Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture –2
Problem Solving Session-1

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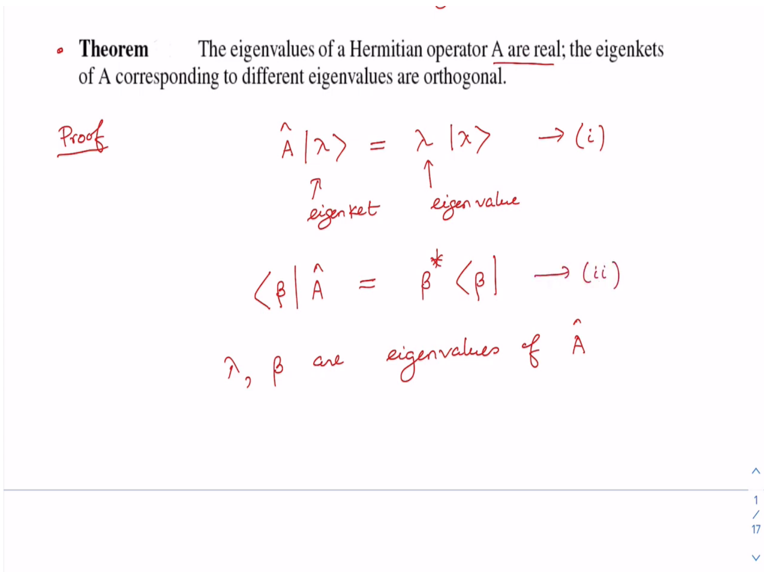

Let us begin this problem-solving session by proving the following theorem. This theorem is very important. It says that the eigenvalues of a Hermitian operator A are real. The eigenkets of A corresponding to different eigenvalues are orthogonal. So, the theorem has 2 parts, and we are going to prove both the parts. Let me start by writing the eigenvalue equation. A is the operator. If it operates on the ket vector lambda, then we get this equation if it is an eigenvalue equation. So, this is the eigenvector, ket lambda is the eigenket, and lambda is the eigenvalue. At the moment we do not know whether lambda is real or not and this is what we are going to prove in the first part of the theorem. Because, A is Hermitian, so, if it operates on a bra vector then we will get it as beta star, bra beta here and lambda beta are eigenvalues of the operator A. Let me label this as equation number 2.

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(iii) − (iv)

$$\boxed{0 = (\lambda - \beta^*)(\langle\beta|\lambda\rangle)} \to (v)$$

$\#$ Say $\lambda = \beta$

Then $\lambda = \lambda^*$ $\Rightarrow$ eigenvalues are real

$\#$ $\lambda \neq \beta$

$$\langle\beta|\lambda\rangle = 0$$

$\Rightarrow$ $|\lambda\rangle$ and $|\beta\rangle$ are orthogonal.

Now let me multiply equation one on the left-hand side by bra beta on both sides. So, from 1 I have beta A lambda. Similarly, I have to multiply the right-hand side as well. Lambda is a number. So, I have here beta lambda. So, let us say this is my equation number 3 and similarly, let me multiply this equation number 2 on both sides from the right by the lambda ket. So, I will have beta A lambda and that would be equal to beta star beta lambda.

So, let me label it as equation number four. Now if I subtract equation 4 from 3, then you will get on the left-hand side 0 and I will have lambda minus beta star, and this is the scalar product of lambda and beta. Now let me consider 2 cases; say lambda is equal to beta. That means the eigenvalues are the same. Lambda is equal to beta. Then, if this equation has to be satisfied, this particular equation has to be satisfied.

Then, I must have lambda is equal to lambda star. This implies that eigenvalues, as you can see, are real. On the other hand, say lambda is not equal to beta. That means the eigenvalues are different. Then if this equation, say if this equation 5 has to be satisfied, then I must have this scalar product of beta and lambda has to be equal to 0 and this implies that this ket lambda and ket beta are orthogonal and this proves the second part of the theorem.

Because the second part of the theorem says that the eigenkets of a corresponding 2 different eigenvalues are orthogonal. Hence, we have proved both the parts.

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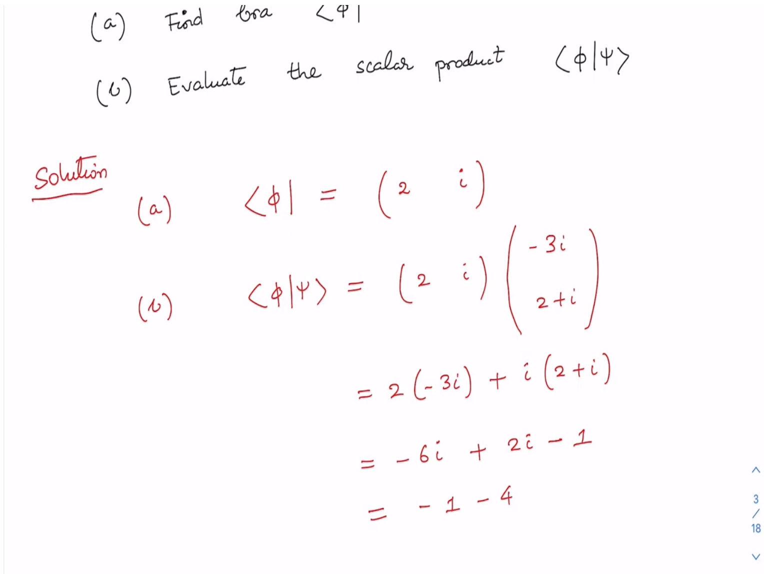

