Quantum Technology and Quantum Phenomena in Macroscopic System
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Lecture - 19
Quantization of Transmission Line – II

Hello, welcome to lecture 14 of the course, this is lecture number 4 of module 2. In the last class, we saw how to quantize a discrete transmission line. In this lecture, we will learn how to write the quantized Hamiltonian in the form of a simple harmonic oscillator Hamiltonian ,also we learn how to quantize a transmission line resonator. So, let us begin.

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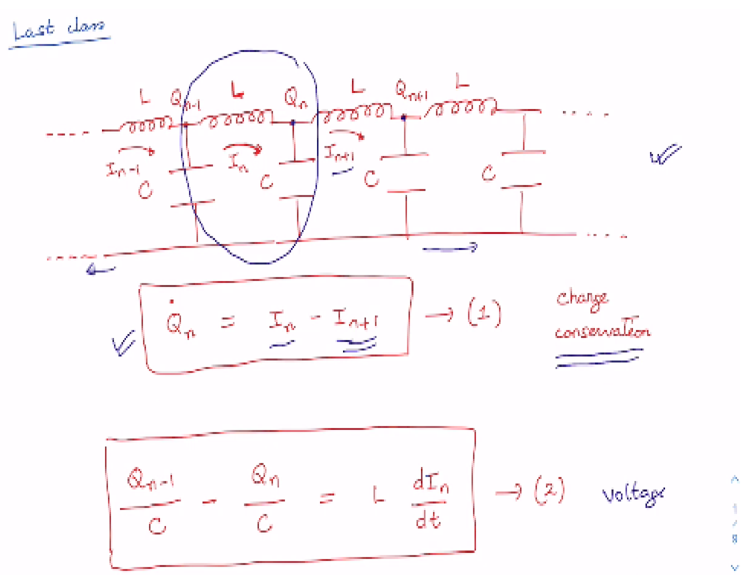

In the last class, we discussed the circuit model photo transmission line, there we consider a collection of LC oscillators as shown in this figure and its set or its branch of LC oscillators are equivalent. And in fact, we have transitional symmetry in the transmission line whether we go towards this direction or in the other direction, we are going to encounter the same kind of branch here and we took there are many, many nodes there say for example, in the nth node we took the charge to be Q 1 and in the (n + 1)th node the charge is taken to be Q n + 1 and so, on.

The current that is flowing towards this node is denoted as nth node as I n and the current flowing towards the node n + 1 is denoted as I n + 1 and based on this model, we got this charge conservation that is for example, here in the nth node, if you look at it the rate of the charge, time rate of change of charge in the nth node is equal to the charge that is flowing into the node and the charge so the current that is flowing into the node and the current flowing out of the node.

So, this is what the charge conservation gives us equation number 1 and equation number 2 is basically the voltage law and this Q n - 1 / C and – Q n / C that is the voltage drop that is happening across these nodes here between nth node and n - 1th node and these voltage drop is responsible for this current I n. And this is what we got is our equation number 2.

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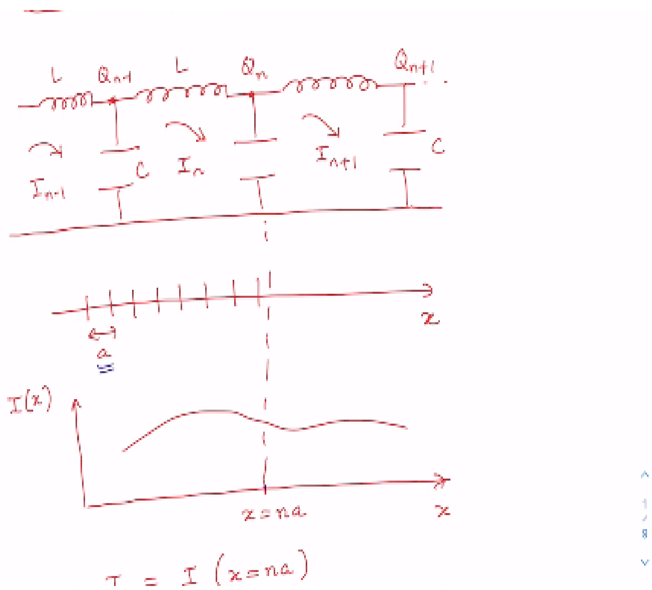

And then we went on to discuss the continuum model for this discrete transmission line .What we have done is that we divided the transmission line into a number of units cell of cell size of length a and then considering the current to be uniform across this or continuous across these transmission lines. Say for example, if we want to know the current at a particular node say n, so, we have to find the current at x = na.

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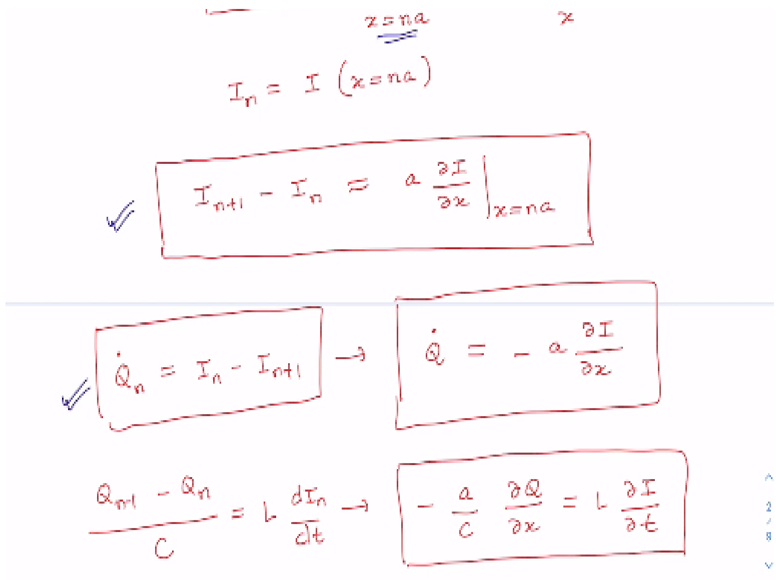

And then using the you know differential calculus as I discussed earlier, so, we can write this equation for difference in current between the nodes and similarly, for this charge conservation equation in the continuum domain we get this equation. And that is how we got even the wave equation for the current. We found that the current flows through the transmission line as a wave with some velocity v.

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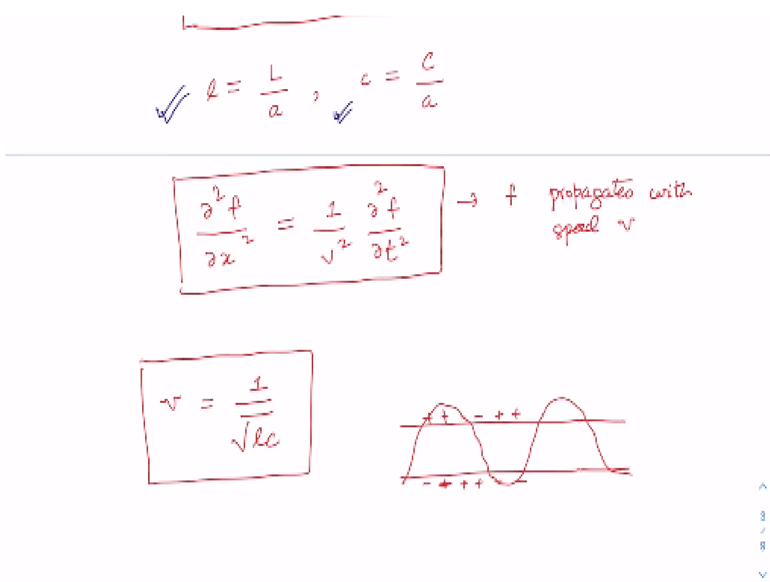

And v is given to be square root of l into c where l is the inductance part unit length and c is the capacitance per unit length.

