Quantum Technology and Quantum Phenomena in Macroscopic Systems
Prof. Amarendra Kumar Sarma
Department of Physics
Indian Institute of Technology, Guwahati.

Lecture – 15
Problem Solving Session-4.

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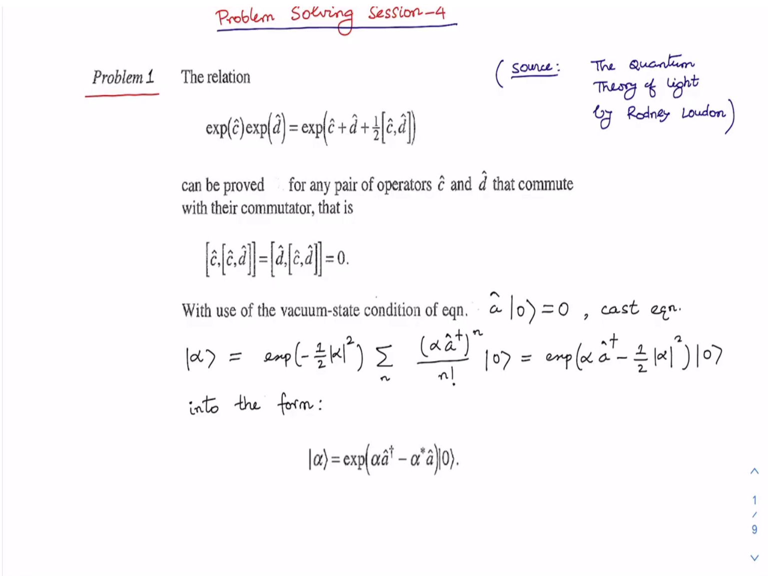

Welcome to problem solving session number 4. In this problem-solving session, we are going to solve problems on the squeezed states, coherent states and even quantization of electromagnetic fields. Most of the problems I have picked up from the classic book, The Quantum Theory of Light by Rodney Louden. So, let us solve the first problem. An operator relation is given and this relation is valid provided the operator c and d satisfy this commutation relations.

And using it also with the vacuum state condition that means when an annihilation operator operates on the vacuum you know that you get 0. Then you have to cast this well-known equation for the coherent state This relation is already all of us know. This relation we have to put it in the form like this. Let us do it.

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Now,

$$\underline{e^{-\alpha^* \hat{a}}}|0\rangle = \left(\underset{=}{1} - \alpha^* \underset{=}{\hat{a}} + \frac{\alpha^{*2}}{2!}\underset{=}{\hat{a}^2} + \cdots\right)\underline{\underline{|0\rangle}}$$

$$= |0\rangle$$

Thus,

$$e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}|0\rangle = e^{\left(\alpha \hat{a}^\dagger - \frac{1}{2}|\alpha|^2\right)}|0\rangle$$

$$= |\alpha\rangle$$

$$\Rightarrow \quad |\alpha\rangle = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)|0\rangle$$

First of all, let me choose operator c and d appropriately. Let me take c to be equal to say alpha a dagger and d to be - alpha star a let us first check whether this choice is all right or not. To do that we have to show that this relations c should commute with the commutator c d. So, let us find it out first. Before I do this let me find out what is c d commutator relation between c and d that would be alpha a dagger here d is equal to alpha star a.

And this I can write as - mod alpha square then I have a dagger a and a dagger a is equal to -1 because a a dagger is equal to 1. So, I will have simply mod alpha square. So, here therefore I have c is equal to alpha a dagger and commutator c d is equal to mod alpha square which is just a number. So, therefore it is equal to zero. So, similarly we can actually prove in the similar way that d commutes with the commutator c d.

So, therefore our choice is perfect c and d is all right. So, therefore we can apply the formula. So, e to the power alpha a dagger e to the power - alpha star a, this formula right, now I am going to apply this formula. So, therefore this would be equal to e to the power alpha a dagger - alpha star a + half of the commutator alpha a dagger - alpha star a and this we can have already we have worked this part.

So, therefore that is equal to mod alpha square. So, therefore I can write it as e to the power alpha a dagger - alpha star a + half of mod alpha square and using this I can therefore write this part. I can write it as e to the power alpha a dagger - alpha star a is equal to e to the power alpha a dagger. I am taking this part to the left hand side then I will have e to the power alpha if I take it to this side then I have e to the power alpha a dagger - half of mod alpha square and I have e to the power - alpha star a.