Now let us consider this problem. It is a simple problem. Consider the following 2 state vectors. Ket psi is a column vector. Minus 3i and 2 + i are its element. And ket phi elements are 2 and -i. You are asked to find out the bra phi and in the second part you are asked to evaluate the scalar product of these 2 state vectors. So, let us do it. It is very simple. So, we will have here the bra phi. It would be simply the row vector. Only the transpose you have to do and then you have to take the complex conjugate.

So, it would be 2, i and so this is simple. This is our bra and then the scalar product would be the matrix multiplication of this bra phi and ket psi. So, bra phi is 2i and ket psi is the column vector. So, you have minus 3i, 2 + i. So, we are going to get 2 into minus 3i + i into 2 + i. So, if I do the math, I will get -6i + 2i -1. So, I will have then -1 -4i. So, that is the scalar product.

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$$U^{-1} = \frac{\mathrm{adj}(U)}{\det(U)} = \frac{(U^c)^T}{\det(U)}$$

$$\det(U) = \frac{1}{\sqrt{2}}\left(-\frac{i}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = -\frac{i}{2} - \frac{i}{2} = \underline{\underline{-i}}$$

$$U^c = \begin{pmatrix} -i/\sqrt{2} & -i/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$(U^c)^T = \begin{pmatrix} -i/\sqrt{2} & -1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$U^{-1} = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix} = U^{\dagger}$$

$\Rightarrow$ U is unitary

Let us now work out this problem. You are asked to show that this given matrix is unitary. Now here this matrix U is unitary if inverse of this matrix U is equal to U dagger which is the Hermitian conjugate of the matrix U. Now U dagger Hermitian conjugate of U would be the complex conjugate you take the complex conjugate of U and then you take the transpose. So, if you take the complex conjugate of U you will get 1 by root 2, 1 by root 2, - i by root 2, then it will be + i by root 2. Then, I have to take the transpose of it and this will give me 1 by root 2, 1 by root 2, -i by root 2, + i by root 2.

So, that is U dagger and what about U inverse? U inverse is equal to adjoint of U divided by determinant of U and what about adjoint of U? Adjoint of U is the cofactor matrix of U and then you take the transpose and then you divide it by determinant of U. Now determinant of U is easy to find. You can see that this would be 1 by root 2. Just look at this expression for U; 1 by root 2 into -i by root 2 - i by root 2 into 1 by root 2.

So, we will get -i by 2 -i by 2 that is equal to simply -i. So, that is determinant of U. Now what about the cofactor matrix of U. So, Uc would be equal to, I hope you know how to do it, -i by root 2, -i by root 2, -1 by root 2 and it will be 1 by root 2. Then, if you take the transpose of it, you will get -i by root 2, -i by root 2, -1 by root 2, + 1 by root 2 and therefore, U inverse would be, you have to divide it by -i that is a determinant and this will give us 1 by root 2, -i by root 2, 1 by root 2, i by root 2.

And if you look at this expression for U inverse and if you look at U dagger, you will see that U inverse is equal to U dagger. That implies that U is unitary.

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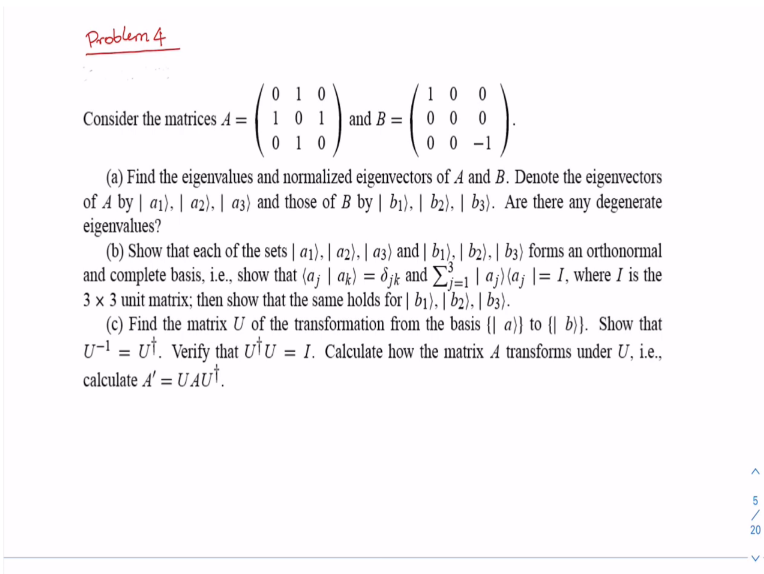

Now let us work out this particular problem. You are given 2 matrices A and B and in the first part of the problem it is asked to find the eigenvalues and normalize eigenvectors of A and B. Denote the eigenvectors of A by ket a, ket a2, ket a1, ket a2, ket a3 and those are B by ket b1, ket b2, ket b3. Are there any degenerate eigenvalues? And part B, you are asked to show that each of the sets a1, a2, a3 and b1, b2, b3 form an orthonormal and complete basis.