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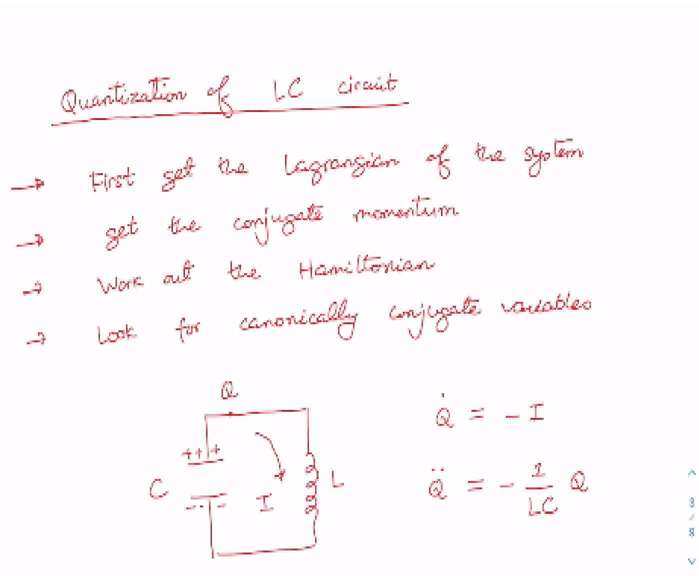

So, this is what we discussed and then we went on to discuss the quantization of the transmission length .For that first we discuss quantization of LC circuit, which is the basic unit cell of the transmission line. The idea was that first to get the Lagrangian of the system that is how we quantize canonical quantization procedure is like that or that, we just you take any classical system you first get the Lagrangian of the system, then find out the conjugate momentum, then work out the Hamiltonian, then look for canonically conjugate variables once you find it.

Then you can write down the commutation relation for these variables and that is how you quantize it. So, we adopted that method for LC oscillator also.

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$$\mathcal{L} = \frac{1}{2} L I^2 - \frac{Q^2}{2C}$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{2} L \dot{Q}^2 - \frac{Q^2}{2C}}$$

$$\frac{\partial \mathcal{L}}{\partial Q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\right) = 0$$

$$\Rightarrow \frac{Q}{C} - L\ddot{Q} = 0$$

$$\Rightarrow \boxed{\ddot{Q} = -\frac{1}{LC} Q}$$

Here we have written down the Lagrangian for the LC oscillator taking charge is the coordinate if we take charge is the coordinate then we get this is our Lagrangian and the fact that this Lagrangian is actually the correct Lagrangian we can get it, we can verify it by putting it in the Lagrangian equation of motion if we do that, then we get this equation of motion and which is well known.

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$$\mathbb{P} = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = L\dot{Q} = -\underbrace{LI} = -\underline{\underline{\Phi}}$$

magnetic flux

$$H = H(\underline{\underline{Q}}, \underline{\underline{\Phi}})$$

$$H = \dot{Q}\mathbb{P} - \mathcal{L}$$

$$= L\dot{Q}^2 - \left(\frac{1}{2} L\dot{Q}^2 - \frac{Q^2}{2C}\right)$$

$$= \frac{Q^2}{2C} + \frac{1}{2} L\dot{Q}^2$$

$$\phi = LI$$

So, the Lagrangian that is chosen is correct and from that once we get the Lagrangian, we can find out the momentum and momentum turns out to be minus or negative of this magnetic flux, magnetic flux phi is L into I and then we can find out the Hamiltonian we have found that this magnetic flux is basically the momentum and charges the coordinate. So, the Hamiltonian is a function of charge and the magnetic flux. So, we get the ultimately the Hamiltonian for the LC oscillator and we find that both the charge and the magnetic flux are canonically conjugate variables because it satisfies the Hamilton's canonical equation of motion.

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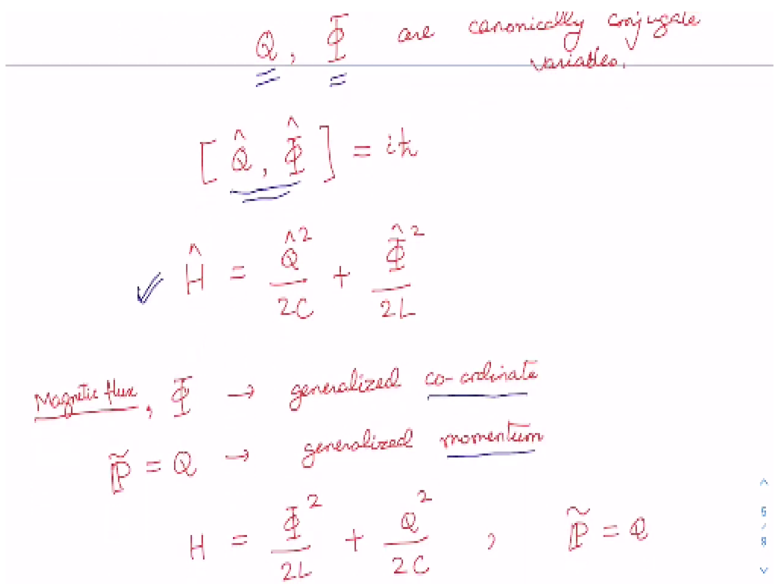

And therefore, we can now do the quantization procedure .We just have to replace it charge in the magnetic flux by the corresponding quantum mechanical operator and this commutation relation we just separate commutation is equal to i h cross and the Hamiltonian accordingly would be written in the operator form. And we discuss another thing that this LC oscillator can be quantized even by taking this magnetic flux as the generalized coordinate and charges the generalized momentum because in Hamiltonian formalism both coordinate and momentum are given equal weightage. So, it is not going to make any difference.

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$$\dot{\Phi} = \frac{\partial H}{\partial \tilde{P}} = \frac{Q}{C}$$

$$\Rightarrow L\dot{I} = \frac{Q}{C}$$

$$\dot{\tilde{P}} = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I$$

$$\Rightarrow \dot{Q} = -I$$

$$\boxed{\mathcal{L} = \frac{C}{2}\dot{\Phi}^2 - \frac{\Phi}{2L}}$$

But as we saw that taking magnetic flux has an advantage over that of charge when we discuss quantization of the transmission line. So, taking this we saw that we can actually quantize the LC oscillator only thing the Lagrangian would be here different.

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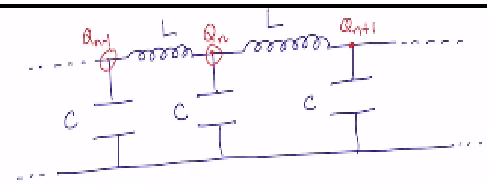

$$\mathcal{L} = \frac{1}{2}LI^2 - \frac{1}{2}\frac{Q^2}{C}$$

$\equiv$ magnetic energy $-$ electrical energy / charging energy

Given Q, find I

$$\dot{Q}_n = I_n - I_{n+1} \quad \text{(charge conservation)}$$

$$\Rightarrow I_n = \dot{Q}_n + I_{n+1}$$

And this Lagrangian is basically this part of this half LI square is the magnetic energy and this Q square by C is called the electrical energy or charging energy. Now, if we actually try to quantize the transmission line taking charge as the coordinate not magnetic flux, then we run into some trouble there also we discussed .For example, if we take, because as you can see from this

Lagrangian that to get the Lagrangian we have to know the current as well as the charge. So, suppose we know charge, then we have to find current I.

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Given Q, find I

$$\dot{Q}_n = I_n - I_{n+1} \quad \text{(charge conservation)}$$

$$\Rightarrow I_n = \dot{Q}_n + I_{n+1}$$
$$= \dot{Q}_n + \dot{Q}_{n+1} + I_{n+2}$$
$$= \dot{Q}_n + \dot{Q}_{n+1} + \dot{Q}_{n+2} + \dot{Q}_{n+3} + \cdots$$

$$\frac{1}{2} L I_n^2 \equiv \text{Infinite sum}$$

→ let us take Φ, magnetic flux as co-ordinate.

And we saw that, if we do that, even in a particular single node, we cannot find out the current because this particular energy term results in an infinite sum and who is we cannot find, and you have to remember that we have to find actually this current, this magnetic energy term it each and every node and even a particular node we are running into problems if we take charge is the coordinate.

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$$= \dot{Q}_n + \dot{Q}_{n+1} + I_{n+2}$$
$$= \dot{Q}_n + \dot{Q}_{n+1} + \dot{Q}_{n+2} + \dot{Q}_{n+3} + \cdots$$

$$\frac{1}{2} L I_n^2 \equiv \text{Infinite sum}$$

→ let us take Φ, magnetic flux as co-ordinate.

$$\phi_n = \int_{-\infty}^{t} V_n(t')\,dt' = \int_{-\infty}^{t} \frac{Q_n(t')}{C}\,dt'$$

$$\Rightarrow \boxed{Q_n = C\dot{\phi}_n}$$

So, therefore, we saw that we have to take magnetic flux as the coordinate. And if we take the magnetic flux as the coordinate, charge is very easy to find out because it will be simply the time derivative to the magnetic flux multiplied by the capacitance, then we will get the charge.