Now if I operate e to the power alpha star a, I am talking about this part, if it operates on the vacuum state I can expand this into this series like this 1 - alpha star a + alpha star square by 2 factorial a square and so on. If it operates on 0 vacuum state you know that when an annihilation operator operates on the vacuum state we will get zero. So, all the other terms will be zero. So, we will be left out it only this.

So, this would be equal to simply the vacuum state zero. So, therefore I can just write that very trivially I can write e to the power alpha a dagger - alpha star a when it operates on the vacuum state. So, this is what I am writing here if it operates on this then you will be left out with e to the power alpha a dagger - half mod alpha square, okay. This operating on this ket 0 and we know that this is nothing but your ket alpha as already we can you can see it from here.

So, this is your ket alpha and therefore we have written it in the required form. So, it implies that I can have now we have k alpha is equal to exponential alpha a dagger - alpha star a ket 0.

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$$= (n+1)\,|n\rangle$$

$$\hat{a}^2\,\hat{a}^\dagger|n\rangle = (n+1)\sqrt{n}\,|n-1\rangle$$

$$\hat{a}^{\dagger 3}\,\hat{a}^4\,\hat{a}^\dagger|n\rangle = (n+1)\,n\,(n-1)(n-2)\,|n\rangle$$

$$\langle n|\,\hat{a}^{\dagger 3}\,\hat{a}^4\,\hat{a}^\dagger\,|n\rangle = (n+1)\,n\,(n-1)(n-2) \longrightarrow (1)$$

Now let us work out this problem. This is actually related to operator algebra. So, we have to prove this relation. It is looking very complicated but let us see. We know that when the annihilation operator operates on the number state this relation is already you know that would be square root of n and n - 1 ket and if the creation operator operates on the number state will get square root of n + 1 and ket n + 1.

So, we have to utilize this relation. So, let us look at this relation first. If we now apply a annihilation operator on a dagger n which would be a square root of n + 1 ket n + 1 and this would be we will get it as n + 1 ket n, again if I apply it by a operator annihilation operator on the whole thing then I will get n + 1 square root of n n - 1 and so on. Actually, we can now repeat this application of these rules for the sequence of these operators.

And I encourage you to do it. You will get a dagger cube a to the power 4 a dagger n if you work it out you will get you should get n + 1 n into n - 1 n - 2 ket n. So, let me say this is my equation number one and from here immediately okay let me say if I now multiply on the left-hand side by bra n then I will get, okay, I will get simply n + 1 into n into n - 1 n - 2. So, this is what I will get.

So, we have basically worked out this part. Now what about this again let us work that out in the similar fashion we can actually work it out.

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$$\langle n|\hat{a}^{\dagger 3}\hat{a}^{4}\hat{a}^{\dagger}|n\rangle = (n+1)\,n\,(n-1)(n-2) \rightarrow (1)$$

$$\hat{a}^{3}\hat{a}^{\dagger 4}\hat{a}\,|n\rangle = (n+3)(n+2)(n+1)\,n\,|n\rangle$$

$$\Rightarrow \langle n|\hat{a}^{3}\hat{a}^{\dagger 4}\hat{a}|n\rangle = (n+3)(n+2)(n+1)\,n \rightarrow (2)$$

$$\langle n|n\rangle = 1$$

$$\boxed{(n+3)(n+2)\langle n|\hat{a}^{\dagger 3}\hat{a}^{4}\hat{a}^{\dagger}|n\rangle = (n-1)(n-2)\langle n|\hat{a}^{3}\hat{a}^{\dagger 4}\hat{a}|n\rangle}$$

And if we work it out do the algebra it is straightforward algebra but a little bit tedious. a dagger 4 a and if you work it out let me see whether everything, yeah this is what I have here. If you work it out you will get n + 3 n + 2 n + 1 n n. And from here if I multiply it by bra n then I will get a cube a to the power a dagger to the power 4 a n that would be equal to n + 3 n + 2 n + 1 into n because we know that this is equal to 1.

Let us say this is my equation number two. Now if I compare equation number one and two you will see that n + 1 into n is common to both. So, if they have to be equal, so, I have to multiply equation 1 by this one and multiply equation 2 with say this one okay if I do that you will see I will get n + 3 into n + 2. So, bra n a dagger cube a to the power 4 a dagger n that is going to be equal to shown on the other side here just this one that would be equal if I multiply this as n - 1 into n - 2 and I will have n a cube a dagger to the power 4 a n. So, I think this is what is asked in the question and we have proved it.

