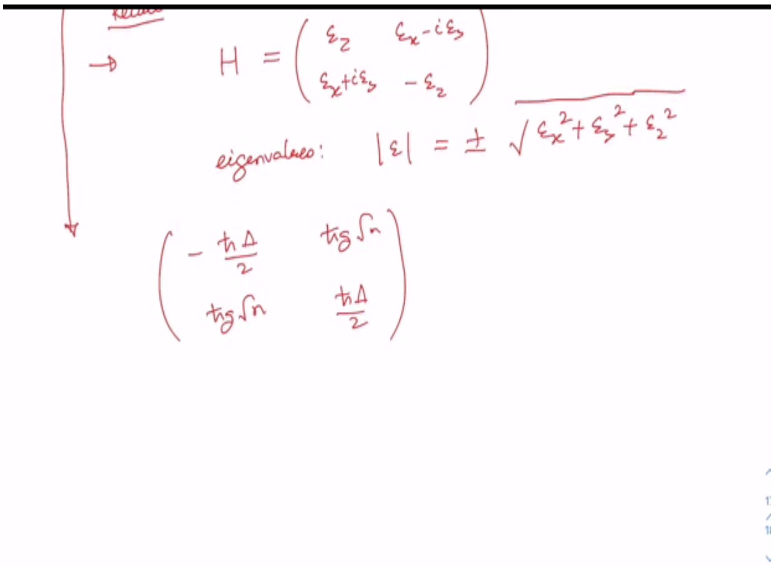

So, let us do that but actually, we need not have to do that, because already from module 1, when we discuss the 2-level atom you just recall that when we write down this Hamiltonian for the 2-level atom epsilon z minus epsilon z epsilon x plus i epsilon y epsilon x minus i epsilon y there, we saw that the Eigen values of this Hamiltonian we worked out and that was mod epsilon that is equal to plus minus square root of epsilon x square plus epsilon y square plus epsilon z square. So, let us use this result directly for this particular part in the Hamiltonian let me write it separately here minus h cross delta by 2 h cross g root n h cross g root n h cross delta by 2.

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Overall

$$E_{\pm} = \hbar\omega n + \frac{\hbar\Delta}{2} \pm \hbar\sqrt{g^2 n + (\Delta/2)^2}$$

$$E_{+} = \hbar\omega n + \frac{\hbar\Delta}{2} + \hbar\sqrt{g^2 n + (\Delta/2)^2}$$

$$E_{-} = \hbar\omega n + \frac{\hbar\Delta}{2} - \hbar\sqrt{g^2 n + (\Delta/2)^2}$$

So, the Eigen values is lambda, let me denote it by lambda so I have plus minus h cross delta by just look at here, h cross delta by 2 whole square plus h cross g root n. So, I know the Eigen values of this Hamiltonian, I know the Eigen values of this Hamiltonian these Hamiltonian so therefore, the overall we have the Eigen values of the Jaynes Cummings Hamiltonian under RWA is this E plus minus is equal to h cross omega n plus h cross delta by 2 plus minus h cross g square n plus del by 2 whole square.

If you were more interested or simply, we have 2 energy eigen value E plus and E minus, E plus refers to h cross omega n plus h cross, del by 2 plus h cross g square n plus delta by 2 whole square and E minus is h cross omega n plus h cross delta by 2 minus h cross g square n plus delta by 2 whole square.

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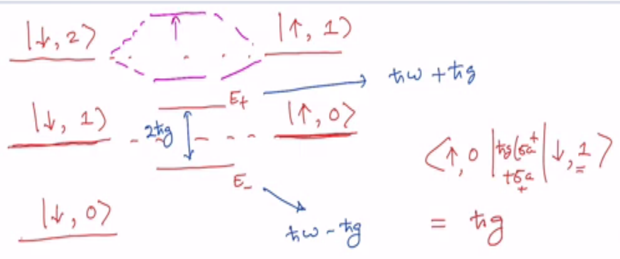
Resonant case $\Delta = 0$

$$E_{+} = \hbar\omega n + \hbar g\sqrt{n}$$

$$E_{-} = \hbar\omega n - \hbar g\sqrt{n}$$

$\vdots$



Now, let us consider the resonance case, it is simple to understand in the resonance case delta detuning parameter is equal to 0. So, therefore, will have this energy level as E plus E minus would be equal to h cross omega n you will have plus h cross g square root of n and here you will have h cross omega n minus h cross g square root of n that is why you see that means the energy level is basically getting splitted into 2 parts.

Let us understand it from the energy level scheme or energy level diagram for the system, we have the atom in the ground state and there are no this is what protons in the field mode. So, this is our ground state then we have atom in the ground state 1 photon in the field mode atom in the ground state 2 photons in the field mode. Now because it is resonance so, in the other case, we have here atom.

I am talking about when there is no coupling the atom in the excited state and there is no photon in the field mode these are now you know horizontally lying because delta is equal to 0. So, atom in the excited state 1 photon in the field mode and so on you can go on. Now when the coupling is switched on this is the same energy level, they have the same energy this is getting split up into 2 parts one is E plus and another one is E minus.

So, if you just look at this particular these 2 states then you see how they are connected this atom in the ground state 1 photon in the field mode and atom in the excited state no photon in the field mode and these interaction term h cross g I just can write it as sigma a dagger plus sigma plus a. This would be if you just look at from here, we have worked out the matrix element this expression you see n is this one. So therefore, from here I can because n is equal to 1 I have to put so this will be simply h cross g.

And if you look at now, these states E plus and E minus so therefore I hope you are getting it so, this is my E plus and this is my E minus. So, E plus let me use this E plus is h cross omega plus h cross g and E minus this is h cross omega minus h cross g. So, therefore, the splitting in the energy level is simply twice as h cross g. Similarly, you can show that this energy states getting spitted into 2 parts E plus and E minus their corresponding energies. So, this splitting would be if you work it out you will get it as 2 into square root of 2 h cross g. Let me stop for today.

In today's class, we saw how quantization of the transmission line resonator leads to the so-called Jaynes Cummings model. And we have also learned that there is no analytical solution to the Jaynes Cummings model. However, we can solve this problem the Jaynes Cummings model Hamiltonian we can solve under the so-called rotating wave approximation. We have written down the matrix form of Jaynes Cummings model under the rotating wave approximation.

In the next class, we will continue our discussion on the Jaynes Cummings model and we will particularly discuss the dynamics associated with the Jaynes Cummings model along with some other features. So, see you in the next class. Thank you.