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→ what about current?

$$\dot{\phi}_{n-1} - \dot{\phi}_n = \frac{Q_{n-1} - Q_n}{C} = L \dot{I}_n$$

$$\Rightarrow \quad \phi_{n-1} - \phi_n = L I_n + \underbrace{\text{constant}}_{=\,0}$$

$$\underbrace{\phi_{n-1} - \phi_n}_{\text{Fluxes at nodes}} = \underbrace{L I_n}_{\text{magnetic flux in coil/branch}}$$

Now, what about the current? Current can also be found very easily, we found that a current would be simply given by the difference of the fluxes at the nodes and thereby we will be able to find out the magnetic, actually the current at a particular node and then we can write down the Lagrangian.

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$$\mathcal{L} = \text{electrical energy} - \text{magnetic energy}$$

$$= \sum_n \left[\frac{C}{2} \dot{\phi}_n^2 - \frac{1}{2} L I_n^2 \right]$$

$$\boxed{\mathcal{L} = \sum_n \left[\frac{C}{2} \dot{\phi}_n^2 - \frac{(\phi_{n-1} - \phi_n)^2}{2L} \right]}$$

- $\phi_{n-1} - \phi_n \rightarrow -a \frac{\partial \phi}{\partial x}$
- $a \sum_n \rightarrow \int dx$ $\quad \left[c = C/a, \; \ell = L/a \right.$

Now, when we took charges our coordinate then the Lagrangian was magnetic energy minus electrical energy, but now, we are taking the reverse. So, it is electrical energy minus magnetic energy. And we got this Lagrangian for the discretized model of the transmission line. And we saw that we can actually get the Lagrangian for the continuum model of the transmission line as well.

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$$\mathcal{L} = \sum_n a\left[\frac{c}{2}\dot{\phi}^2 - \frac{a}{2L}\left(\frac{\partial \phi}{\partial x}\right)^2\right.$$

$$\Big\downarrow \; a \to 0$$

$$\mathcal{L}_{continuum} = \int dx \left[\frac{c}{2}\dot{\phi}^2 - \frac{1}{2l}\left(\frac{\partial \phi}{\partial x}\right)^2\right]$$

$$P_n = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = c\dot{\phi}_n = Q_n$$

$$\mathcal{H} = \sum_n P_n \dot{\phi}_n - \mathcal{L}$$

So, taking the usual procedure I saw that when in the limit a tends to 0 that means when cell size is vanishing, then we get the continuum model. And but actually for quantization purposes, as I said, that we are going to take the discretized model along with some boundary conditions, that is what we are going to discuss today and is taking this discrete model it is very straightforward to do the quantization of the discrete model, we just have to find out the momentum and obviously, momentum turns out to be the charge here and then we can write down the Hamiltonian.

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$$= \sum_n \left\{ \frac{P_n^2}{2C} + \frac{(\Phi_{n-1} - \Phi_n)^2}{2L} \right\}$$

Quantization

$$\Phi_n \rightarrow \hat{\Phi}_n$$

$$P_n \rightarrow \hat{P}_n$$

$$[\hat{\Phi}_n, \hat{P}_n] = i\hbar$$

And Hamiltonian is simply this. And we know that this momentum and this P n actually this magnetic flux and the current are canonically conjugate variables. So, therefore, quantization is straightforward. So, this coordinate we take now write it as an operator and momentum has a operator here and then we can improve this quantization rule.

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$$[\hat{\Phi}_n, \hat{P}_n] = i\hbar$$

More generally, $[\hat{\Phi}_n, \hat{P}_{n'}] = i\hbar\, \delta_{nn'}$

$$[\hat{\Phi}_n, \hat{P}_{n'}] = 0 \qquad n \neq n'$$

$$\rightarrow \boxed{\hat{H} = \sum_n \frac{\hat{P}_n^2}{2C} + \frac{(\hat{\Phi}_{n-1} - \hat{\Phi}_n)^2}{2L}}$$

And thereby, we can quantize the ,you know, the transmission line descript transmission line model there, but as I said at the end of the class that this Hamiltonian is not still in useful form we have to we can write it in a more useful form using the normal coordinates and that is what now, we are going to discuss. Because of translational invariance, we can immediately guess that normal modes would be some kind of plane waves.

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$$\boxed{\hat{\mathcal{H}} = \sum_n \frac{P_n^2}{2C} + \frac{(\phi_{n-1} - \phi_n)^2}{2L}}$$

e^{ikx}

Normal modes can be assumed to be plane waves

$k \rightarrow$ wave vector

Say e to the power ikx, where k is the wave vector and x is the position on the transmission line. So, normal modes can be assumed to be or taken to be plane waves and we want that this k wave vector, k is the wave vector and this wave vector we want it to be discretized, it is just like the same strategy we have applied when we have quantized electromagnetic wave if you can remember from a first module where because of boundary conditions these wave vector k got discretized here also we are going to adopt the same strategy.

And to make it discretized we are going to adopt periodic boundary conditions and in this regard transitional invariance of the transmission line is going to help because you know, in the, in the transmission line whether we go towards you know we go towards right or towards left it does not matter because it is a infinite transmission line and every branch in the transmission line is equivalent.

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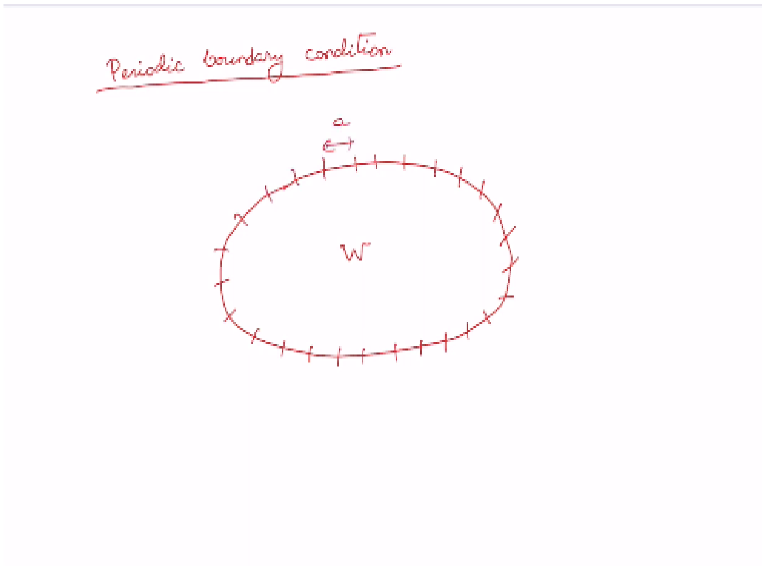

So, because of all these things ,mainly because of the transitional invariance, we can take our transmission line or imagine our transmission line to be some kind of a ring here, say ring of circumference w and this transmission line has to be of course, now divided into some units cell size is ,cell having a length a. So, let us divide it into a number of units cell size a and we want that this transmission line has to support a plane wave.

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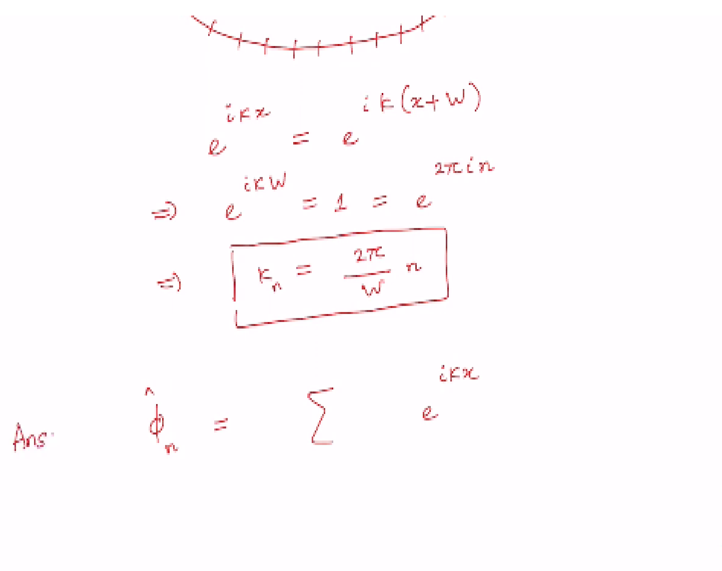

And it has to satisfy this condition ,periodic boundary conditions such as pi at any point x say, any point x which is equal to x = n into a. It has to be equal to e to the power ikx + w. So, let us say you have, this is your point x then if you come back to this point you will go in this direction again come back to this point x. So, you should have e to the power of ikx = e to the power of ikx into x + w. So, this implies that I must have e to the power ikw = 1 which I can write as e to the power of 2 pi i into some integer.

So, immediately you see that our k would be equal to 2 pi / w into some integer. So, therefore, this wave vector k is now getting discretized. Now, our strategy would be as follows we have to find an answers for our magnetic flux, we want to decompose into a number of plane waves, basically we want to express our magnetic flux as a sum of plane waves, superposition of plane waves, see e to the power ikx and we are going to put these answers, we are not going to guess the answers, we are going to put these answers in this Hamiltonian.