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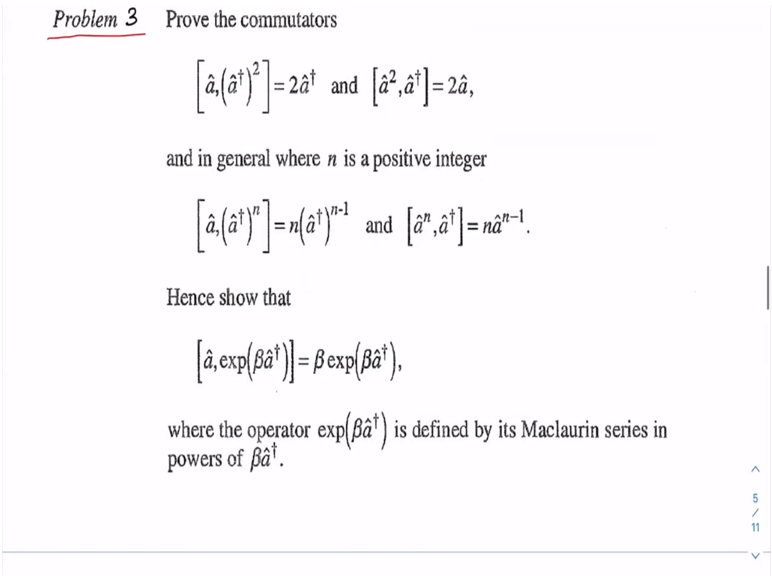

Now let us work out this particular problem. Here, you are asked to prove this commutation relation and then for the general case where n is a positive integer this is what you have to prove and finally you are asked to establish this commutation relation. So, let us do that. It is related to operator algebra.

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Solution

$$[\hat{a}, \hat{a}^{\dagger 2}] = \hat{a}^{\dagger}\underbrace{[\hat{a}, \hat{a}^{\dagger}]} + \underbrace{[\hat{a}, \hat{a}^{\dagger}]}\hat{a}^{\dagger}$$

$$= \hat{a}^{\dagger} + \hat{a}^{\dagger}$$

$$= 2\hat{a}^{\dagger}$$

$$[\hat{a}^2, \hat{a}^{\dagger}] = \hat{a}\underbrace{[\hat{a}, \hat{a}^{\dagger}]}_{=1} + \underbrace{[\hat{a}, \hat{a}^{\dagger}]}_{=1}\hat{a}$$

$$= 2\hat{a} \quad //$$

So, first one this is simple a a dagger squared communication relation we have to work out. So, we can write it as a dagger a dagger plus a a dagger a dagger. Now we know that a a dagger is equal to 1. So, therefore we will have a dagger + a dagger. So, therefore this has to be two a dagger, okay. So, this one is proved. Similarly, the second one also we can prove. Here we have a square a dagger that we can write as a a a dagger + a a dagger a and again because a a dagger this commutation is equal to 1 this is equal to 1. So, we will have 2 a. So, this is also proved.

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$$[\hat{a}, (a)] = ?$$

We notice:

$$[\hat{a}, \hat{a}^{\dagger 2}] = 2\hat{a}^{\dagger}$$

$$[\hat{a}, \hat{a}^{\dagger 3}] = [\hat{a}, \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}^{\dagger}]$$

$$= \hat{a}^{\dagger}\underbrace{[\hat{a}, \hat{a}^{\dagger 2}]} + \underbrace{[\hat{a}, \hat{a}^{\dagger}]}_{=1}\hat{a}^{\dagger}\hat{a}^{\dagger}$$

$$= 2\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}$$

$$\boxed{[\hat{a}, \hat{a}^{\dagger 3}] = 3\hat{a}^{\dagger 2}}$$

Now we have to find out this general relation where a a dagger to the power n where n is a positive integer. So, what is this? So, let us proceed as follows. First of all, we notice that which already we have proved a a dagger square is equal to 2 a dagger. Now let us work out a a dagger cube. This we can write it as a a cube. There are 3 a daggers multiplication is there and these are operators.

So, order matters. This one I can write it as a dagger a a dagger square + a a dagger a dagger a dagger and already we have worked it out and this guy is 2 a dagger, right. So, therefore we will have here twice of a dagger square and here we have a a dagger is equal to 1. So, therefore we will have a dagger square. So, this will give us 3 a dagger square. So, we have a a dagger cube is equal to 3 a dagger square.