And finally, you are asked to find the matrix U, the part c, of the transformation from the basis A to B and you have to show that U inverse is equal to U dagger and thereby verify this, and calculate how the matrix A transform under U. let us do it!

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$$|a_3\rangle = \frac{1}{2}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Eigenvalues of B

$$b_1 = 1$$
$$b_2 = 0$$
$$b_3 = -1$$

$$|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |b_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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So, to do the first part let us work out the eigenvalues of A by solving the characteristic equation. So, we have A is equal to 0, 1, 0, 1, 0, 1, 0, 1, 0 and you know the characteristic equation. You just have to solve this determinant that is, say, here it is minus lambda, 1, 0, 1, minus lambda, 1, 0, 1, minus lambda. This is equal to 0. And if we solve it, we will get the equation as lambda into lambda square -2 is equal to 0. This implies that I have 3 eigenvalues one is lambda 1 is equal to 0, lambda 2 is equal to root 2, and lambda 3 is equal to minus root 2 and because it is eigenvalue of A, so, let me denote lambda 1 is equal to 0 as a1, lambda 2 as a2 and lambda 3 as a3. So, what about eigenvectors? So, eigenvectors or eigenvector corresponding to eigenvalue a1 is equal to 0. I will show you the procedure how to do it. You just have to solve A a1. This is the eigenvalue equation. Eigenvalue is a1. And a1 is a column vector. This ket is a column vector and A is 0, 1, 0, 1, 0, 1, 0, 1, 0 and let me take the elements of the column vector as a11, a12, a13 and a1 is equal to 0 eigenvalue 0.

So, here I have a11, a12, a13 and this is going to give me the equation say, a12 is equal to 0 because, what you are going to have here is 0, 0, 0, a12 is equal to 0, a11 + a13 is equal to 0 and a12 is equal to 0. So, this implies that if I take a11 is equal to 1 then a13 has to be equal to -1. So, therefore, the column vector ket a1 would be a11 is equal to 1, a12 is 0, a13 is equal to -1 and say N is the normalization factor.

So, because I must have to have a1 is equal to 1. Because this is eigenket, it has to be normalized. So, this implies that I will have 1 by N square, 1, 0, -1, 1, 0, -1 is equal to 1. So, if you solve it you will see that you will get N square is equal to 2. So, therefore N is equal to root 2. So, therefore this ket a1 is equal to 1 by root 2, 1, 0, -1. Similarly, you can get a2 and a3 and a2 would turn out to be half 1, root 2, 1 and a3. If you do it in the similar way you will get it as half 1, minus root 2, 1.

Exactly the same way you can find out eigenvalues of B. You will find that there would be 3 eigenvalues. You will get b1 is equal to 1, b2 is equal to 0, and b 3 is equal to -1. I do not want to repeat the calculations here. It is the similar way you do. Please verify that you will get ket b1 as 1, 0, 0. It is already normalized. Ket b2 is equal to 0, 1, 0 and ket b3 is equal to 0, 0, 1. As you can see that none of the eigenvalues of A and B are degenerate here. Now in the second part of the problem part b.

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$$\sum_{i=1}^{3} |a_i\rangle\langle a_i| = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, we have now done part a. In part b you have to show that a1 a2 a3 as well as b1 b2 b3 form complete basis as well as an orthonormal basis. So, first of all let us remind ourselves that the completeness condition is given by this say, lambda i lambda i is equal to identity. So, in the case of a1 the set of this eigenkets a1, a2, a3 forms a complete basis if we have ket a1 bra a1 + ket a2 bra a2 + ket a3 bra a3 is equal to the identity matrix.

Let us show it. You have ket a1 bra a1. So, ket a1 is 1 by root 2, 1, 0, -1 and bra is going to be a row vector. So, you will have 1 by root 2, 1, 0, -1. So, if I do the math you will get half here and you will get 1, 0, -1, 0, 0, 0, -1, 0, +1. Similarly, this one you will get ket a2 outer product of a2 a2 you will get it as 1 by 4, 1 root 2, 1 root 2, 2, root 2. Please verify these things yourself. Simple calculations this is what you will get.