That is what we are going to do and then we are going to demand does these Hamiltonians look like a simple harmonic oscillator Hamiltonian like h cross omega a they got a + half where a is there any relation operator and a dagger is the creation operator. So, this is our strategy.

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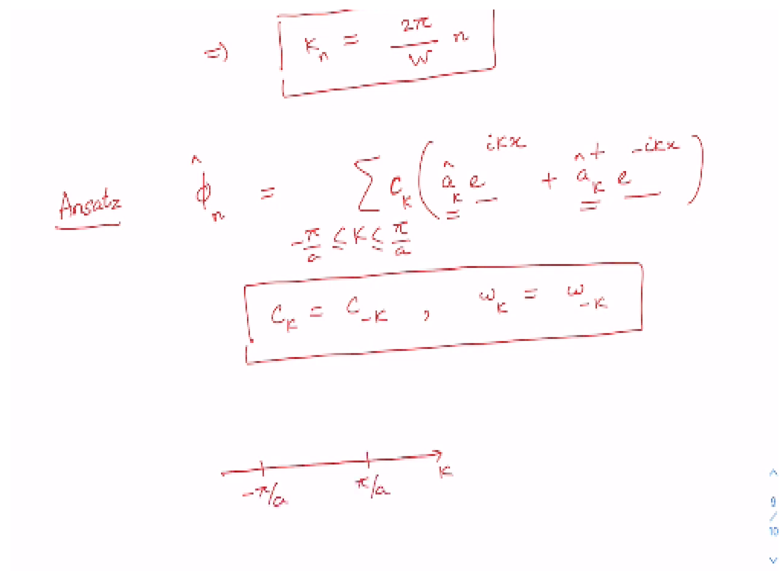
So, I have to put here this annihilation operator a k and because this magnetic flux is Hermitian so, I must have a term like this a k dagger e to the power - ikx. Now, e to the power ikx this term is dimensionless. This is dimensionless .Now, annihi-, creation operator and annihilation operator are dimensionless. So, therefore, I must have to multiply it by an appropriate constant say k for the kth mode and I am taking the sum over all the modes k.

And this constant c k has the dimension of magnetic flux and this constant we are going to determine later and also because of transitional invariance whether we go towards right or towards left it does not matter. So, we must have to have c k should be equal to c - k. And for the same reason, because of transitional invariance we must have omega k would be equal to omega - k. Another important point to note here that this k lies between -pi / a to + pi / a, this could be understood as follows, say our k is lying between - pi / a to + pi / a.



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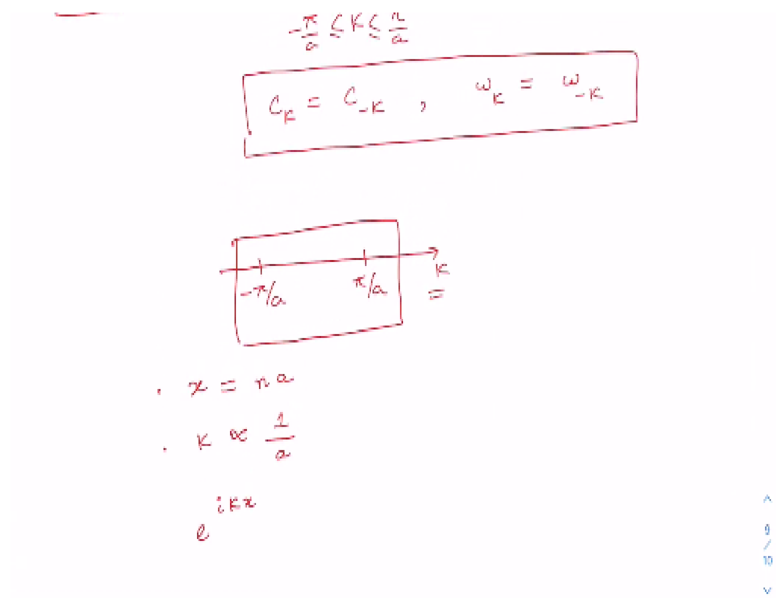

Say we take a large k value beyond this range that we are taking c k is larger than pi / a and we evaluate it at some discrete value of x and we know that x is equal to an integer n into a. So, x is scaled as a and on the other hand, the k is inversely proportional to the cell size. So, therefore, if we take a large scale that is equivalent to considering a plane wave at k, because you see e to the power ikx because of these relations, so, it does not going to make any difference. So, therefore, we can take k to lie between - pi / a to + pi / a.

So, now, we have these answers for this magnetic flux or the coordinate and we have to put it in this Hamiltonian but, but we need to know the answers for this momentum and that is easy to get because we know that this momentum P n is related, we have actually worked it out earlier that this is related to the magnetic flux because by these expressions. So, P n is equal to , we did it right here.

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$$\boxed{\hat{P}_n = c\,\dot{\hat{\phi}}_n}$$

$$\hat{a}_k \propto e^{-i\omega_k t}$$

$$\hat{P}_n = \sum_k c A_k \left(-i\omega_k\right)\left\{\hat{a}_k e^{ikx} - \hat{a}_k^{\dagger} e^{-ikx}\right\}$$

$$\hat{H} = \sum_n \frac{\hat{P}_n^2}{2C} + \frac{\left(\hat{\phi}_{n-1} - \hat{\phi}_n\right)^2}{2L}$$

So, P n = c into phi n where phi n dot. So, this is what we have and phi n is this, now there may be a confusion because I have taken c as my capacitance. So, let me take this constant difference. So, let us say this is a not c so this would be a. Now I have to take the time derivative of this magnetic flux. Now you see the presence of these annihilation operator and the creation operator here. And you know from Module 1 that these annihilation operator, it actually varies with time.

It is pre-evolution get like this e to the power i omega k t. So we can exploit this, when we exploit this, then this momentum operator, we can write it like this. So, let us if we just take the time derivative, then we will get it c into A k. That is the constant now, we are taking it - i omega k, this is sum over all k, and we have a k operator e to the power ikx - a k dagger e to the power - ikx. So, we have both now this momentum operator as well as this coordinate phi at our disposal, we need to insert a coordinate phi n and the momentum P n into this Hamiltonian.

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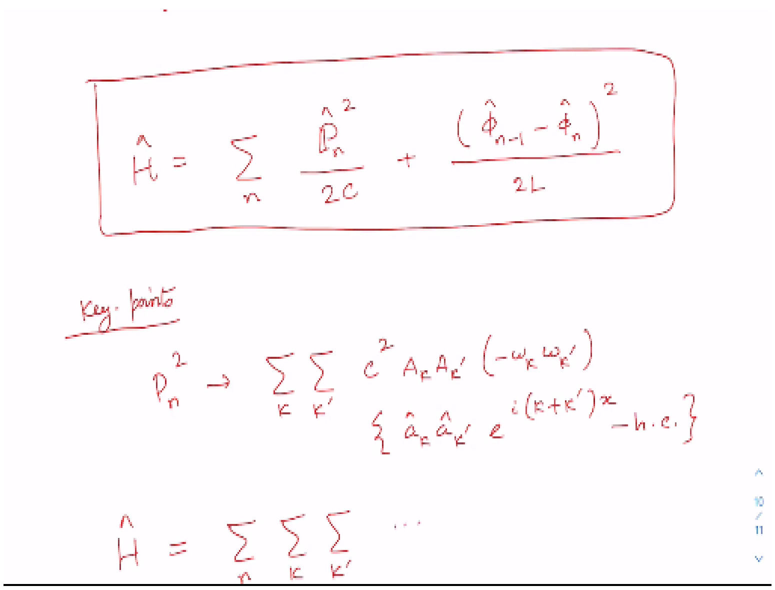

Now, this involves a little bit of straightforward but tedious calculations. So, I am going to avoid giving you the details of the calculations but we can work out the details in a problem solving session later. Now, here are the key points of the derivations. If you see this Hamiltonian here, we have to take P n square. Now, while taking P n square, we are going to encounter 2 sums over k, say P n square, if I take it then I will have sum over k and then another sum over k dash and we will get terms like c square A k A k dash - omega k omega k dash.

And then we will have term a k a k dash e to the power i k + k dash x and then Hermitian conjugate. So, therefore, in the Hamiltonian will have 3 sums 1 would be sum over n another would be sum over k and then sum over k dash. However, things get simplified because of an orthogonality condition.