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$$= \left[\hat{a},\ 1+\beta\hat{a}^{+}+\frac{1}{2}\beta^{2}\hat{a}^{+2}+\frac{1}{6}\beta^{3}\hat{a}^{+3}+\cdots\right]$$

$$= \underbrace{\left[\hat{a},1\right]}_{=0}+\left[\hat{a},\beta\hat{a}^{+}\right]+\left[\hat{a},\frac{1}{2}\beta^{2}\hat{a}^{+2}\right]+\cdots$$

$$= 0+\beta+\frac{1}{2}\beta^{2}\,2\hat{a}^{+}+\frac{1}{6}\beta^{3}\underbrace{\left[\hat{a},\hat{a}^{+3}\right]}+\cdots$$

$$= \beta+\beta^{2}\hat{a}^{+}+\frac{1}{2}\beta^{3}\hat{a}^{+2}+\cdots$$

$$= \beta\underbrace{\left(1+\beta\hat{a}^{+}+\frac{1}{2}\left(\beta\hat{a}^{+}\right)^{2}+\cdots\right)}$$

$$= \beta\, e^{\beta\hat{a}^{+}}$$

Let us do another one. We have, say, a a dagger to the power 4. what about this? Again, in the similar way we can write it as a you have 4 a dagger that are multiplied and this we can write as a dagger a a dagger to the power 3 and here I can write it as a a dagger and a dagger cube and a a dagger cube already we have worked out from here. So, we will have 3 a dagger to the power 4 and a a dagger is equal to 1. So, this would be this would be 3 a dagger square.

So, it is cube and here we have a dagger cube. So, this will be 4 a dagger cube. So, as you can see from here okay if you examine all these relations. So, this way you can go on and on and therefore I can conclude that a a dagger to the power n would be equal to n a dagger to the power n – 1. Similarly, you can prove applying the same procedure a degree to the power n a dagger this commutation relation you can prove it to be n a dagger a to the power n – 1.

Finally, we are asked to establish the relationship we know what is this computational relation is given and we are asked to prove a with e to the power beta a dagger. So, what is this? We have to prove. e to the power beta a dagger we can write it as 1 + beta a dagger + half beta a dagger whole square + 1 by 3 factorial beta a dagger cube and so on. Therefore, the commutation a e to the power beta a dagger this we can write as a. I have here 1 + beta a dagger + half beta square a dagger square + 1 by 6 beta cube a dagger cube and so on.

So, which I can write as a 1 + a beta a dagger a half beta square a dagger square and so on. So, first of all this is a number. So, this commutation relation will give me 0 plus this one will have beta into a a dagger. So, it would be simply beta, the second one. And the third one I can have half beta square a a dagger square commutation between a a dagger square which already we know that would be twice of a dagger and then I think we'll have again 1 by 6 beta cube a a dagger cube.

So, what we will have here is beta + beta square a dagger then we will have a a dagger cube that would be 3 a dagger square. So, therefore I will have here a half beta cube a dagger square and this I can now write as beta 1 + beta a dagger + half beta a dagger whole square and therefore this expression I can write as beta e to the power beta a dagger. So, this is what was asked to prove. Let us see, yes. So, therefore we have proved this relation.

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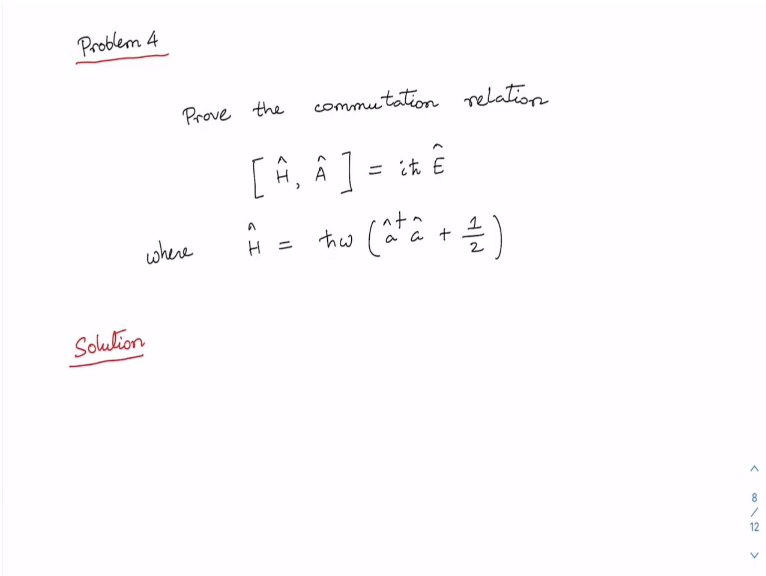

Now let us work out this particular problem where you are asked to prove the commutation relation between the Hamiltonian of the electromagnetic field and its vector potential. Let us work it out. Before I do that let me remind you about the operator form of various quantities. For example, the electric field operator for a traveling electromagnetic wave which we have done in our class, in our lecture as this.

