Quantum Technology and Quantum Phenomena in Macroscopic Systems
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Lecture - 18
Quantization of Transmission Line-1.

Hello, welcome to lecture 3 of second module of the course. In this lecture, we will continue our discussion on transmission line and we will see how we can quantize a discrete transmission line, so let us begin.

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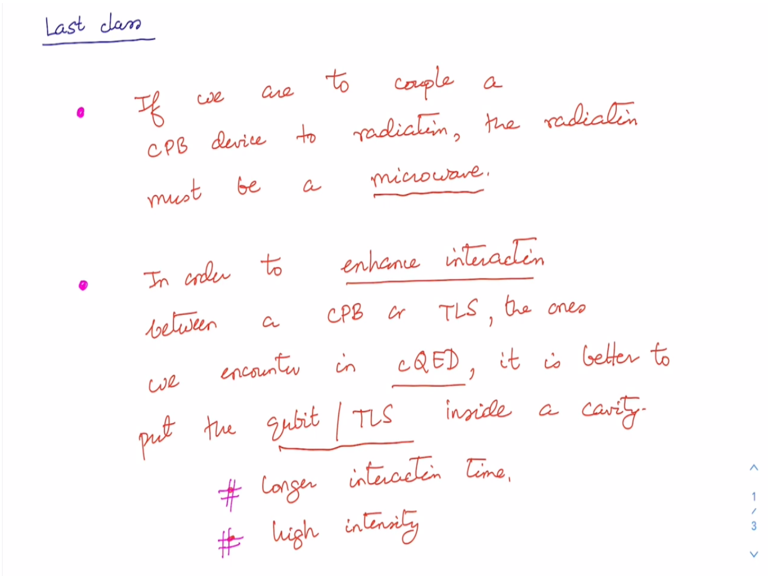

In the last class, we completed our discussion on Cooper pair box. There we saw that if we are to couple a cooper pair box device to radiation, the radiation must be a microwave and in order to enhance interaction between a cooper pair box or 2-level system, the kind of 2-level system that we encountered in circuit quantum electrodynamics, it is better to put the qubit or the 2-level system inside a cavity. Because it will ensure long interaction time between the Cooper pair box and the radiation and high intensity of the radiation inside the cavity

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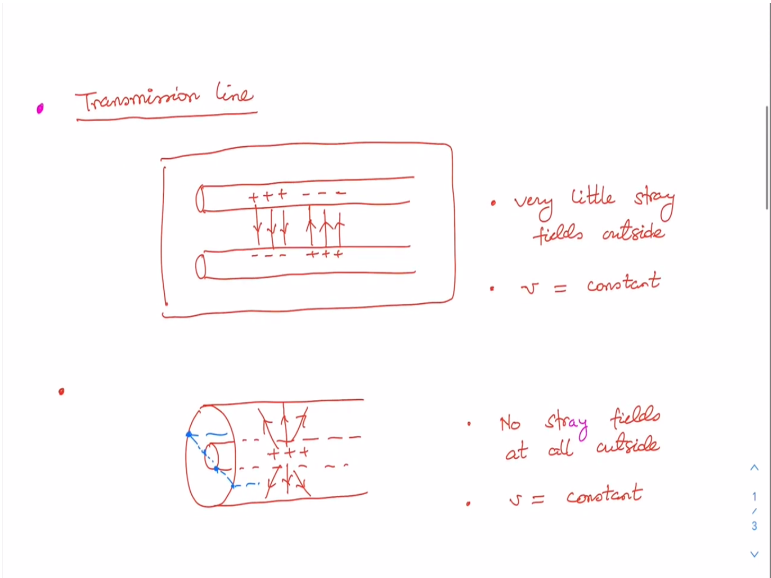

And we can have a microwave cavity using the so-called transmission line and we discussed about transmission line. It can be constructed by considering say two metallic wires parallel to each other or a coaxial cable or two concentric cylinders, metallic cylinders that is also a transmission line. The advantage of this coaxial cable is that there will be hardly any stray field at all outside and that is advantageous and this speed of the signal in both the cases is going to be constant, it is not dependent on either wavelength or the frequency.

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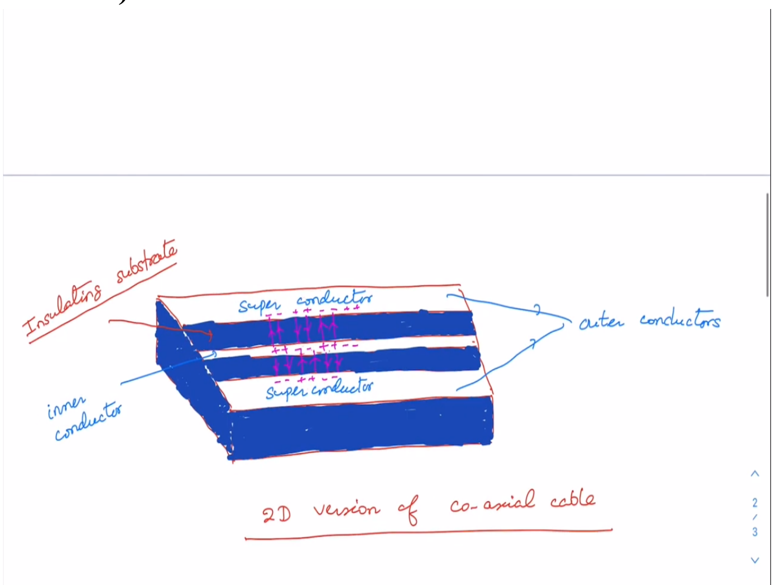

And we also learned that how this coaxial cable model which is a 3-dimensional one can be utilized to design a 2-dimensional version of the transmission line on a chip, this we discussed.

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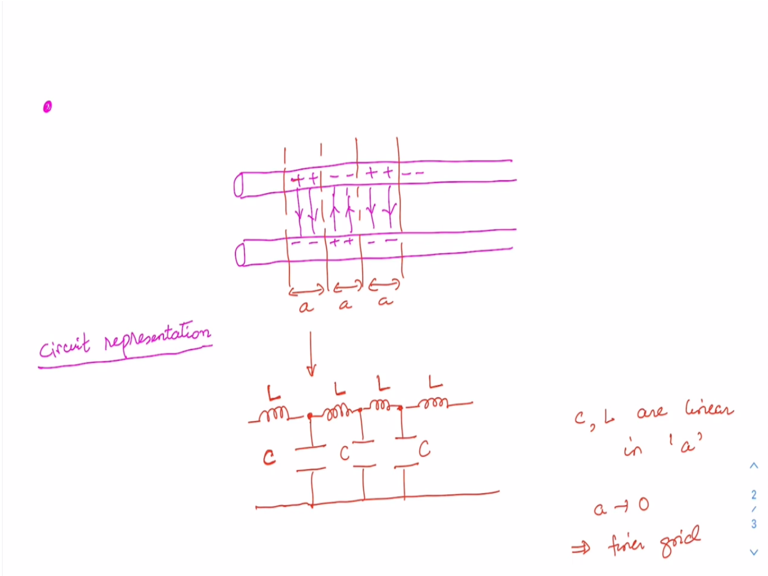

And finally, we discussed how this transmission line can be represented in circuit, using a circuit, for that we use these 2 parallel wires model. The idea was that we divided these parallel wires into a number of unit cell size of length a and each cell comprising of a capacitance and inductance and this capacitance and inductance are linear in the cell size and ultimately our goal would be to make the cell size smaller and smaller tends to 0. So that it gives us a finer grid.