And ket a3 bra a3 will get as 1 by 4, 1, minus root 2, 1, minus root 2, 2, minus root 2, 1 minus root 2, 1. So, now if you sum all these things, that is, i is equal to 1 to 3, ai ai. You will get 1, 0, -1, 0, 0, 0, -1, 0 ,1 + 1 by 4, 1, root 2, 1, root 2, 2, root 2, 1, root 2, 1 + 1 by 4, 1, minus root 2, 1, minus root 2, 2, minus root 2, 1, minus root 2, 1. So, just you have to sum it up. In fact, if you sum it up you will get finally as 1, 0, 0, 0, 1, 0, 0, 0, 1.

So, that is the identity matrix. Therefore, we proved that this set of eigenvectors a1 a2 a3 forms a complete basis.

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$$A' = U A U^{+}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+\sqrt{2} & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1-\sqrt{2} \end{pmatrix}$$

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Following the same procedure similarly, you can prove that the set of eigenvectors b1 b2 b3 also form a complete basis and you will get it as identity matrix. Also, you can show that, very straightforward, a1 is already normalized. These are normalized a2 and a3, they are already normalized to one. Another thing you can show that a1 and a2 they are orthogonal, by this say ket a1 bra a1, this part would be 1 by root 2 1, 0, -1 and ket a2 was half and you have 1, root 2, 1. So, if I do the math here you will get 2 root 2 and you will get it as one then you have here 0, -1. So, clearly you see you will get it as 0. So, similarly you can show a1 a3 is equal to 0, a2 a3 is equal to 0. So, therefore, you see they form an orthonormal basis as well. Same is the case for bis: b1, b2, b3.

Now finally in the third part of the problem you are asked to find the matrix U of transformation from the basis a to b. Let us do that. Let me just remind you how we can write the basic thing that you are going from the basis a to b and let me just take one component say ai under the transfer, I can write it as ai here and then summation bj bj right because this is identity and therefore I can write it as sum over j bj ai. This is just a number and then I can write here it is bj.

So, this I can write as j and this is my transformation matrix Uji bj where Uji is equal to you are going from the basis ai to bj. So, therefore the matrix U of transformation would be b1 a1, b1 a2, b1 a3, here you have b2 a1, b2 a2, b2 a3, b3 a1, b3 a2, and b3 a3. So, you have to find out all the elements. So, let me write it as a11 is equal to b1 a1. So, b1 this is 1, 0, 0 row vector, a1 is 1 by root 2, 1, 0, -1. If you do the math you see that you are going to get it as 1 by root 2. Similarly, you can find out what is U12 that is, your b1 a2 that is equal to 1, 0, 0, a 2 is half, 1, root 2, 1.

So, this is going to give you simply a half and this way you can find out all the elements. U13 you will get it as half. U14, there is no 4. So, you will have it is a 3 by 3 matrix. So, you will have U21 is equal to 0, U22 is equal to 1 by root 2, please verify all these things yourself, U23 is equal to -1 by root 2, U31 you will get it as -1 by root 2, U32 you will get us a half, and U33 you will get is a half.

So, therefore this matrix you can write as 1 by root 2, half, half. You just have to write down the all the elements. You will get here 0, 1 by root 2, -1 by root 2, -1 by root 2, half, half. So, that is the matrix of transformation. Then, finally you are asked to show, you can easily show now that adopting the same procedure as that we have done in the problem three you can find out the U inverse and you can easily find out what is U dagger.

Then it will turn out to be equal to U dagger and then I am not going to do it but it is simple multiplication then you can show that U dagger U will turn out to be an identity a matrix that is, you will get it as 1, 0, 0, 0, 1, 0, 0, 0, 1. Finally, the matrix A transform under this unitary transformation as follow: A dash would be equal to U A U dagger. So, let me quickly do it. U is equal to we have 1 by root 2, half, half, 0, 1 by root 2, -1 by root 2, -1 by root 2, half, half, and A is 0, 1, 0, 1, 0, 1, 0, 1, 0 and U dagger would be equal to 1 by root 2, 0, -1 by root 2 and you will have half, 1 by root 2, half and you will have half, -1 by root 2, half.

So, if you do the multiplication then finally you will get the answer as half 1 + root 2, 1, 1, 1, -2, -1, 1, -1, 1 minus root 2. Please verify all these things yourself. So, I have just given you the procedure how to do these things but if you do the calculation yourself then you will be able to learn it quickly.