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- Orthogonality condition

$$\sum_n e^{i(k+k')x} = \delta_{k,-k'} \frac{W}{a}$$

If $k = -k'$, $e^{i(k+k')x} = 1$

So, $\sum_n \{ \quad \}$ = Total no. of cells

$$= \frac{W}{a}$$

Let me write that first then I will explain because of this orthogonality condition .Say , because of this orthogonality condition sum over n e to the power i k + k dash x would be equal to delta k - k dash w / a, w is the circumference of the ring when we have taken the transmission line imagine as a ring and a is the cell size.Let me explain this orthogonality condition. So, say if we take k = - k dash then you will see that e to the power i k + k dash x = 1. So, therefore, we will have this sum over n and then this would be simply total number of cells in the transmission line. And that would be equal to length of the transmission line divided by the cell size.

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$$= \frac{W}{a}$$

If $k \neq -k'$

$$\sum_n e^{i(k+k')x} = 0$$

$$\downarrow$$

$$\frac{1}{a}\int \underline{dx\, e^{i(k+k')x}} = \frac{2\pi}{a}\,\delta(k+k')$$

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On the other hand, if k is not equal to -k dash then this sum over n e to the power i k + k dash x would be equal to 0 if I you can quickly understand it by going over to the continuum limit if I go to the continuum limit, continuum domain then this summation sign would be replaced by an integral then this would be dx 1 by integral dx e to the power i k + k dash x. Now you know this is nothing but the Dirac delta function. So, you have 2 pi / a delta k + k dash and you know that when k is not equal to -k dash this term would be equal to 0. So, that is how all this information that just I discussed is contained in this orthogonality condition .

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$$\rightarrow \quad \hat{H} = \sum_{k} \cdots \left(\hat{a}_k^{\dagger} \hat{a}_k + \frac{1}{2} \right)$$

$$\rightarrow \quad a_k a_{k'} \, \frac{W}{a} A_k^2 \underbrace{\left\{ -\frac{C}{2} \omega_k^2 + \frac{2}{L} \sin^2\left(\frac{ka}{2}\right) \right\}}_{= 0}$$

$$\Rightarrow \quad \omega_k^2 = \frac{4}{LC} \sin^2\left(\frac{ka}{2}\right)$$

So, due to this orthogonality condition, we'll end up in a Hamiltonian where we have a single sum over k and there would be other it will be terms in the Hamiltonian. So, one of the terms in the Hamiltonian would be like this say a k a k dash w / a A k square - c / 2 omega k square + 2 / L sin square ka / 2. Now, clearly a term like this containing bilinear combination of annihilation operator is not wanted, as we expect that our Hamiltonian is going to have this kind of a form that would be a k dagger a k kind of form we are expecting in after doing all the calculations.

So, therefore, a term like this is we have to actually avoid and we can actually get rid of this kind of a term, if I make this term here to be equal to 0, who is actually imply choosing our frequency omega k as this you can immediately make it out from this expression here that omega if I choose omega k square = 4 by LC sin square ka / 2, then we will be able to get rid of this term containing bilinear combination of this annihilation operators.

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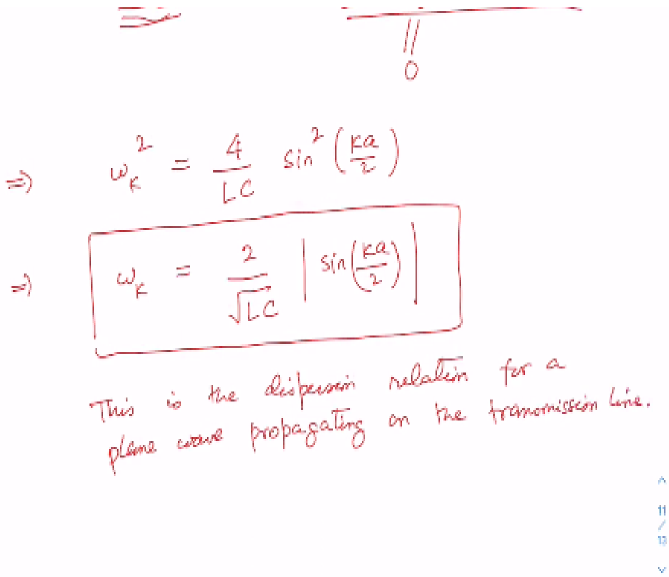

And because omega k is positive, so omega k, I can write it as this magnitude would be 2 / square root of LC mod sin ka / 2. In fact, this is the dispersion relation for plane wave propagating on a transmission line. So, this is the dispersion relation for a plane wave propagating on the transmission line, let us plot the so, called w-k diagram based on this relation for the transmission line.

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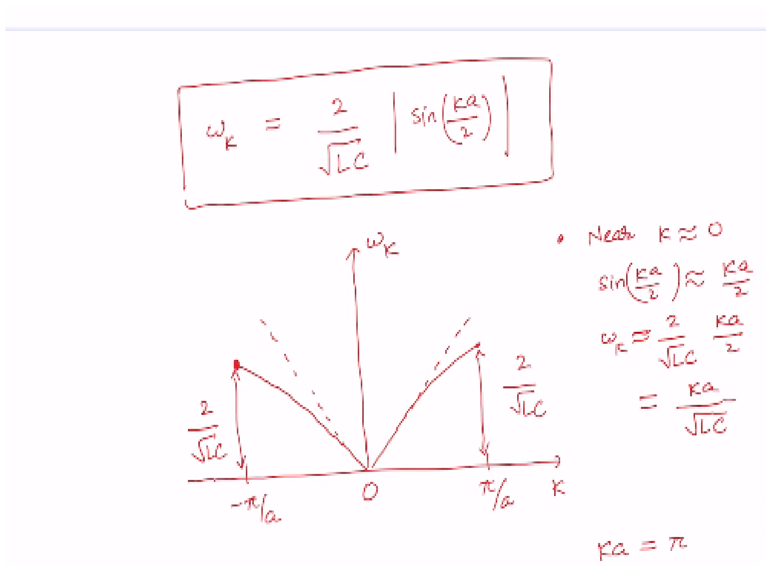

Let me take my k along the x axis and frequency along the y axis ,k is lying between say -pi / a to +pi / a as I discussed earlier all other cases are equivalent. Now, near k = 0, near k = 0 I can write sin ka / 2 to be equal to simply ka / 2, then I have omega k = 2 / square root of LC ka / 2 or ka / square root of LC. So, clearly omega is linear in k. So, therefore, near k = 0 this omega k curve would be linear, we will have a linear curve here.

And again if you look at ka = pi, then you will have omega k would be equal to 2 / square root of LC. From this expression ka / 2 would be pi / 2 so, that will be 1. So, you will have here say, you have this height here would be 2 / square root of LC. Similar is the case here at this other in here you will have this height would be 2 / square root of LC. So, I can expect. So it will deviate from this linear line and it will go like this and similarly here it will be, you will have so, you have this would be linear near k = 0, but as we deviate from k = 0. It will have a plot like this it will deviate from the straight line.

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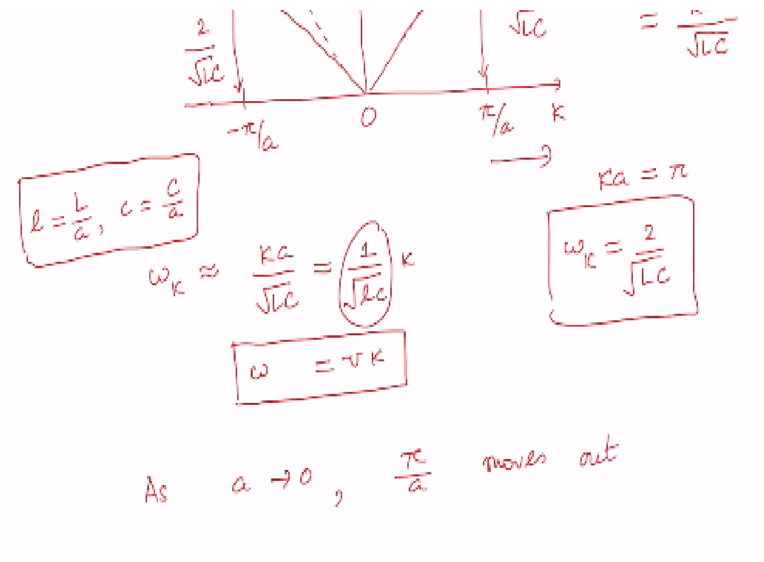

So, omega k would be the height would be this much. So, clearly in the linear region as you can see omega k = ka / square root of LC who is I can actually write it as 1 / square root of lc into k and if you remember this expression is nothing but the velocity of the wave and omega k in the continuum , we actually this is what we got the earlier analysis shows that the wave travels with a constant speed v because l and c they are the l is the inductance power unit length of the transmission line.