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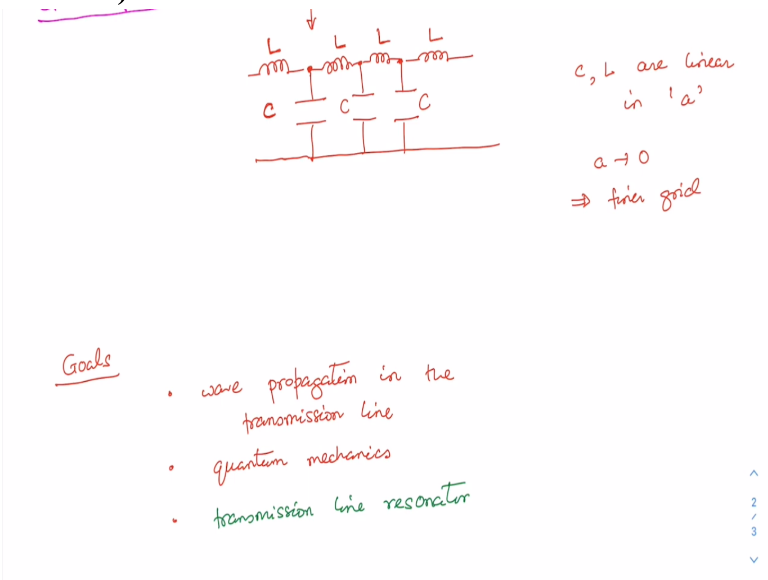

And our goal today would be to discuss wave propagation in the transmission line and basically to get a wave equation and convince you that indeed this model or this kind of setup supports transmission of signal and then we are going to quantize the transmission line. Finally, if time permits then we will start discussing transmission line resonators. So, before I go further, let me first remind you about the so-called LC circuit which is the most elementary unit cell of transmission line.

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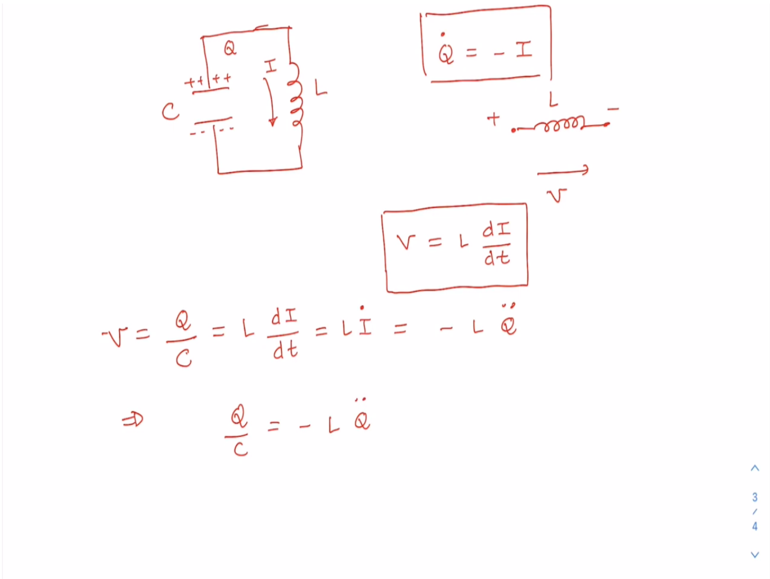

So, let us remind ourselves. We have an inductance and we have this capacitance here. And first of all, let me introduce the sign convention for the charge. Let us say the upper plate of this capacitor has positive charge and lower plate has negative charge say the charge Q. Here it is represented by this Q and say the current I, current I flow in the direction as I swing here, then as you can see that this charge has to decrease with time.

So, Q dot I can write it as -I because charge is decreasing as the current is flowing. And on the other hand, if we look at the voltage drop across the inductance then it will be given by suppose this is the inductance and the voltage is applied along these directions from this positive to negative, then this voltage drop across this inductance L would be L dI dt, because it will take a bit of time for a current to flow through it when you apply the voltage.

So, this is a well-known expression all of you know. On the other hand, the voltage across the capacitor is given by charge divided by the capacitance C and which is obviously equal to L dI dt and or I can just write it as LI dot which I can further write because Q dot = -I here using these I can write it as -L Q double dot.

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$$V = \frac{Q}{C} = L\frac{dI}{dt} = L\dot{I} = -L\ddot{Q}$$

$$\Rightarrow \quad \frac{Q}{C} = -L\ddot{Q}$$

$$\Rightarrow \quad \ddot{Q} + \frac{1}{LC}Q = 0$$

$$\Rightarrow \quad \boxed{\ddot{Q} + \omega^2 Q = 0}\ ; \qquad \omega^2 = \frac{1}{LC}$$

So, I get a differential equation for the charges that is Q / C = -L Q double dot or I just write Q double dot + 1 / LC Q = 0. As you can easily see that this is a second order differential equation for charge. It is an equation of motion for charge in LC circuit and which we can write as Q double dot + omega square Q = 0, where obviously omega square = 1 / LC. So, this is a well-known equation all of we know. This equation tells us that charge Q oscillate simple harmonically in an LC circuit, now, let us go back to our transmission line.

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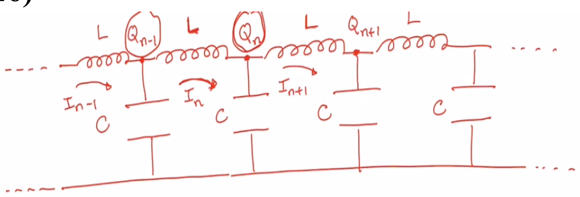

$$\boxed{\dot{Q}_n = I_n - I_{n+1}} \longrightarrow (1) \quad \text{charge conservation}$$

$$\boxed{\frac{Q_{n-1}}{C} - \frac{Q_n}{C} = L\frac{dI_n}{dt}} \longrightarrow$$

Say there is a charge at every node and there is a current flowing between the nodes, say charges Q, denoted as Q n in the nth node and in n + 1nth node the charge is denoted as Q n + 1 and here we are in the n - 1th node charge is denoted as Q n - 1 and so on. And first actually we consider the discretized version and then we will go over to the continuum version by assuming that the charges vary very slowly as a function of position.

Let the current flowing between the node n be denoted by I n that means, say current that is flowing towards the node n, let me denoted by I n and then current flowing towards node n - 1 be denoted as I n - 1 and current that is flowing towards the node n + 1 let me denoted it by I n + 1 and so on. Now, you see the charge building up at a node is determined by current flowing into it and out of it.

So, we can invoke the so, called charge conservation at a particular node. If I invoke it for this nth node here, then I can write this charge conservation as, say, rate of change of charges at the nth node is equal to that is the current flowing towards this nth node and the current flowing out of it. So, this is one equation we obtain that is this equation let me denoted it as equation number 1. This equation is basically referring to the charge conservation principle.

On the other hand, this current I n, this current I n is driven by the voltage drop between these node n and n - 1 and this voltage drop would be here in this n - 1th node, now voltage drop would be Q n - 1 divided by C. On the other hand, here at the nth node it would be Q n divided by C. So, this is the voltage drop difference in the voltage and that has to be equal to the current through this the voltage through this inductor L here. And that would be L dI n / dt. So, we get another equation for using this voltage drop thing.