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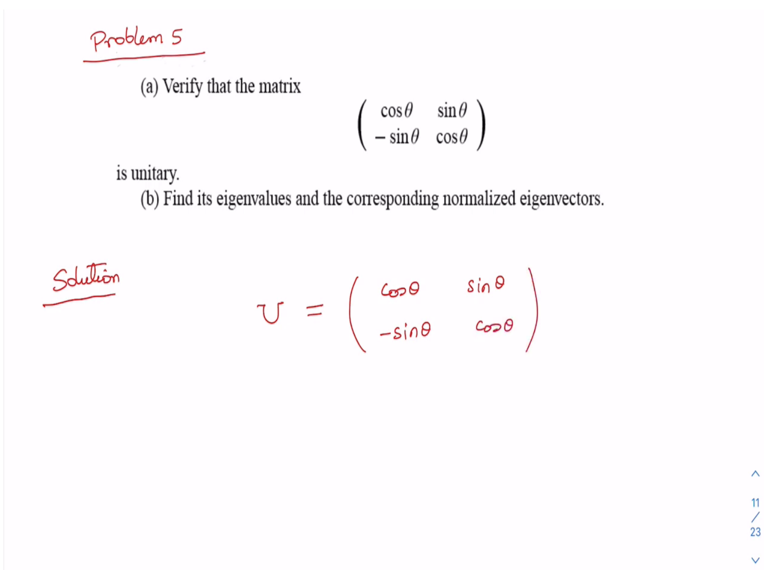

Now let us work out this problem. This is a well-known transformation matrix in 2 dimensional. I have to verify that this is unitary and also you are asked to find out the eigenvalues and the corresponding normalized eigenvectors. So, let us do it. Let me say this matrix let me leave it as

U. This is equal to cos theta, sine theta, minus sine theta, cos theta. To show that this is unitary, you know that we have to show that the inverse of the matrix is equal to its hermitian conjugate that is, U dagger.

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$$\Rightarrow \quad (\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\Rightarrow \quad \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

$$\left.\begin{array}{l} \lambda_1 = e^{i\theta} \\ \lambda_2 = e^{-i\theta} \end{array}\right\} \text{eigenvalues}$$

So, if U inverse is equal to U dagger then U is unitary. First of all, let me find out U inverse. Already we know how to do it but let me repeat again. That would be U inverse is equal to adjoint of the matrix U divided by the determinant of U and adjoint of U is equal to cofactor matrix of U and then you take the transpose divided by determinant of U. Determinant of U is simple to find out. That would be here as you can see this is square theta minus minus sine square theta.

So, determinant is simply one. Cofactor matrix of U as you can see this would be it is cos theta. So, this is cos theta. So, therefore, you'll have here cos theta, and you will have sine theta, and here you will have minus sine theta, and you will have here cos theta. Under transpose of this will give you cos theta, sine theta, minus sine theta, cos theta. what about U dagger? That you take the Hermitian conjugate and then take the transpose. So, this would be simply cos theta, minus sine theta, sine theta, cos theta. So, as you can see U inverse this is equal to U inverse.

So, this implies that U inverse is equal to U dagger. Which clearly shows that U is unitary. All right. Now second part, part b, let us find out the eigenvalues. Eigenvalues of U. To do that you just have to solve the characteristic equation cos theta minus lambda, sine theta, minus sine theta, cos theta minus lambda. This determinant this equal to 0 then you will see that you have lambda minus cos theta or cos theta minus lambda whole square + sine square theta is equal to 0.

So, this is going to give us the quadratic equation lambda square minus twice lambda cos theta + 1 is equal to 0. So, this implies that lambda is equal to 2 cos theta + - 4 cos square theta -4 which means I have cos theta + -i sine theta. That means I have lambda 1 is equal to e to the power i theta and lambda 2 is equal to e to the power -i theta. So, these are the eigenvalues these are our eigenvalues. Now, what about the eigenvectors?

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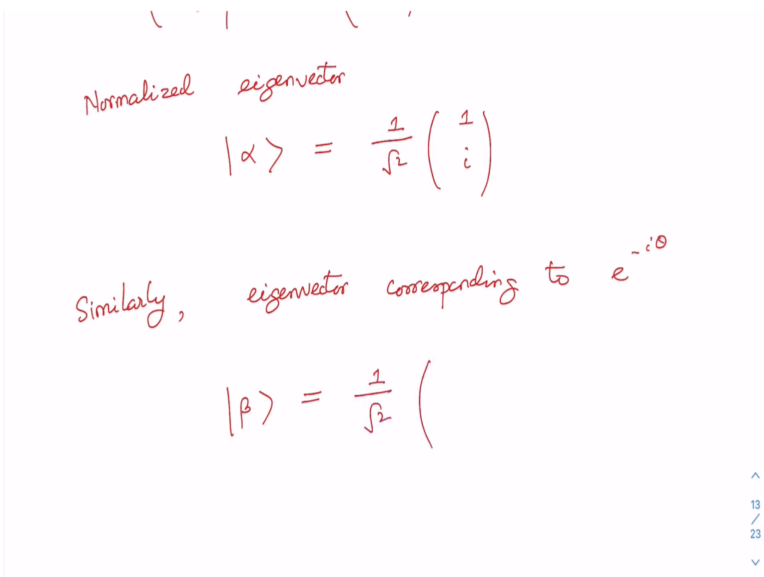

So, first of all eigenvector corresponding to the eigenvalue lambda 1 is equal to e to the power i theta. Let us work it out. So, say cos theta, sine theta, minus sine theta, cos theta. let me denote the eigenvector by this column matrix say, its elements are alpha 1, alpha 2 and eigenvalue is e to the power i theta alpha 1, alpha 2. So, from here I get alpha 1 cos theta + alpha 2 sine theta is equal to e to the power i theta alpha 1. Only one equation is sufficient to solve this problem. I will have here alpha 1 cos theta + i alpha 1 sine theta.