And c is the capacitance per unit length. So, and they are independent of the size of the unit cell. So, in fact, when we are near k = 0 or that means, in the when we are actually near very small wave number k or large wave length as expected we are in the continuous limit. So, as a tends to 0 is the cell size tends to 0 pi / a, as a tends to 0 pi / a moves out as you can see these will move out and similarly this side and will follow linear dispersion curve, this linear curve would go on extending is the scale size tends to 0.

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$$\hat{H} = \sum_k \frac{2W}{a} A_k^2 \left\{ \frac{c}{2}\omega_k^2 + \frac{2}{L}\sin^2\left(\frac{ka}{2}\right) \right\} \left(\hat{a}_k^{\dagger} \hat{a}_k + \frac{1}{2} \right)$$

$$-\frac{c}{2}\omega_k^2 + \frac{2}{L}\sin^2\left(\frac{ka}{2}\right) = 0$$

$$\Rightarrow \frac{2}{L}\sin^2\left(\frac{ka}{2}\right) = \frac{c}{2}\omega_k^2$$

Now, let us look at the other terms of the Hamiltonian. Now, we will have another term in the Hamiltonian there to look like this that will be sum over k 2w / a A k square and we have this term c / 2 omega k square + 2 / L sin square ka / 2 and this would be multiplied by a k dagger a k + half and this is the kind of term we wanted because we have you see, we have even this is zero point fluctuation term is also there and this is the desired term because of the presence of this a k dagger a k.

And using that relation that, we, dispersion relation that we obtained this relation that we have because of this relation, in fact this relation I can let me write it here because, so, these relations c / 2 omega k square + 2 / L sin square ka / 2 = 0, I can replace the sin square ka / 2 or these implies I can write the 2 / L sin square ka / 2 = c / 2 omega k square. So, if I put this expression here.

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$$\Rightarrow \quad \frac{2}{L}\sin^2\left(\frac{ka}{2}\right) = \underline{\underline{\frac{C}{2}\omega_k^2}}$$

$$\hat{H} = \sum_k \frac{2W}{a} A_k^2\, C\, \omega_k^2 \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right)$$

$$= \sum 2Wc\, A_k^2\, \omega_k^2 \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right)$$

$$\hbar\omega_k = 2Wc\, A_k^2\, \omega_k^2$$

Then I will get my Hamiltonian has sum over k 2w / a A k square C omega k square a k dagger a k + half. And I can write it as more simpler form 2w you see, this C that is the capacitance and this is a so, I can write it a c capacitance per unit length ,A k square omega k square a k dagger a k + half. So, we are almost there. Now, let us choose h cross omega k let us take it says that that means, this is going to give us a choice for a k let me put it is 2 wc A k square omega k square.

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$$\Rightarrow \quad \boxed{A_k = \sqrt{\frac{\hbar}{2cW\omega_k}} = \sqrt{\frac{\hbar}{2C_{full}\,\omega_k}}}$$

$$\boxed{\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right)} \qquad \boxed{\sqrt{\frac{\hbar}{2m\omega}}}$$

Which means that I am taking my A k to be equal to h cross divided by twice c w omega k under root. So, if I take my this parameter A k like this or it is simply h cross divided by twice C, CW is C full here is the capacitance of the full transmission line and I have omega k that is the frequency. So, this should actually remind you of h cross by twice m omega term when we discuss quantization of mechanical harmonic oscillator. So, finally, therefore, we obtain using choosing my A k to be like this.

So, I get my Hamiltonian is sum over h cross omega k a k dagger a k + half. So, clearly when quantized ,when quantized ,our transmission line also look like a collection of independent harmonic oscillators. So, this concludes our discussion on quantization of transmission line.Let us write down the expression for voltage as it is a physical quantity and our Cooper pair box or artificial atom is coupled via voltage

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$$\dot{\hat{\phi}}_n = \text{voltage}$$

$$\hat{\phi}_n = \sum \sqrt{\frac{\hbar}{2 C_{full} \omega_k}} \left(\hat{a}_k e^{ikx} + \hat{a}_k^{\dagger} e^{-ikx} \right)$$

So,
$$\hat{V}(x) = \sum (-i\omega_k) \sqrt{\frac{\hbar}{2 C_{full} \omega_k}} \left\{ \hat{a}_k e^{ikx} - \hat{a}_k^{\dagger} e^{-ikx} \right\}$$

$$\hat{a}_k(t) = \hat{a}_k(0)\, e^{-i\omega_k t}$$

Please recall that the rate of change of magnetic flux phi, the rate of change of magnetic flux gives us the voltage this we discussed earlier and we have our magnetic flux as this, this is the expression we get h cross that is our constant A, this constant we have to put there, then we will have h cross by twice C the full capacitance omega k square root and then we have a k e to the power ikx + a k dagger to the power - ikx. So, this is the magnetic flux.

So, the voltage would be the just a time derivative of this magnetic flux expression and it would be simply - i omega k h cross by twice C full omega k square root and here I have a k e to the power + ikx + a k dagger e to the power - ikx ,there will be a minus sign here and we have actually exploited the fact that annihilation operator is time evolution is given by a k 0 e to the power - i omega k t. Now, while we have done the quantization using discrete model, it is perfectly valid for the continuum model also

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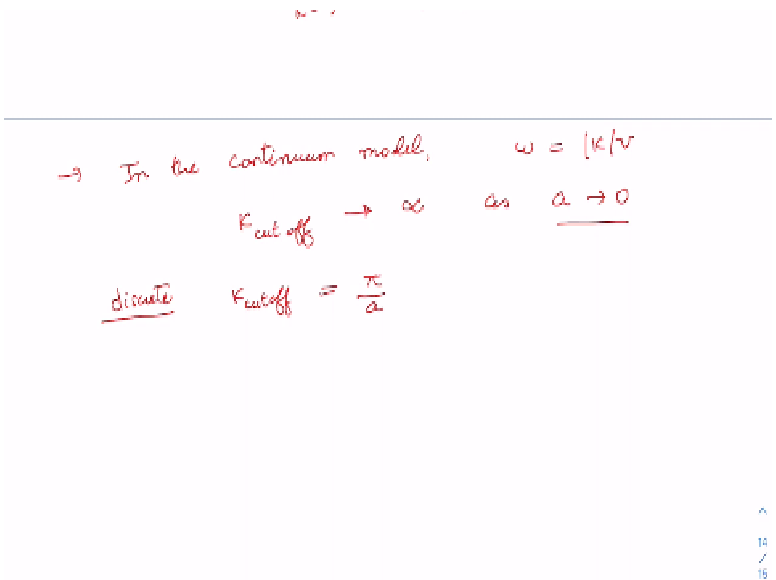

Because the only thing is that in the continuum model, so, in the continuum model, our cutoff frequency cutoff wave number is good. We will actually go to infinity as a tends to 0. By the way you know that the k cutoff for the discrete for the discrete case, discrete model is simply pi / a + - pi / a I am just giving you the on this one. So in the other side, it is - pi / a. So as a tends to 0, you can see that we will, the cutoff frequency will tend to infinity.

And we will have omega in the continuum model, we will have omega = k into v. So, now, let us do the quantization of the transmission line resonator. And this is important because we want our artificial atom to interact with the microwave very strongly. Hence, we put our artificial atom inside the resonator. So, how we can get a resonator, we can get a resonator by just taking a transmission line and then cut the transmission line at 2 points.

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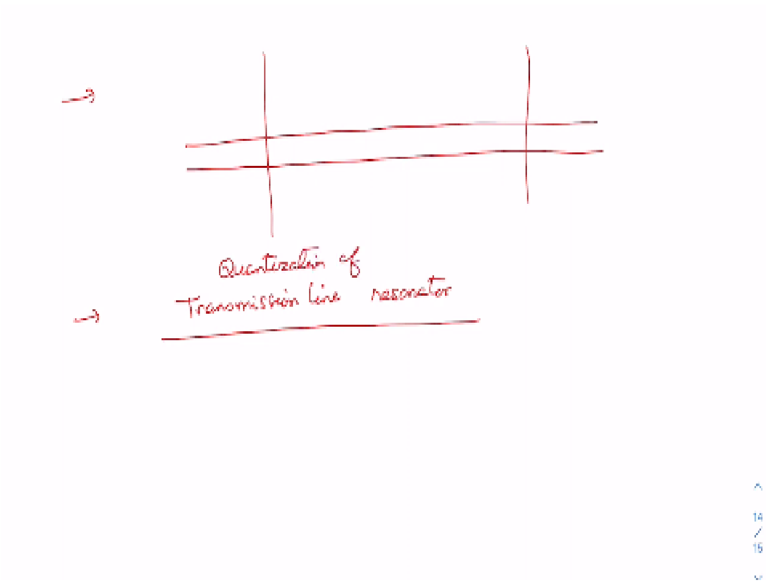

So you just take this transmission line and if we cut it at 2 points, then we are going to get a resonator. Now, clearly, we cannot use periodic boundary conditions now as we can no longer assume the resonator to form a kind of ring as we could do that in the case of infinite transmission lines. So, we are now going to discuss transmission line resonator in quantization of transmission line resonator.