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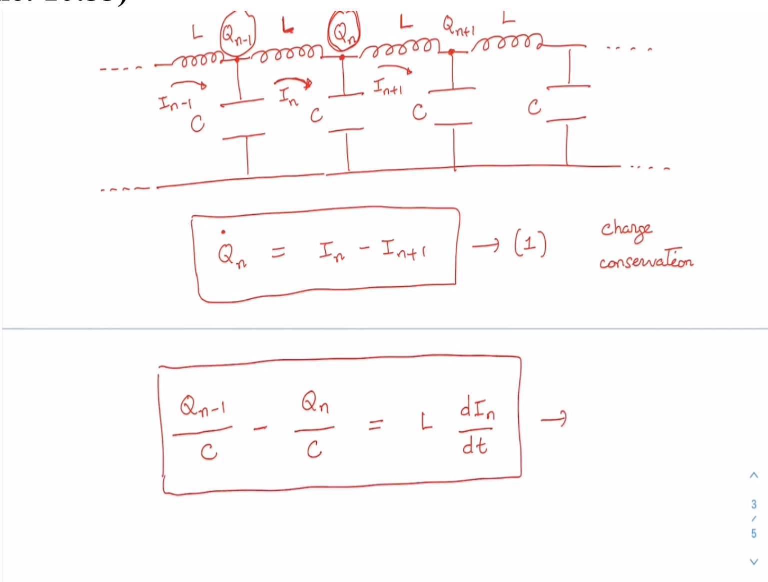

The current, let me reiterate again that this current I n is driven due to the voltage drop at node n - 1 and n. So, now, let us see how these 2 equations look or appear in the continuum domain to do that, let us say let me draw the transmission line once again here this discretized version let me say I have like this, this is let me first draw it and I will explain. So, I have this nth node that is Q n here and this is Q n - 1 current is flowing towards the nth node here. And here it is Q n + 1 so, this is I n + 1 and so on, this is I n - 1 and this is inductances L capacitances is C.

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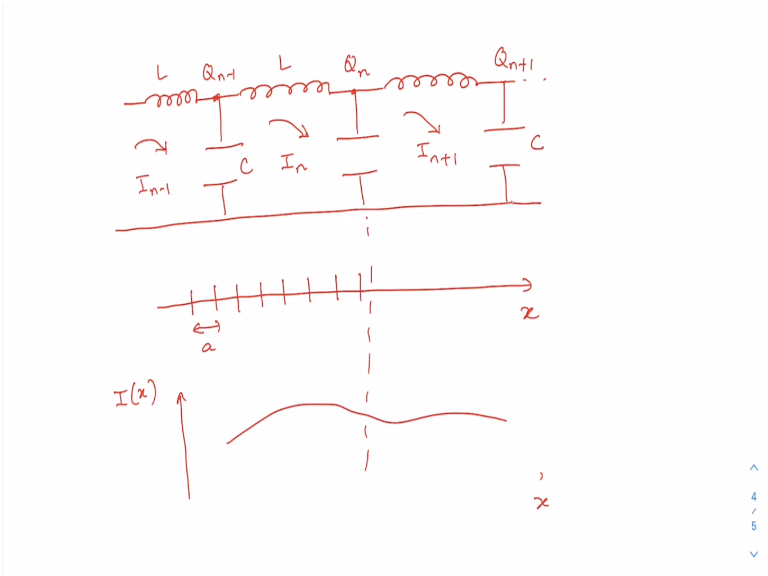

Let us say in the continuum limit as we have learned earlier that we can actually divide this transmission line into a number of cells say along this x direction if I just break up this transmission line into several small number of cells each cell of size say a. So, this is along the x direction I am taking. Now, as a tends to 0 I can assume that as the cell size is getting smaller and smaller, let us say I have a current flowing through this transmission line that is smooth like this and this is a function of position.

Now, if I am interested in knowing the current and the nth node let us say at this particular point if I let us say something like this. Or something like this, this particular point in the nth node I have to evaluate the current and this will basically refer to x = n into a and then current at nth node is simply current that is measured at the point x = na.

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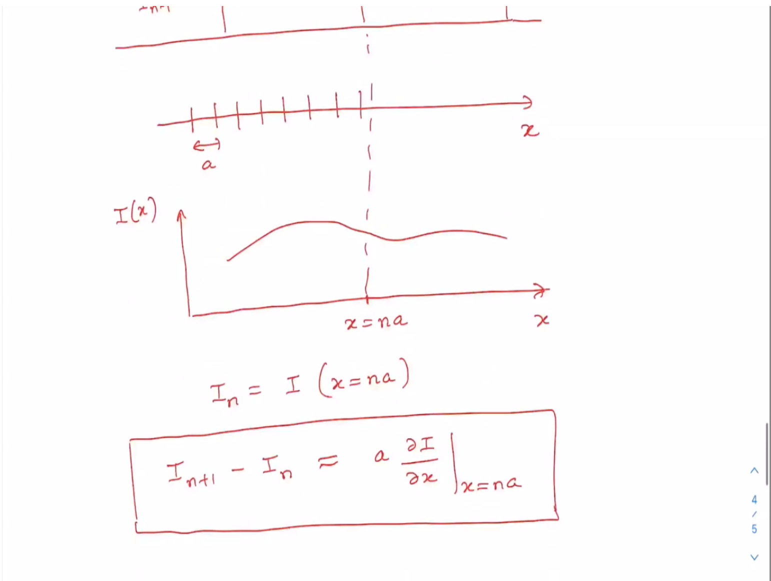

So, we can write actually the difference between the current in the node n + 1, the difference between the current at n + 1 and I n at nth node that will be nothing but I can write it as it will be nearly equal to the slope of the current at the position x = na. So, I think you are getting the idea here, I am just invoking or taking this, when I am writing this expression, I am using my knowledge of differential calculus. So, actually using this prescription, I can write my equation number 1 as well as equation number 2 in the continuum limit that means when a tends to 0 as follows.

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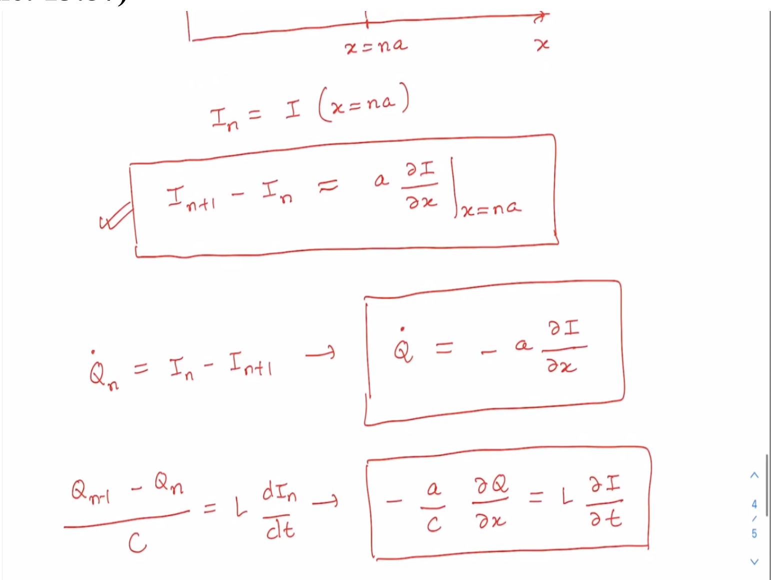

Let us say I have this charge conservation equation Q n dot I n - I n + 1. So, this would be in the continuum limit this would be Q dot = - a del I del x. And I have actually used this expression while writing it. So, this is what I have in the continuum domain and similarly, the voltage drop equation which was Q n + 1 - Q n I think or I can just this was Q n - 1 - Q n / C = L dI n / dt, this again I can write in the continuum limit as -a / C del Q / del x = L del I del t, I think you are getting the idea here. So, these are the equations that I am writing in the continuum limit.