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Solution

$$\hat{E} = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,\vec{\eta}\left[\hat{a}\,e^{i\vec{k}\cdot\vec{r}} - \hat{a}^{\dagger}\,e^{-i\vec{k}\cdot\vec{r}}\right]$$

$$\hat{B} = i\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}}\,(\vec{k}\times\vec{\eta})\left[\hat{a}\,e^{i\vec{k}\cdot\vec{r}} - \hat{a}^{\dagger}\,e^{-i\vec{k}\cdot\vec{r}}\right]$$

$$\# \quad \hat{A} = \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}}\,\vec{\eta}\left[\hat{a}\,e^{i\vec{k}\cdot\vec{r}} + \hat{a}^{\dagger}\,e^{-i\vec{k}\cdot\vec{r}}\right]$$

$$\hat{E} = -\frac{\partial\hat{A}}{\partial t}, \qquad \hat{a} \longrightarrow \hat{a}\,e^{-i\omega t}$$

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So, i h cross omega divided by twice epsilon 0 v square root. It is propagating along say its electric field direction is say eta and we have a e to the power i k dot r - a dagger e to the power - i k dot r. So, that is the electric field and the magnetic field we know the expression was i into h cross divided by twice epsilon 0 v omega square root k cross eta and we had a e to the power i k dot r - a dagger e to the power - i k dot r.

We have not worked out the expression for the vector potential but vector potential can also be very easily worked out and in fact I urge you to verify it. You will find that the expression for the vector potential would be of this type. It would be h cross divided by twice epsilon 0 v omega square root eta vector then a e to the power i k dot r + a dagger e to the power - i k dot r.

So, these are our required expressions. In fact you can verify that this expression is all right for the vector potential because you know that under Coulomb's gauge we wrote our electric field operator as its time derivative of the vector potential and while verifying it you just always keep in mind that this annihilation operator varies in time as a e to the power - i omega t. If you keep these things in mind and if you put it in this expression you will be able to get the expression for the electric field.

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$$= -\hbar \underbrace{\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, \vec{\eta} \left[\hat{a}\, e^{i\vec{k}.\vec{r}} - \hat{a}^{+} e^{-i\vec{k}.\vec{r}} \right]}$$

$$= -\hbar \frac{\hat{E}}{i}$$

$$\Rightarrow \boxed{\left[\hat{H}, \hat{A}\right] = i\hbar \hat{E}}$$

So, please try to verify it. Now let me proceed with the given problem. So, here we are asked to find out the commutation relation between the Hamiltonian and the vector potential. So, we have the Hamiltonian is h cross omega a dagger a. I am doing it for single mode but this can easily be done for multi mode case also. So, here this is the Hamiltonian and the vector potential. Let me write down the full expression h cross divided by twice epsilon zero v omega eta vector.

And I have a e to the power i k dot r + a dagger e to the power - i k dot r. Actually, if you see it essentially this problem is to just to find out the commutation relation between a dagger a and a and also this one with this and then this one with this one. So, therefore we have to just find out these two expressions and then you will be able to work it out very simply. So, this one a dagger a a. So, this would be equal to a dagger a a +, I can write it as a dagger a a which is obviously 0.

So, let us not bother about it. So, this will simply give us -a and similarly, here I will get it as a dagger a a dagger and this will give me +a dagger okay this is what I will have. Therefore, I can have the commutation relation between the Hamiltonian and the vector potential would be equal to, I can just take these things out h cross omega, I can take out okay let us do it. So, h cross we will have h cross omega here h cross divided by twice epsilon zero v omega under root and this eta vector.

Then I have here, I have already known these two relationships. So, I have to utilize it here then you will see that I will get here - a e to the power i k dot r from the first term and from the second one I will get a dagger e to the power - i k dot r, okay. So, I have then I can write it as - h cross. h cross omega let me take inside h cross omega twice epsilon 0 v under root eta vector this I can write as a e to the power i k dot r - a dagger e to the power - i k dot r.