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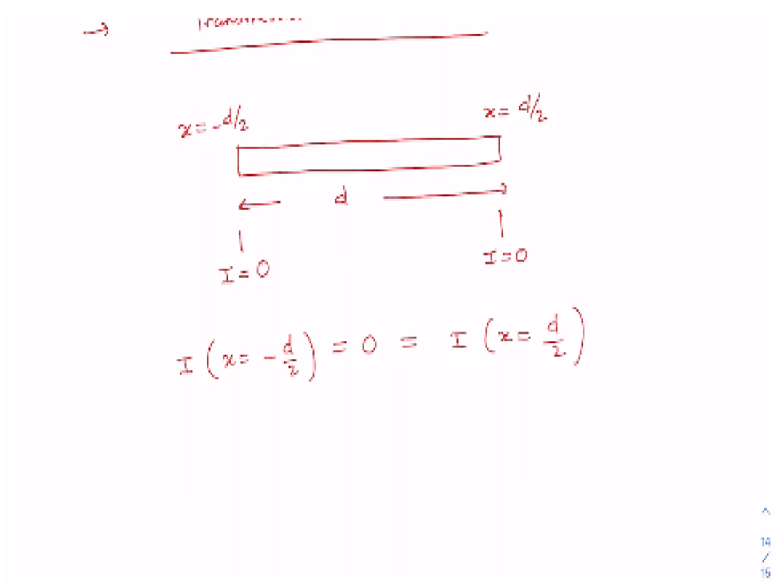

So, for that first of all, we need appropriate boundary conditions, as you see that we cannot use the periodic boundary conditions, but we can still get boundary conditions very easily because of the fact that suppose we have a transmission line of length d and its endpoints are, say is located at say x = - d / 2 and another end is at x = +d / 2 and the endpoints quite clearly the current has to be 0 at this end and the current at x = d / 2 has to be equal to 0. So, we get the boundary condition like this that current at x = -d / 2 = 0 and current x = +d / 2 has to be 0.

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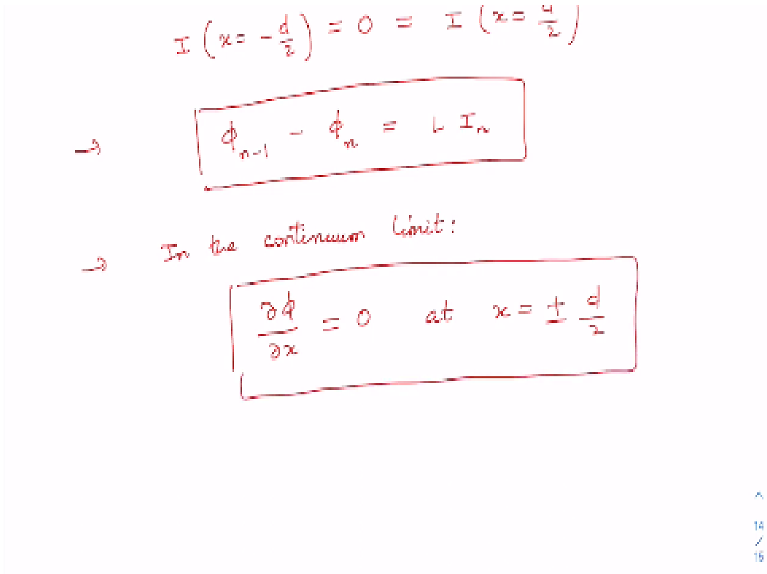

This means that because of the relation that we discussed earlier, that the difference in the magnetic flux at 2 node points in the transmission line is simply given ,we can get the current from this expression and now in the continuum limit, in the continuum limit, we go to the continuum limit, we can write this expression as del phi del x and that has to be equal to 0 at x = + - d / 2. So, this is going to serve as an important condition and it will be of very useful when we are going to quantizing the transmission line resonator **.**

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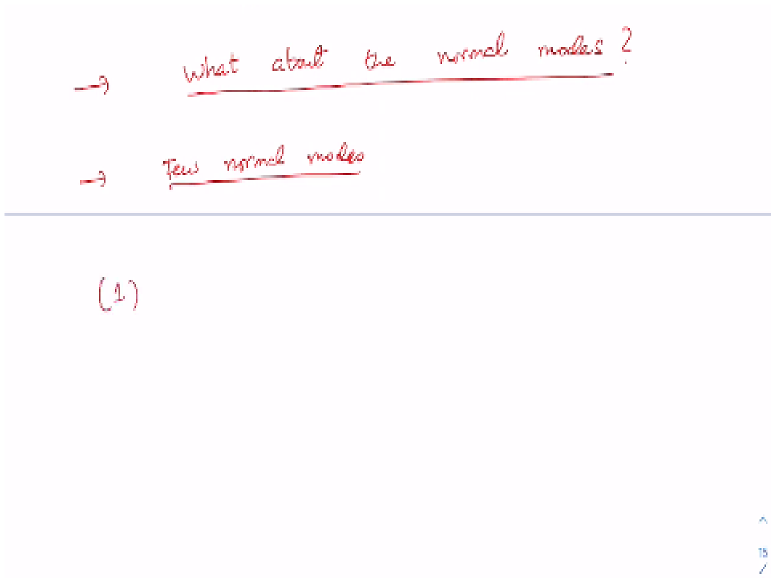

Now, the next thing is what about the normal modes? First of all, let us discuss what kind of shape of normal modes we can have. In the infinite transmission line we had plane waves. However, here we have standing waves, and these standing waves are maybe sine or cosine function. So, let us examine first few of the normal modes. So let me say few normal modes. Let us now discuss.

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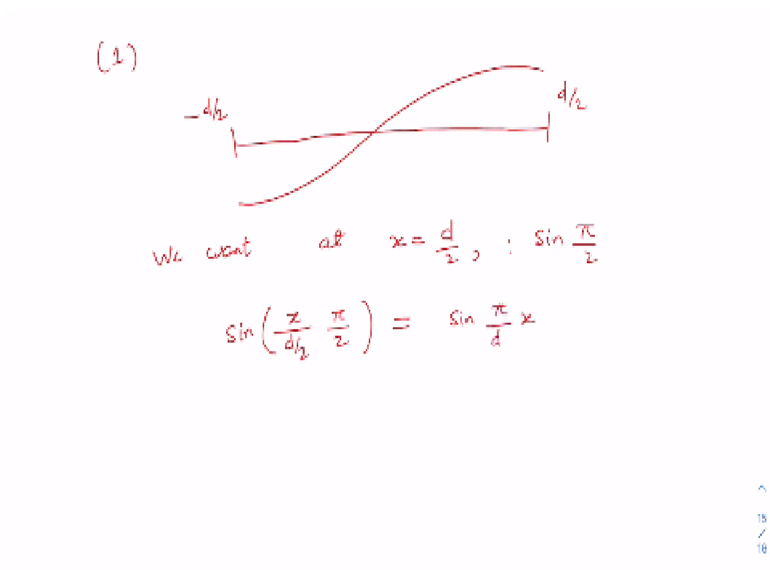

Let us say, we have a sine function. Suppose, here the current at this end not I am going to now I am going to sorry not current we are going to talk about normal modes. So, suppose the normal mode has a shape like this. So, this is what I have that means normal mode means the magnetic flux, we are going to express it in terms of normal modes. So, suppose we have a shape like this now, quite clearly this says a shape of a sine function and we want at say x = d / 2 I want my sine function to have this form say sin pi / 2.

So, therefore, how I can write the shape of this function So, at x = d / 2 I want it to be pi / 2 so, my mode function would have this form sin pi / d x.