Now, let us write down the wave equation in the continuum limit, we can easily get it from this discrete version of the equations. For that, I simply need to consider, this equation number 2 here.

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$$\frac{Q_{n-1}}{C} - \frac{Q_n}{C} = L\frac{dI_n}{dt}$$

$$\Rightarrow \frac{1}{C}\left[\dot{Q}_{n-1} - \dot{Q}_n\right] = L\frac{d^2 I_n}{dt^2}$$

$$\Rightarrow \frac{1}{C}\left[I_{n-1} - I_n\right)$$

If I consider that equation say Q n - 1 / C – Q n / C = L dI n dt. Let me take time derivative of this equation on both sides, then what I am going to get is this 1 / C. I have Q n - 1 Q this one and then Q n dot = L d 2 I n dt 2. I can get rid of the charges by using this equation that I have written earlier that charge conservation that is 1 / C Q dot n - 1 so, that is I n - 1 - I n. I hope you can see that. This is I have for this part.

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$$\Rightarrow \frac{1}{C}\left[\dot{Q}_{n-1} - \dot{Q}_n\right] = L\frac{d^2 I_n}{dt^2}$$

$$\Rightarrow \frac{1}{C}\left[\left(I_{n-1} - I_n\right) - \left(I_n - I_{n+1}\right)\right] = L\frac{d^2 I_n}{dt^2}$$

$$\Rightarrow \frac{1}{C}\left[I_{n+1} + I_{n-1} - 2I_n\right] = L\frac{d^2 I_n}{dt^2}$$

And for the other part we can write I n - I n + 1 and the whole thing is equal to L d 2 I n dt 2, these you can further write it as 1 / C I n + 1 + I n - 1 - 2 I n = L d 2 I n dt 2.

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$$\Rightarrow \frac{1}{C}\left[\left(I_{n-1}-I_n\right)-\left(I_n-I_{n+1}\right)\right] = L\frac{d^2 I_n}{dt^2}$$

$$\Rightarrow \frac{1}{C}\underbrace{\left[I_{n+1}+I_{n-1}-2I_n\right]} = L\frac{d^2 I_n}{dt^2}$$

$$\frac{a^2}{C}\frac{\partial^2 I}{\partial x^2} = L\frac{\partial^2 I}{\partial t^2}$$

Now, you can recognise that in the continuum limit this part is actually second order derivative of current with respect to position. So, in the continuum limit we can write it as this whole thing as a square / C del 2 I del x 2, that is equal to L del 2 I del t 2.

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$$\Rightarrow \boxed{\frac{\partial^2 I(x,t)}{\partial x^2} = lc\frac{\partial^2 I(x,t)}{\partial t^2}}$$

$$l = \frac{L}{a}, \quad c = \frac{C}{a}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}} \rightarrow f \text{ propagates with speed } v$$

Or I can write it as del 2 I del x 2 = LC / a square del 2 I del t 2. This further I can simplify by writing as del 2 I which is a function of position and time now, del x 2 is equal to l and c del 2 I delta t 2 I which is a function of position and time. And here this l is inductance part unit length and c is the capacitance per unit length and these quantities are actually the physical quantity of the transmission line and they do not depend on our discretization independent of that.

And now, you can see that this equation is nothing but the wave equation just recall that the wave equation that you must have learned earlier somewhere is given by this form that is second order in space and second order in time where f is a disturbance which is a function of position and time and this disturbance this equation simply tells that f propagates or travels propagates with speed v. So, comparing this equation with our equation that I obtained for current.

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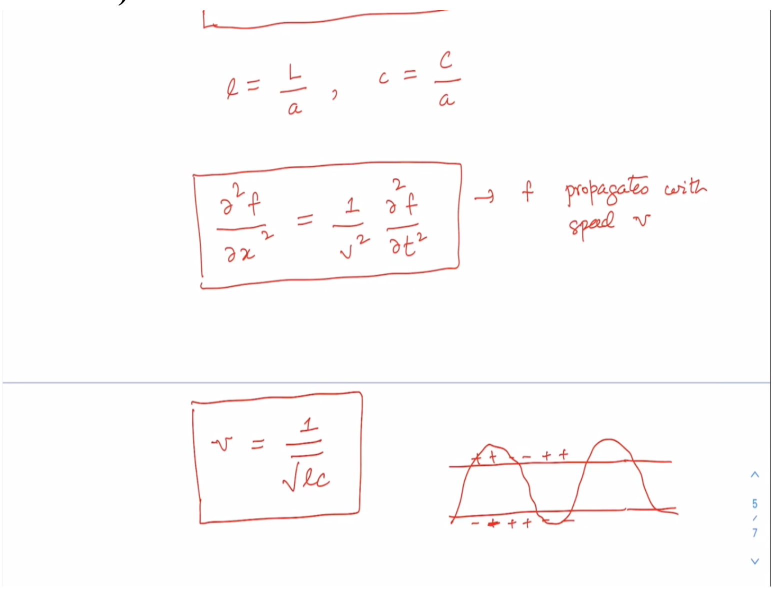

It simply says that current propagates like a wave in a transmission line with speed 1 divided by square root of l c. So, this is the speed of a current signal in a transmission line and I think and because that this both l and c are constant. So, this speed is basically a constant quantity do not depend on frequency or wavelength. So, in the transmission line we can create a disturbance of disturbance so, charges like this.

And as I explained earlier so, in this whole system we will be able to get a propagation of a signal in the form of current. And now because we have gotten classically, we have got that the current propagates like a wave in a transmission line that means the current can carry information. So, the next problem is to how to quantize these transmission line how to quantize this particular problem and this is what we are going to do now, but before actually we do that.

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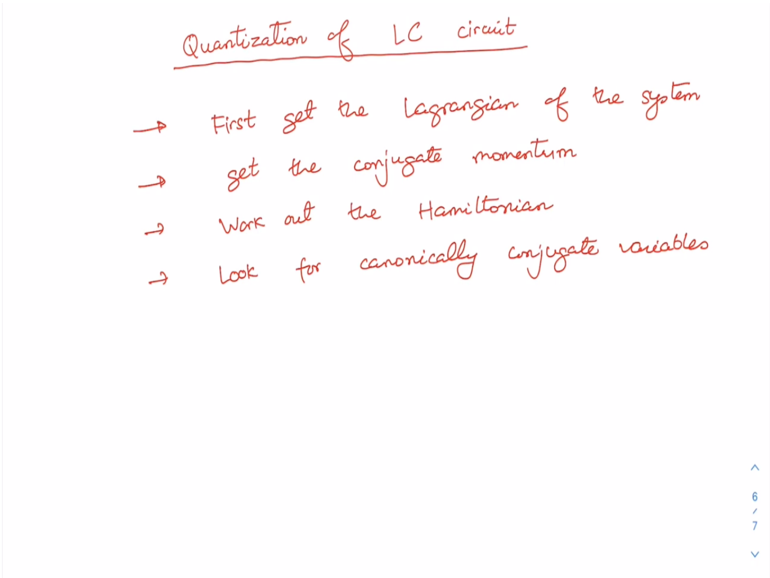

Let us see how we can quantize LC circuit or LC oscillator which is the basic building block of a transmission line. So, the topic that we are now going to discuss is quantization of LC circuit. And this is going to prepare you for the bigger thing that is the quantization of transmission line which we are going to discuss next. In the previous module we learned general procedure for the canonical quantization of a classical system.