So, this is what I have now if you look at the expression for the electric field, okay, this one if you see then you will know that I have - h cross and the whole this expression is electric field I will get it I just have to multiply it by i and divided by i right. So, I hope you get the idea here. So, this one would be - h cross E electric field operator divided by i this whole thing is that therefore I have here i h cross the electric field operator.

So, I think this is what was asked to prove, yes indeed. So, therefore the commutation relation between the Hamiltonian and the vector potential is this. So, what it means physically is that the energy operator for the electromagnetic field and the vector potential do not commute.

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$$S^\dagger(\xi)\,\hat{a}\,S(\xi)$$

$$= \hat{a} + \xi\,\hat{a}^\dagger + \frac{1}{2!}\hat{a}\,|\xi|^2 + \frac{1}{3!}|\xi|^2\xi\,\hat{a}^\dagger + \cdots$$

$$= \hat{a}\left(1 + \frac{1}{2!}|\xi|^2 + \cdots\right) + \hat{a}^\dagger\left(\xi + \frac{1}{3!}|\xi|^2\xi + \cdots\right)$$

$$= \hat{a}\left(1 + \frac{1}{2!}r^2 + \cdots\right) + \hat{a}^\dagger e^{i\theta}\left(r + \frac{1}{3!}r^3 + \cdots\right) \Big| \; \xi = re^{i\theta}$$

$$= \hat{a}\cosh r + \hat{a}^\dagger e^{i\theta}\sinh r \quad //$$

Let us now work out this problem. You are asked to derive this transformation property for this squeezing operator. In fact, we have discussed it in the class but did not prove it. So, let us prove it here. We know that we have taken our squeezing operator S Xi as e to the power half Xi a dagger square - half Xi star a square then S dagger Xi would be equal to e to the power half Xi star a square - half Xi a dagger square.

Now we have to simplify S dagger Xi a Xi. So, going by this formula, given formula, let us take the operator A to be the small a annihilation operator and B let me take it as half Xi a digger square - half Xi star a square, then we have this formula e to the power - B A e to the power B that is equal to A + A B + 1 by 2 factorial A B commutator then B then we will have term like 1 by 3 factorial. I will have A B B B here.

Like this let us evaluate term by term. First of all, if I evaluate A B this I will have capital A is equal to the small a and B is equal to half Xi a dagger square - half Xi star a square because a commutes with a square. So, we do not have to border that part. So, you have to border about this part. So, this I can write as half Xi a a dagger square which we know that this would be half Xi and this is 2 a dagger. So, therefore this would be simply xi a dagger. Now next term we have A B this commutator with B let us evaluate it and this would be Xi a dagger and B is half Xi a dagger square - half Xi star a square again we have to border about a dagger and a square this part only because a dagger and a square a dagger square will commute.

So, this I can write as minus half let me take this side and Xi into Xi star. So, that would be mod psi square and we will have a dagger a square okay. This we can work out. This would be – half Xi square. So, here I can write it as a dagger a a + a a dagger a a dagger a is equal to – 1. Similarly, here so therefore I will have it as simply a mod Xi square okay. So, let me work out another term that would be say A B commutator with B this already we know this commutator of this whole thing with the operator B.

Let us work it out this would be a modes Xi square and then here I have B is half Xi a dagger square - a half Xi star a square. So, I will get here it is as half mod Xi square Xi a a dagger square and this will give me more Xi square Xi a dagger. So, now let us gather all the terms S daggers Xi a S Xi. So, if I gather the terms, first term is a the second term is Xi a dagger third term is 1 by 2 factorial a mod Xi squared 4th term would be one by 3 factorial mod Xi square Xi a dagger + like this. So, we'll have various terms.

Now if I take a common then I have 1 + 1 by 2 factorial mod Xi square I will get an idea about the series basically here and I will have a dagger if I take common then I have Xi + 1 by 3 factorial mod Xi square psi in terms like this. So, I know that Xi is equal to it is a complex quantity that would be mod Xi which is actually r. Xi is equal to r e to the power i theta. So, mod Xi is equal to r.

So, therefore I can write a 1 + 1 by 2 factorial r square and so on. And here I will have a dagger e to the power i theta r + 1 by 3 factorial r cube and so on. So, you can immediately recognize that here I have a cos hyperbolic r and here I have a dagger e to the power i theta sine hyperbolic r. So, hence we have proved the relation..