So, if I take consider both sides then I have alpha 2 sine theta is equal to i alpha 1 cos theta.

Sorry, you have here sine theta. This means that if I take alpha 1 is equal to 1 then alpha 2 has to be equal to i. But this has to be normalized. So, you know how to do the normalization. So, normalize eigenvector would be, let me denote the eigenvector as ket alpha, 1 by root 2 1, i.

Similarly, you can show exactly by the similar process. Similarly, eigenvector corresponding to e to the power -i theta you can show that the normalized eigenvector would be ket beta is equal to 1 by root 2 1, –i.

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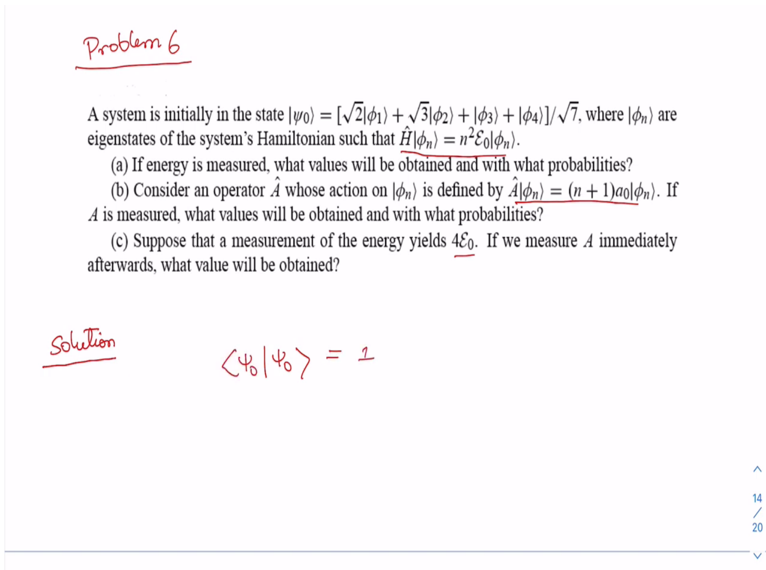

Now let us work out this problem. It is given a system, initially in the state psi 0. It is written as a superposition of eigenstates phi 1, phi 2, phi 3, phi 4, where phi ns are eigenstate of the systems Hamiltonian such that the Hamiltonian satisfies this eigenvalue equation. Now in part a of the problem, if energy is measured what values will be obtained and with what probabilities. In part b, consider an operator A whose action on phi n is defined by this eigenvalue equation. If A is measured what values will be obtained and with what probabilities?

Finally, in part c, suppose that a measurement of energy is 4E0. If we measure A immediately afterwards what value will be obtained. So, let us do it. One thing you can immediately notice is that this state psi 0 is normalized. Right? That is quite obvious if you add it up you will see that this is the state psi 0 is already normalized.

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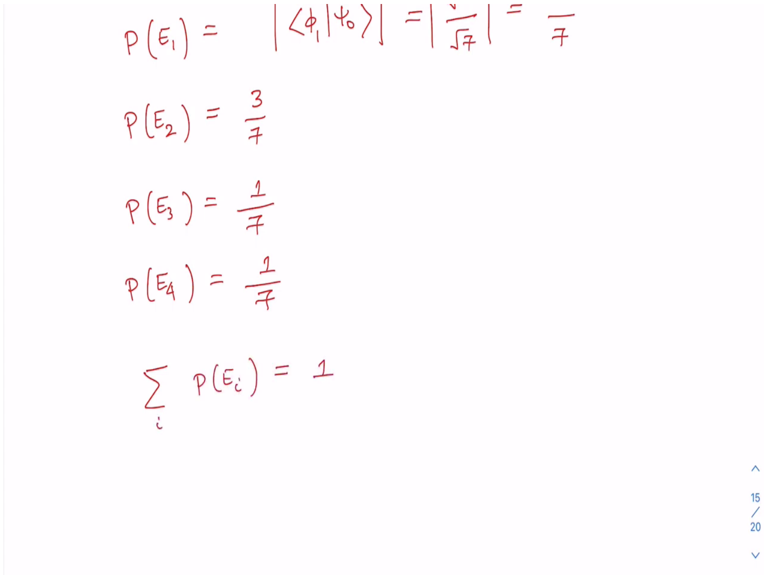

Now, if energy measurement is done then it is going to give us the value En. So, what that would be? This implies that En would be equal to phi n, H phi n is given, n square E0 phi n. Because phi ns are eigenstates, therefore, you will get n squared. This is just a number phi n phi n. This is equal to one. So, therefore En is equal to n square E0. This is the energy eigenvalue one we will get.