The procedure, is as follows: First get the Lagrangian, Lagrangian of the classical system then get the conjugate momentum actually get the conjugate momentum and once you get the conjugate momentum, then work out the Hamiltonian or find the Hamiltonian of the classical system. And then, look for the canonically conjugate variables. Look for canonically conjugate variables the canonically conjugate variables are the ones that satisfy the Hamiltons canonical equation of motion.

And finally, replace the Poison's brackets by commutation bracket and write the conjugate variables as well as the Hamiltonian in the operator form.

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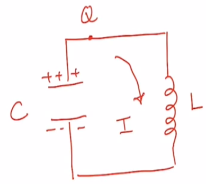

$$\dot{Q} = -I$$

$$\ddot{Q} = -\frac{1}{LC} Q$$

$$\mathcal{L} = \frac{1}{2} L I^2 - \frac{Q^2}{2C}$$
$$= \frac{1}{2} L \dot{Q}^2 - \frac{Q^2}{2C}$$

Now, here exactly the same procedure we can apply for quantization of our LC circuits. So, we have this inductor with inductance L and this capacitor with capacitance C, we assume that the upper plate is positively charged and the lower plate is negatively charged upper plate has a charge Q and the current is flowing in this direction and just we have discussed already that Q dot = -I.

And also, we have worked out the equation of motion of the LC circuit Q double dot = -1 / LC Q. Now, because we know this equation of motion, so we can straightforwardly guess the Lagrangian of the system that is L = half LI square. You see I am using before them L here, this is for the Lagrangian and this L represents the inductance - Q square / 2C. Q here is the generalised coordinates. So, I have half L Q dot square - Q square / 2 C.

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$$\Rightarrow \mathcal{L} = \frac{1}{2} L \dot{Q}^2 - \frac{Q^2}{2C}$$

$$\frac{\partial \mathcal{L}}{\partial Q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\right) = 0$$

$$\Rightarrow -\frac{Q}{C} - L\ddot{Q} = 0$$

$$\Rightarrow \boxed{\ddot{Q} = -\frac{1}{LC} Q}$$

To check if this is indeed the correct Lagrangian, we can actually we must obtain the equation of motion from the Lagrangian equation of motion, the Lagrangian equation of motion with Q as our generalised coordinate is this del L del Q here - d of dt del L del Q dot = 0. Now, if you put this Lagrangian expression here in this equation, then you will see the first term is going to give you simply Q / C - then the second term is going to give you L Q double dot = 0. So, quite clearly you will get Q dot double dot = 1 / LC into Q and this is indeed the equation of motion. So, the Lagrangian that we have written is absolutely correct.

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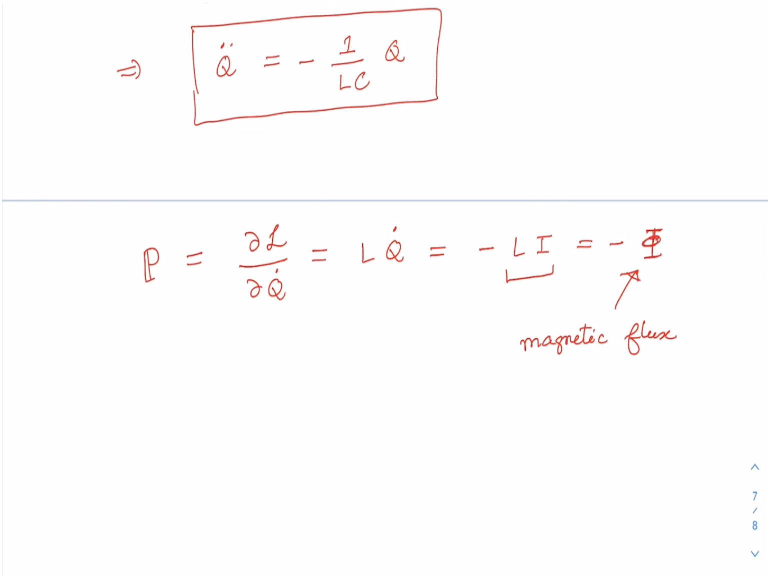

Now, as regards the canonically conjugate momentum, so P = del L del Q dot and it turn out to be simply LQ dot and in terms of current I can write it as LI. So, maybe you recognise that LI is nothing but the magnetic flux in the circuit which can I can represent by the symbol. So, phi here is magnetic flux. So, when I take charge as my generalise coordinate the generalised momentum or conjugate momentum is minus of the magnetic flux in the circuit gives the conjugate momentum in the problem.

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$$H = H(\underline{Q, \Phi})$$

$$H = \dot{Q}P - \mathcal{L}$$

$$= L\dot{Q}^2 - \left(\frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C}\right)$$

$$= \frac{Q^2}{2C} + \frac{1}{2}L\dot{Q}^2$$

$$\Rightarrow \boxed{H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}}$$

$$\phi = LI$$

So, the Hamiltonian now is a function of the generalised coordinate Q and momentum phi and we know that the Hamiltonian is written like this Q dot P generalised coordinate the velocity basically and P, generalised momentum, minus the Lagrangian. So, P = LQ dot so if you put here then you will get LQ double dot Q dot square - L is half L Q dot square - Q square / 2 C. So, immediately you will get this equation you will get Q square / 2C + half LQ dot square. So, this is what you will get and you can in fact write from here.

Because Hamiltonian is a function of charge Q and this magnetic flux, so, better we write it like this Q square / 2C + phi square / 2L, because Q dot square = I and we have phi = LI so, I have just utilised it and so, this is our Hamiltonian.

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$$= \frac{Q^2}{2C} + \frac{1}{2}L\dot{Q}^2$$

$$\Rightarrow \boxed{H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}}$$

$$\phi = LI$$

$$P = -LI$$
$$= -\Phi$$

$$H = \frac{Q^2}{2C} + \frac{P^2}{2L}$$

$$\dot{Q} = \frac{\partial H}{\partial P}$$
$$= \frac{P}{L}$$

So, what about the canonically conjugate variables here obviously, Q and phi are the canonically conjugate variable but to ensured that, let us say this canonical equation of motion gives us Q dot = del H del P that is the momentum and now, momentum let me write once again that this is equal to LI or instead of -phi. So, in fact, better I can write these Hamiltonian as H = Q square / 2C + this P square / 2L so, therefore, del x del P is simply going to give me P / L.

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$$\dot{Q} = \frac{\partial H}{\partial \mathbb{P}} = \frac{\mathbb{P}}{L} = \dot{Q} \qquad H = \frac{Q^2}{2C} + \frac{\mathbb{P}^2}{2L}$$

$$\dot{\mathbb{P}} = -\frac{\partial H}{\partial Q} = -\frac{Q}{C}$$

$$\Rightarrow \boxed{L\ddot{Q} = -\frac{Q}{C}}$$

And P / L is nothing but Q double dot, because P = LQ, that is what I have written here, L Q dot so therefore, so that is correct and other one there is momentum P dot = -del H del Q and this is going to give me from this Hamiltonian I will get - Q / C and because P = LQ, so it is LQ double dot = -Q / C so, we get the equation of motion so quite clearly we are obtaining the equation of motion and correct equation of motion so.