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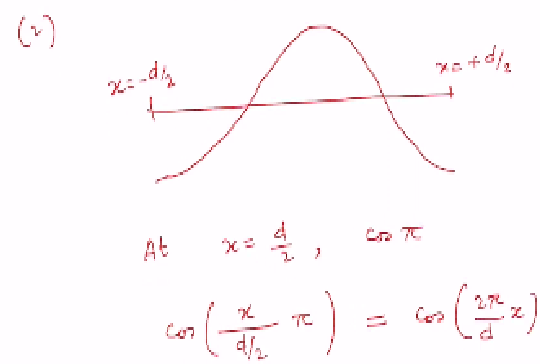

Similarly, let me consider estimate another one, let us say my normal mode now has this kind of phase shape. Say so, it is a cosine with here at this end x = -d / 2 and x = +d / 2. So, this is negative -1 say .So, therefore, again at x = d / 2 I want it to have this form say, cos pi. So, therefore, my shape of the function I can write it like this x, x = d / 2 I want it to be pi. So, my mode function would look like it is 2 pi / d x.

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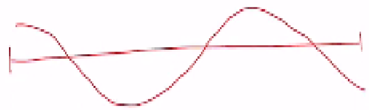
$$\cos\left(\frac{x}{d/2}\,\pi\right) = \cos\left(\frac{\pi}{d}x\right)$$

(3)



$$\sin\left(\frac{x}{d/2}\,\frac{3\pi}{2}\right) = \sin\left(3\,\frac{\pi}{d}\,x\right)$$

Let us do another one say, this time we have a mode of this type, say, we have this kind of mode shape and here it is going to be of this form and sin it will be x d by, x = d / 2 I now want it to have 3 pi / 2 values. So, the mode function will look like this sin 3 pi / d x. So, this way we can go on writing shape of many many modes.

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$$\rightarrow \text{modes}, \quad \phi_n(x) = \begin{cases} \sin\left(n\,\frac{\pi}{d}\,x\right), & n = 1, 3, 5, \ldots \\ \cos\left(n\,\frac{\pi}{d}\,x\right); & n = 2, 4, 6, \ldots \end{cases}$$

$\rightarrow$ orthogonality condition :

But in general the shapes of the modes are represented by these functions. So, I have say, sin n pi / d x for n = 1, 3, 5 and so on and I have cosine of n pi / d x and that would be for n = 2, 4, 6 and so on. So, this is our modes that means, the shape of the mode function would look like this. So, clearly these modes are orthogonal because these are sine and cosine functions and we have these orthogonality conditions.

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$$\boxed{\int_{-d/2}^{d/2} \phi_n(x)\, \phi_{n'}(x)\, dx = \frac{d}{2}\, \delta_{n,n'}}$$

$$\boxed{\hat{\phi}_n(x) = \sum_{n=1}^{\infty} A_n\, \underline{\phi_n(x)\left(\hat{a}_n + \hat{a}_n^{\dagger}\right)}}$$

So, orthogonality condition that we are going to use is this orthogonality condition that would be limit from -d / 2 to +d / 2 and I have one function and then I have phi n dash x dx that would be equal to d / 2 delta n, n dash. So, these are very easy to understand and I hope all of you are getting it. Now, let us make our answers.Let us assume our flux magnetic flux are the coordinate to be of this type phi n of x is equal to it is now some of our all the modes.

So, sum of these modes,, this is our mode functions just remember I am using the same kind of a symbol here, but I am not putting a cap here. So, this is the mode function ,shape of the mode function as I discussed here. So, this is what I have here and these I am going to express in terms of normal coordinates of a harmonic oscillator. So, normal coordinates of a harmonic oscillator I can write it as a n + a n dagger these are the annihilation and creation operators of the harmonic oscillator.

And then, because this is dimensionless it has to be multiplied by constant which we are going to determine later on. So, exactly the same procedure that we adopted for the case of the transmission line ,the same procedure now, only thing is that the mode function the magnetic flux the answers is now rather than plane wave now, we are having a standing wave and with different boundary conditions.

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$$\hat{P}_n = c\,\dot{\hat{\phi}}_n$$

In the continuum limit:

$$\hat{\pi}(x) = c\,\dot{\hat{\phi}}(x)$$

$$= \sum A_n \phi_n(x)\,(-i\omega_n)\,(\hat{a}_n - \hat{a}_n^{\dagger})$$

Then next quantity we require is the momentum. So, for the discrete case you will know that the momentum P n is equal to this capacitance into phi n dot, but in the continuum limit we are going to use the continuum limit case now, in the continuum limit from classical physics, we know that the momentum we can write it as c into phi dot x, c is the capacitance per unit length and then this would be I will write it as A n phi n x - i omega n a n – a n dagger.

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$$\mathcal{H}_{continuum} = \int \left(\pi\,\dot{\phi}(x) - \mathcal{L}\right) dx$$

$$\hat{H} = \int_{-d/2}^{d/2} \left[\frac{\hat{\pi}^2}{2c} + \frac{1}{2l}\left(\frac{\partial \hat{\phi}(x)}{\partial x}\right)^2 \right] dx$$

$$\boxed{\hat{H} = \sum_n \hbar\omega_n \left(\hat{a}_n^{\dagger}\hat{a}_n + \frac{1}{2}\right)}$$

Now, taking the idea from classical Hamiltonian for the continuum case so, this is in the continuum case the classical Hamiltonian would be like this momentum into the coordinate then minus the Lagrangian. So, this is what we have. So, therefore, we can write down the Hamiltonian, quantum Hamiltonian like this it would be minus integration from -d / 2 to +d / 2 and this now this momentum would be replaced by an operator.

So, in fact, you are going to get it pi square / 2 this is c, capacitance per unit length +1 by this is very straightforward and easy to understand. With so much of training already, I think you are easily going to get it and this l is the inductance per unit length and this would be delta x phi of x whole square and dx. So, again I am going to avoid the detailed calculations, maybe we will do the detailed calculations in a tutorial problem or a problem solving session later on then I will give more detailed calculations there.

Now, using the answers and the boundary conditions ,a similar calculation procedure we can have as we did in the case of the infinite transmission line and then if we do that finally, our Hamiltonian is going to take this simple form of the simple harmonic oscillator Hamiltonian h cross omega n a n dagger a n + half. So, this is what finally, even in the case of the transmission line resonator we are going to see that it is also a collection of independent harmonic oscillators.

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- $A_n = \sqrt{\dfrac{\hbar}{c d \omega_n}} = \sqrt{\dfrac{\hbar}{C_{full}\,\omega_n}} \qquad C_{full} = c d$
- $\omega_n = k_n v = v\left(\dfrac{n\pi}{d}\right), \quad v = \dfrac{1}{\sqrt{lc}}$
- $d \approx 1\ \text{cm}, \quad \omega \sim \text{GHz}$

Now, while we get this in this case our this constant A n we take it as we have to choose it like this, it is h cross c there is capacitance per unit length into the length of the resonator into the frequency and we can write it as h cross divided by the full now, this is the capacitance of the resonator, the full capacitance of the resonator into omega n. So, C is full a capacitance is nothing but c into d.

So, and the dispersion relation we actually got in the process is a simple one,that would be omega n = k n v who is I can write it as v into n k n is pi / d this is what I have and where this v is the velocity of the wave and that is nothing but square root of n into c and typically for the transmission line resonator is say, it is on the order of 1 centimeter or so, then the frequency would turn out to be in the range of Giga Hertz.

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$$\hat{V}(x) = \hat{\phi}(x)$$

$$= \sum_n i\omega_n A_n \phi_n(x) \left(\hat{a}_n^{+} - \hat{a}_n\right)$$

$$\left\{ i\left(\hat{a}_n^{+} - \hat{a}_n\right) \right\} \longrightarrow \left(\hat{a}_n + \hat{a}_n^{+}\right)$$

$$\boxed{\hat{V}(x) = \sum_n \omega_n A_n \phi_n(x) \left(\hat{a}_n + \hat{a}_n^{+}\right)}$$

The fluctuating voltage we can obtain this follows ,the fluctuating voltage in this case would turn out to be users have to take the time derivative of the flux expression and you were going to get i omega n A n phi n of x a n dagger - a n. Now, many times rather than writing this expression i into a n - a n because you see this is actually related to the momentum and you know the Hamiltonian formalism momentum and coordinate have equal weightage and therefore, we can easily make transformations.

So, rather than writing it in this form we can make a transformation are choose our annihilation operator such that we can write this expression as simply a n + a n dagger and this is what we are going to use later in this course. And therefore, the fluctuating voltage is going to be written like omega n A n phi n of x a n + a n dagger, the reason I am giving so much stress on this fluctuating quantities is due to the fact that we are going to couple our Cooper pair box set of voltage is when we are going to put it inside the resonator , artificial atom.

That atom is going to get coupled to the voltage v of x. Let me stop here for today. In this lecture, we learned how to quantize an infinite transmission line. And also we learned how to quantize your transmission and resonator. In the next class, we will see how to couple a 2 level system or an artificial atom into a transmission line resonator. In the process we are going to learn about the famous Jaynes Cummings model. So see you in the next class. Thank you.