So, what one can get? One can get values E1 would be equal to E0, E2 would be equal to 4E0, E3 would be equal to 9E0, and E4 would be equal to 16E0. Then with what probabilities? Well, the probability of getting the energy E1 would be; you are in the state psi 0, then you are going to the state phi 1. Because phi 1 corresponds to the energy state, it will give you in the state E1. Energy would be E 1 there.

So, its probability would be mod square and as you can see from here this expression psi 0. Actually, let me write psi 0 here again. Psi 0 is root 2 by root 7 phi 1 and then we have term root 3 by root 7 phi 2 then we have 1 by root 7 phi 3 and 1 by root 7 phi 4. So, you see immediately that this would be root 2 by root seven mod square and therefore this would be simply 2 by 7. Similarly, probability of getting the energy E2 would be 3 by 7, probability of getting the energy E3 would be 1 by 7, and probability of getting the energy E4 would be 1 by 7.

As you can see that if you actually sum up all the probabilities P(Ei), obviously, you should get it to be equal to 1.

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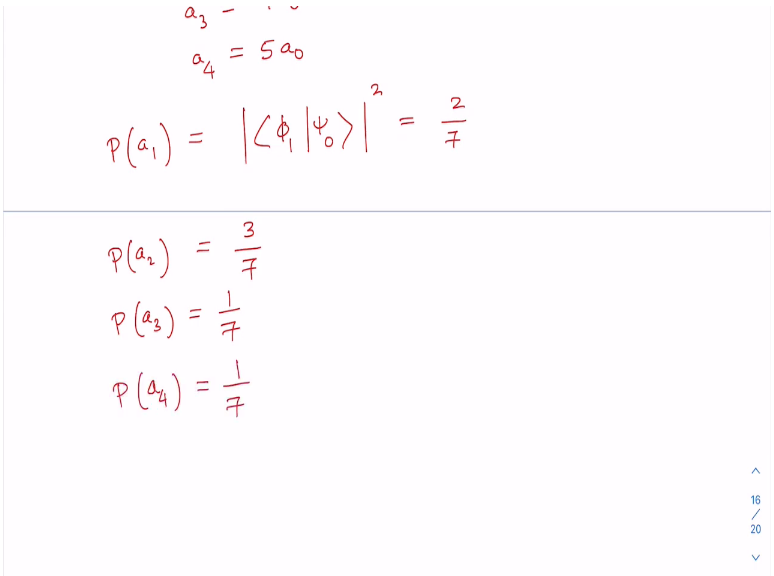

Now let us do the second part of the problem, problem part b. You are given the operator A which satisfy this eigenvalue equation A phi n is equal to n + 1 a0 phi n. So, if a measurement of A is made then you will get phi n A phi n and as you can see this would be let me say this is a n and a n would be equal to n + 1 a0. So, what are the values one can get? One can get say a1 is equal to 2a0, a2 is equal to 3a0, a3 is equal to 4a0, and a4 is equal to 5a0.

Now, what are the corresponding probabilities? So, probability of getting the value a1, if you want to have a1 then you have to be in the state phi 1. You were initially interested psi 0 you have to go from psi 0 to phi 1 then you take the mod square and as you can see from here, from this one, that would be nothing but 2 by 7. Similarly, you will have probability of getting the value a2 would be 3 by 7, probability of getting the value a3 would be 1 by 7, and probability of getting a4 would be 1 by 7.

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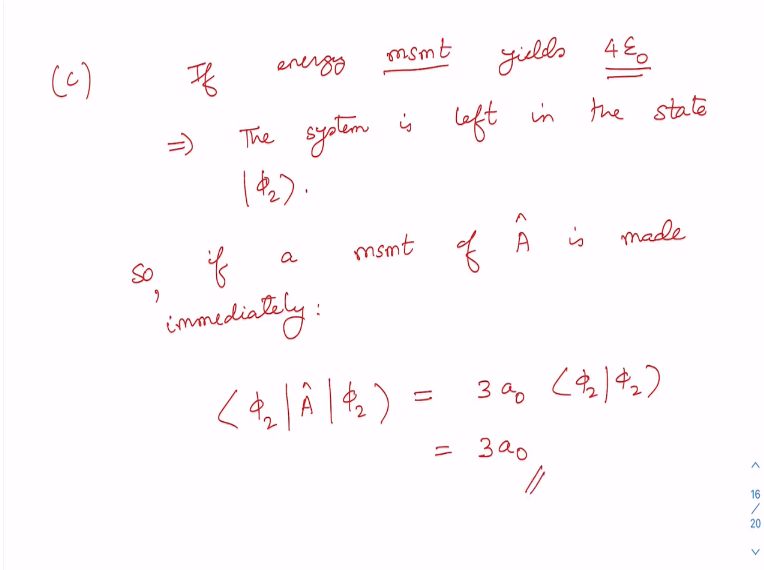

Now let me do the final part of the problem. Here, suppose that a measurement of energy yields 4E0. If we measure A immediately afterwards what value will be obtained? So, if you get the energy 4E0 that means it corresponds to the fact that the system is in the state phi 2. So, if energy measurement is 4E0 for E0. This implies the system is left in the state phi 2 immediately just let me say it is the phi 2.