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$$\dot{\mathbb{P}} = -\frac{\partial H}{\partial Q} = -\frac{Q}{C}$$

$$\Rightarrow \boxed{L\ddot{Q} = -\frac{Q}{C}}$$

Q, Φ are canonically conjugate variables.

$$[\hat{Q}, \hat{\Phi}] = i\hbar$$

So, Q and phi are charge Q and this magnetic flux phi are canonically, canonically conjugate variables in the problem conjugate variables. So, therefore, we can invoke quantization now, what we have to do we just have to replace this Q by operator Q and phi by operator phi and that will be equal to i h cross and these Hamiltonian we write it as here we have written so, that will be Q square / 2 C + 5 square / 2L. So, this is the way we can quantize LC oscillator.

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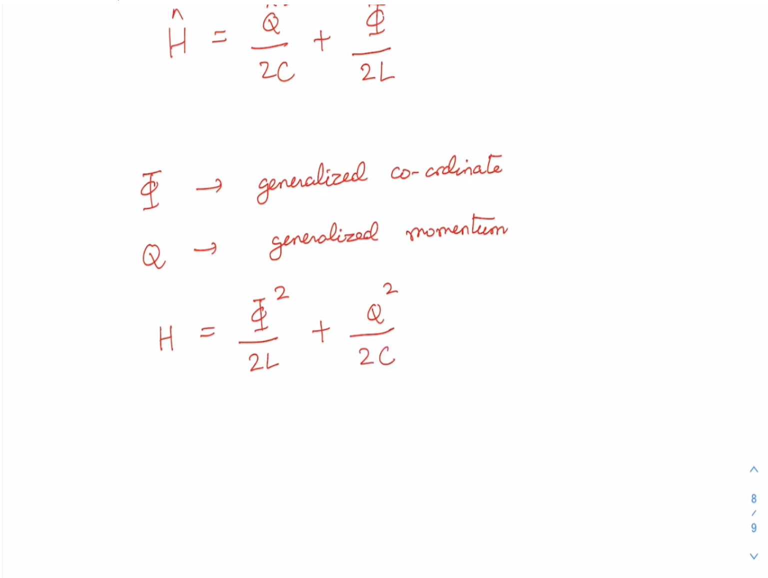

You may actually know that in the Hamiltonian formulism both generalised coordinate and momentums are given equal weightage. In fact, the Hamiltonian formalism is quite flexible. So, we can quantize the LC oscillator taking magnetic flux is our generalised coordinate rather than the charge. So, if we take phi as our generalised coordinate and Q as generalised momentum. So, if we do that, only thing is that we have to be careful about the commutation relation here, but anyway, if I take because if I take phi not –phi then that is correct.

The commutation relation would remain intact. So, I encourage you to do this and you will indeed get the same expression as your Hamiltonian that would be phi square / 2L + Q square divided by 2C that would be the Hamiltonian.

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magnetic flux $\Phi \rightarrow$ generalized co-ordinate

$\tilde{P} = Q \rightarrow$ generalized momentum

$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}, \quad \tilde{P} = Q$$

$$\dot{\Phi} = \frac{\partial H}{\partial \tilde{P}} = \frac{Q}{C}$$

Only difference here is that this time the magnetic flux phi, this magnetic flux phi is now our generalised coordinate and charge is our new generalise momentum. In fact, let us check whether indeed this is the correct one. So, if I put it in the canonical Hamiltons canonical equation of motion, so phi is my generalised coordinates. So, therefore phi dot = del H del P tilde let me write for our new generalise momentum which is nothing but the charge. So, therefore, you will get it to be simply Q / C here because remember here generalised momentum is nothing but charge Q.

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$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}, \quad \tilde{P} = Q$$

$$\dot{\Phi} = \frac{\partial H}{\partial \tilde{P}} = \frac{Q}{C}$$

$$\Rightarrow L\dot{I} = \frac{Q}{C}$$

$$\dot{\tilde{P}} = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I$$

$$\Rightarrow \dot{Q} = -I$$

And phi is magnetic flux So, therefore, it is LI dot that is Q / C and this is correct because this we already know. Another thing is about the momentum P dot = - del H, the generalised coordinate is phi and that would be from here we get it as phi / L, again this is nothing but -I and because P is the generalised momentum that is P tilde that is your Q so, Q dot = - I. This is also we know that this is correct.

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$$\Rightarrow L\dot{I} = \frac{Q}{C}$$

$$\dot{P} = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I$$

$$\Rightarrow \dot{Q} = -I$$

$$\boxed{\mathcal{L} = \frac{C}{2}\dot{\Phi}^2 - \frac{\Phi}{2L}}$$

Now, by the way in this case the Lagrangian would be turned out to be C / 2 phi dot square - phi / twice L. When I take magnetic flux as my generalised coordinate, this is the Lagrangian that I should take. And in fact, we will see that while quantizing transmission line, we should take magnetic flux as our generalised coordinate not charge Q and we will discuss about it.

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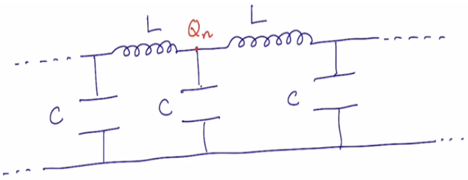

$$\mathcal{L} = \frac{1}{2}LI^2 - \frac{1}{2}\frac{Q^2}{C}$$

$$\equiv \text{magnetic energy} - \text{electrical energy / charging energy}$$

Now, going back to our transmission line, let us propose Q n charge as our coordinate and see what happens. As you know the Lagrangian for the basic unit cell for this transmission line which is the LC circuit the Lagrangian is written like this L = half here this L represents the inductance half LI square - half Q square / C. In fact, this first term is actually called magnetic energy; this is the magnetic energy term. And the second term is the electrical energy or it also called a charging energy term, charging energy. So, we need to write down the charging energy and the magnetic energy for the full transmission line.

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$\equiv$ magnetic energy $-$ electrical energy / charging energy

Given Q, find I

$$\dot{Q}_n = I_n - I_{n+1} \quad \text{(charge conservation)}$$

$$\Rightarrow I_n = \dot{Q}_n + I_{n+1}$$
$$= \dot{Q}_n + \dot{Q}_{n+1} + I_{n+2}$$

Now, given the charge distribution given the charge distribution Q, we have to find the current I to write down the Lagrangian for transmission line. Now, we know that from the charge conservation that rate of change of charge at the node n, Q n dot is equal to current that is flowing into the node n the current flowing out of the node. So, this we know from charge conservation, we can use this we can use these expressions to find out the current head node n. So, I can write I n = Q n dot + I n + 1. Now, I can write express I n + 1 as again, you can see from this expression itself, I can write it as Q n + 1 dot here + I n + 2.

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$$\Rightarrow I_n = \dot{Q}_n + I_{n+1}$$
$$= \dot{Q}_n + \dot{Q}_{n+1} + I_{n+2}$$
$$= \dot{Q}_n + \dot{Q}_{n+1} + \dot{Q}_{n+2} + \dot{Q}_{n+3} + \cdots$$

$\frac{1}{2} L I_n^2 \equiv$ Infinite sum

$\rightarrow$ Let us take Φ, magnetic flux as co-ordinate.

And then I can again write it as Q n dot + Q dot n + 1 + Q dot n + 2. And, in fact, I will get another term I n + 2. So, we will get Q dot n + 3 plus so, we will get a series like this. So, as you can see, from this we can try to find out that current, but problem here is that we cannot write down the magnetic energy term because of this expression here from the current expression as you can see that this results in infinite sum and that means, that we are running into a problem.

If we choose charge as our coordinate we will not be able to write down the energy term in the Lagrangian or the Hamiltonian and mind it you have to note that we have to find magnetic energy at each node at this node n and then we have to find it at this node as well n + 1nth node and n – 1nth node and so on. And already we are running into trouble even with a single node only. So, therefore choosing charge as the coordinate is not a good idea while we try to write down the Lagrangian for the transmission line, now then the next option is what about the magnetic flux. So, now, let us take magnetic flux now, this is our magnetic flux as our coordinate.

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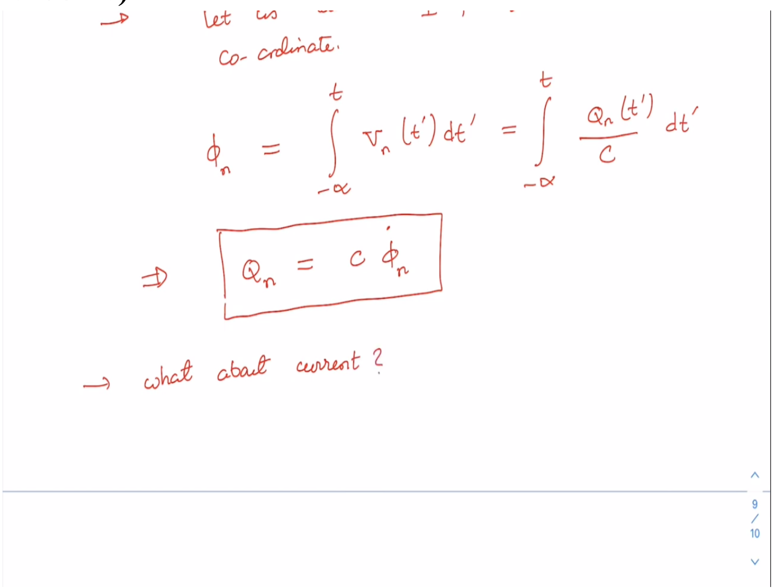

So, we know that the magnetic flux at a particular node n is a time integral over voltage. So, phi n I can write it as say - infinity to sometime t and that is the voltage at the node n and this is the integral. This gives us the magnetic flux and we know that the voltage at node n I can write it as Q charge at the node n divided by the capacitance dt dash. So, from here you can see that.

I can write charge at the node n as simply capacitance into time rate of change of the flux, magnetic flux. So, given the flux phi n at node n, we know the charge Q n at node n, and we need to take simply the time derivative of the flux then we will be able to know the charge. Now, what about the current? Whatever the current because that is where we run into trouble when we took charges our coordinate.

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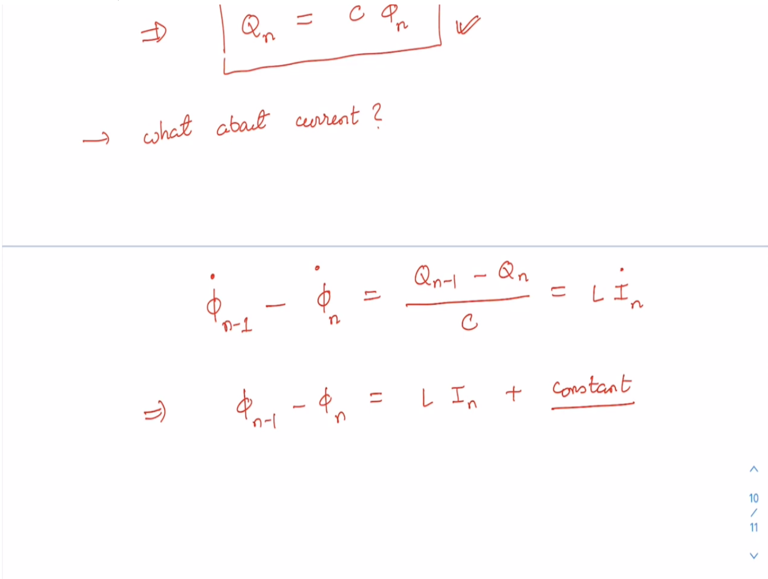

Now, let us consider fluxes at node n here at this node as well as say n - 1th node and take their difference and then take the time derivative. So, what I mean to say is this take the flux at the node n - 1 and take the flux at node n and then takes the time derivative here. So, if I do that from this expression, you can see that I can write it as Q n - 1 - Q n / C, you can easily see this and what is this quantity? This quantity is nothing but the voltage drops across these 2 nodes.

And this voltage drop is responsible for the current I n and these I can write LI n dot. So, we have an equation connecting the time derivative of the magnetic flux to the time derivative of current. So, if we integrate it then we will get phi n - 1 – phi n = LI n + some integration constant. This integration constant can be taken to be 0 as in the infinite past say there was nothing current and fluxes were 0.

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$$\underbrace{\phi_{n-1} - \phi_n}_{\text{Fluxes at nodes}} = \underbrace{L I_n}_{\text{magnetic flux in coil / branch}}$$

$$\mathcal{L} = \text{electrical energy} - \text{magnetic energy}$$

$$= \sum_n \left[\frac{C}{2} \dot{\phi}_n^2 - \frac{1}{2} L I_n^2 \right]$$

So, therefore, I can take this constant to be 0 and then we will simply have phi n – 1, that is the difference of magnetic fluxes at the nodes = L into I n. In fact, this is the fluxes at nodes and this is simply the magnetic flux in the coil or in the branch. Magnetic flux in this branch. So, what we get is that we can obtain the current just by taking the differences of magnetic flux while we can get charges by taking the time derivative of the flux.

Now, we can write down the Lagrangian for the full transmission line. So, the Lagrangian is because now I have taken the magnetic flux as my coordinate. So, the first term would be here now, the electrical energy., electrical energy. Just remembered when I take charge as my coordinate I got magnetic energy as the first term and the electrical energy as the second term. Now, here it would be the reverse, so, it would be electrical energy minus magnetic energy.

When I take magnetic flux as my coordinate and that would be I have to sum up all these, I have to find this energy difference at all nodes. So, at a particular node n, I have this C / 2 electrical energy C / 2 phi n dot square, that is basically the current, charge actually, this is the charge term. And then I have minus half LI n square that is the current that is a magnetic energy part.

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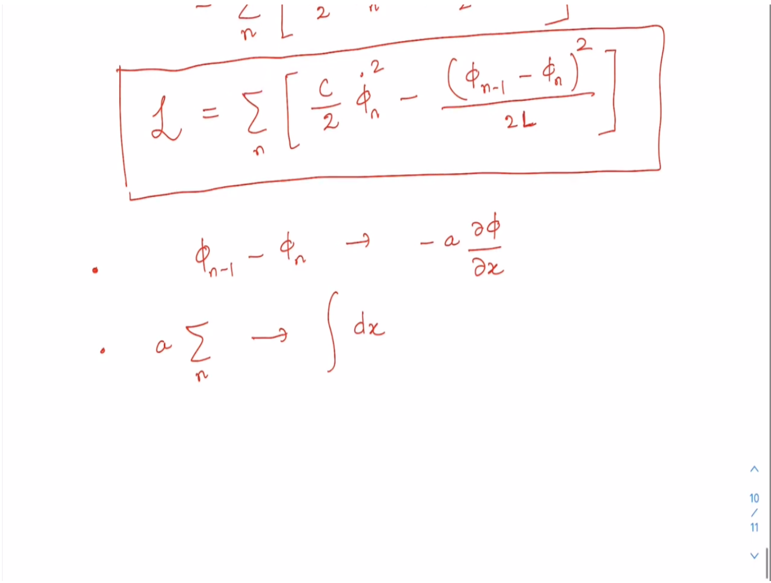

And these I can write as C / 2 phi n dot square minus, now I can express this current in terms of the magnetic flux as phi n - 1 – phi n whole squared divided by 2L and here this curly L is the Lagrangian and this L is our inductance. So, this is the Lagrangian for the discrete version of the transmission line. This is the Lagrangian for the discrete version of the transmission line. But it is straightforward to get the continuum version of the transmission line.