So, if a measurement of A is made immediately after the measurement of energy which is yielding 4E0 then one will get in the state phi 2. So, therefore you will get phi 2 A phi 2 and this in fact will be 3a0 phi 2 phi 2 which is normalized. So, you will get simply 3a0. So, that is the answer.

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$$\Rightarrow \quad -\left(a^2-\lambda^2\right)-a^2=0$$

$$\Rightarrow \quad \lambda^2 = 2a^2$$

$$\Rightarrow \quad \lambda = \pm\sqrt{2}\,a$$

$$E_1 = \sqrt{2}\,a$$

$$E_2 = -\sqrt{2}\,a$$

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Let us now work out this problem. The Hamiltonian operator for a 2-state system is given by this Hamiltonian, where a is the number with dimension of energy. You are asked to find out the energy eigenvalues and the corresponding energy eigenkets. Let us do it. So, first of all this problem can be done very easily if we can write the matrix form of the Hamiltonian because this is a 2-stage system.

So, the matrix would be a 2 by 2 matrix and its elements would be of this form. So, let me write here. First one would be 1 H 1, 1 H 2, 2 H 1, 2 H 2. Now you can very easily work out these elements. For example, 1 H 1 would be you have 1 here the Hamiltonian a 1 1 -2 2 + 1 2 + 2 1 and then here 1. So, as you can see this will simply give you a, because 1 and 2 would be orthogonal to each other.

So, similarly you can work out 1 H 2 and that also, as you can see, would be simply a. You will have 2 H 1 would be equal to a and 2 H 2 would be equal to minus a. Therefore, the Hamiltonian would be in matrix form. It would be a, a, a, minus a. So, you now just have to find out the eigenvalues of this matrix. Because the Hamiltonian corresponds to energy, so eigenvalues would be energies. So, for eigenvalues you have to solve the characteristic equation.

So, a minus lambda, a, a, minus a minus lambda is equal to 0. If you solve it you will get a minus lambda minus a + lambda minus a square is equal to 0 and you will get minus a square minus lambda square minus a square is equal to 0. So, therefore, lambda square is equal to 2a square. So, you have lambda is equal to plus minus square root of 2a because they correspond to energy eigenvalues. So, let me say one energy eigenvalue is E1 and that is square root of 2a and other one is minus square root of 2a. So, these are the energy eigenvalues.

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$$\Rightarrow \quad N = \frac{1}{2\left(1-\frac{1}{\sqrt{2}}\right)^{1/2}}$$

Thus,

$$|E_1\rangle = \frac{1}{2\left(1-\frac{1}{\sqrt{2}}\right)^{1/2}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

Sly,

$$|E_2\rangle = \frac{1}{2\sqrt{1+\frac{1}{\sqrt{2}}}} \begin{pmatrix} -1 \\ \sqrt{2}+1 \end{pmatrix}$$

So, next is what about the energy eigenket corresponding to E1. Let me find it. So, we have to set this eigenvalue equation. The Hamiltonian is a, a, a, minus a and let us say the eigenvector has this column vector. Its elements are e1, e2. Eigenvalue is root 2a and the vector is e1, e2. So, this is going to give us, only one equation is sufficient, a e1 + a e2 is equal to square root of 2 a e1. This means I can have 1 minus square root of 2 e1 + e2 is equal to 0.

Therefore, I can choose e1, e2 this way. If I choose say e1 is equal to 1 then e2 has to be root 2 -1 then only this equation would be satisfied. Now this is what the eigenvector is. But this has to be normalized. So, let me denote the eigenvector by this ket E1 and this is the normalization factor. So, I have 1 root 2 -1 and because it has to be normalized so E1, E1 is equal to 1. So, this will give us N square 1, root 2 -1, 1, root 2 -1 is equal to 1.

So, you will get N square 1 root 2 -1 whole square. So, this will give me 2 + 1 -2 root 2 is equal to 1. So, this means I have N square is equal to 1 divided by 4 -2 root 2 which I can write as 1 by 4 1 minus 1 by root 2. This means N is equal to 1 by 2 1 -1 by square root of 2 half. So, therefore we get energy eigenvector correspond to the energy eigenvalue E1 as one by 2 1 -1 by root 2 to the power half, and here I have 1 root 2 -1. Similarly, please verify it.

So, for the other one that would be ket E2 equal to 1 divided by 2 square root of 1 + 1 by root 2, you will have here -1 root 2 + 1. So that is the answer!