What we just have to take here if we go to the continuum, these difference in fluxes I can express it should be actually converted to a special derivative, a special derivative we can write it as when we go to the continuum domain these kinds of things we discussed already that would be say del pi / del x and this sum I have to convert it to an integral. In fact, if you look at observe the transmission and carefully spacing between the nodes is a. That is the unit cell size and we have to take sum of our different nodes and these I can write it as integral dx.

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$$\cdot \quad a\sum_n \longrightarrow \int dx \qquad \left[\begin{array}{l} c = C/a \\ l = L/a \end{array}\right.$$

$$\mathcal{L} = \sum_n a\left[\frac{c}{2}\dot{\phi}^2 - \frac{a}{2L}\left(\frac{\partial\phi}{\partial x}\right)^2\right]$$

$$\Big\downarrow \; a \to 0$$

$$\mathcal{L}_{continuum} = \int dx \left[\frac{c}{2}\dot{\phi}^2 - \frac{1}{2l}\left(\frac{\partial\phi}{\partial x}\right)^2\right]$$

So, then this Lagrangian. this part of discrete transmission line, I can write in this form also, a into c is the capacitance per unit length. Just recall that c is capacitance per unit length and then we have a l that will be inductance per unit length that is what we are going to utilize here. So, this is c here, c phi dot square - a / twice L del phi del x whole square. And because as in the limit when a tends to 0 I can write for the continuum case, for the continuum data of the transmission line, I can replace the sum by integral and I can write dx.

And the bracketed term would be c / 2 phi dot square - 1 / 2 l, l is the inductance per unit length del phi del x whole square. So, this is the Lagrangian for the continuum model.

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$$\mathcal{L} = \sum_n \left[\frac{C}{2}\dot{\phi}_n^2 - \frac{(\phi_{n-1} - \phi_n)^2}{2L}\right]$$

$$P_n = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = C\dot{\phi}_n = Q_n$$

$$\mathcal{H} = \sum_n P_n \dot{\phi}_n - \mathcal{L}$$

$$= \sum_n \left\{\frac{P_n^2}{2C} + \frac{(\phi_{n-1} - \phi_n)^2}{2L}\right\}$$

But, for our quantizing purpose, we will take the discrete model rather than the continuum. So, we will consider this Lagrangian note this continuum one and along with some boundary conditions. Now, the conjugate momentum is, from this Lagrangian we can write it as P n, the momentum del L of phi n dot phi n, that is the magnetic flux is our coordinate. So, from it we can see that that would be simply c into phi n dot.

And this is nothing but the charge Q the Hamiltonian for the discrete transmission line would be the it would be P n momentum into coordinate time derivative of the coordinate minus the Lagrangian and you can easily work it out and if you work it out, you will find that this would be simply P n square / 2C + phi n - 1 - phi n square divided by twice L.

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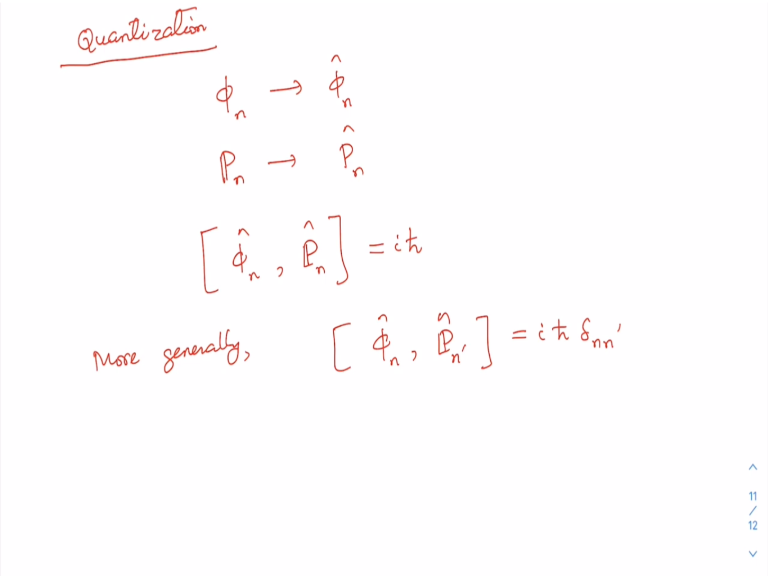

So, as we have already discussed about the quantization of LC circuit, earlier taking the magnetic flux as our coordinate and charge is our momentum, we found that these are canonically conjugate variables. So, therefore, here also that is what we will find in the case of transmission line as well. So, now, we are ready for quantization. So, while we quantize it, we now simply replace its coordinate phi n by the corresponding operator and the momentum P n we replace it by the momentum operator.

And then write down the Heisenberg uncertainty relation in this computation form phi n dot cap P n and = i h cross. Actually, more generally we can write it more generally we can write phi n and P n dash = i h cross delta n n dash.

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$$\left[\hat{\phi}_n, \hat{P}_{n'}\right] = 0 \qquad n \neq n'$$

$$\rightarrow \quad \hat{\mathcal{H}} = \sum_n \frac{\hat{P}_n^2}{2C} + \frac{\left(\hat{\phi}_{n-1} - \hat{\phi}_n\right)^2}{2L}$$

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In fact, the coordinates and the momentum belonging to different nodes say phi n the commutation relation between phi n and momentum P n if this n and n dash are not equal that means, that belong to different nodes then this commutation relation would be = 0. So, as you can see, now, we have this quantum mechanical Hamiltonian that we have to write in this operator form.

That will be = P n square / 2C + phi n - 1 - phi n cap whole square / 2L. This is quadratic in position, quadratic in position and quadratic in momentum and it should remind you about the so-called harmonic oscillator, simple harmonic oscillator Hamiltonian. Also, you should note that this Hamiltonian is actually translationally invariant because the transmission line that we have taken this transmission line is translationally invariant.

Each cell in the transmission line is equivalent to one another. And in fact, this is what we are going to exploit that means the translational invariance of the transmission line we are going to exploit to write the Hamiltonian in terms of normal coordinates by choosing appropriate normal modes. And we are going to discuss it in the next class.

In this class we learned about transmission line and we saw that indeed, a transmission line supports a wave in the form of a current. And then we also learned how to quantize an LC oscillator which is the building block of a transmission line. And it turns out that taking the magnetic flux as the coordinate rather than the charge is more beneficial, if we are interested in quantizing a transmission line. In fact, taking the Hamiltonian for the discrete transmission line, we saw how it can be quantized. However, this Hamiltonian can be written in a more useful form using so called normal coordinates and that is what we are going to do in the next class. So, see you in the next class. Thank you!