Quantum Technology and Quantum Phenomena in Macroscopic Systems

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Lecture – 13
Problem Solving Session-3

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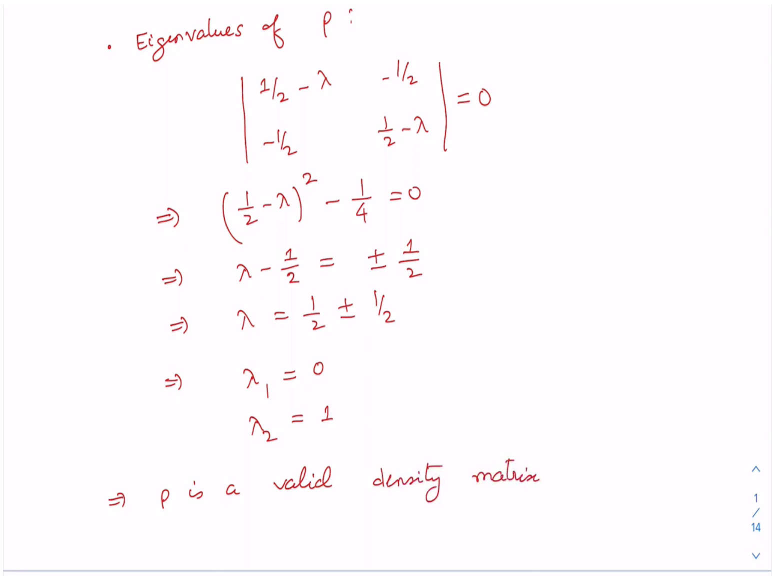

In this problem solving session, we will solve problems related to density operators mainly. So, the first problem, you are asked to verify, if the given matrix rho is a valid density matrix and if so then find the corresponding normalized state. So, let us do it. We know that the matrix rho would be a valid density matrix, provided it satisfy certain condition for example the matrix rho has to be Hermitian ,this is has to be satisfied.

Another condition is the trace of rho has to be equal to 1 and eigenvalues of rho has to be real. Let us check one by one whether these three conditions are satisfied by the given matrix. As I see that the elements here are real. So, finding rho dagger is trivial because we have to first take the complex conjugate and then take the transpose because complex conjugate would be the same one and then if we take the transpose then we will get half minus half minus half and half. So, this is exactly the same as rho.

So, therefore the matrix given matrix is Hermitian. Secondly trace of rho, the sum of the diagonal elements, half plus half that is equal to 1. So, this is also satisfied. Now let us find out the eigenvalues of rho. So, eigenvalues of rho to find it out we have to set the characteristic equation, we have you have to set up the characteristic equation. So, we can easily set it up that would be determinant of say half minus lambda minus half minus half a half minus lambda that is equal to 0.

And if we solve it we will get half minus lambda whole square minus 1 by 4 is equal to 0 or we have lambda minus half is equal to plus minus half. So, therefore what we have is lambda is equal to half plus minus half. So, this is going to yield us eigenvalues lambda 1 is equal to say 0 and lambda 2 is equal to 1. So, as you see that eigenvalues are real. So, definitely it implies that rho is a valid density matrix is a valid density matrix.

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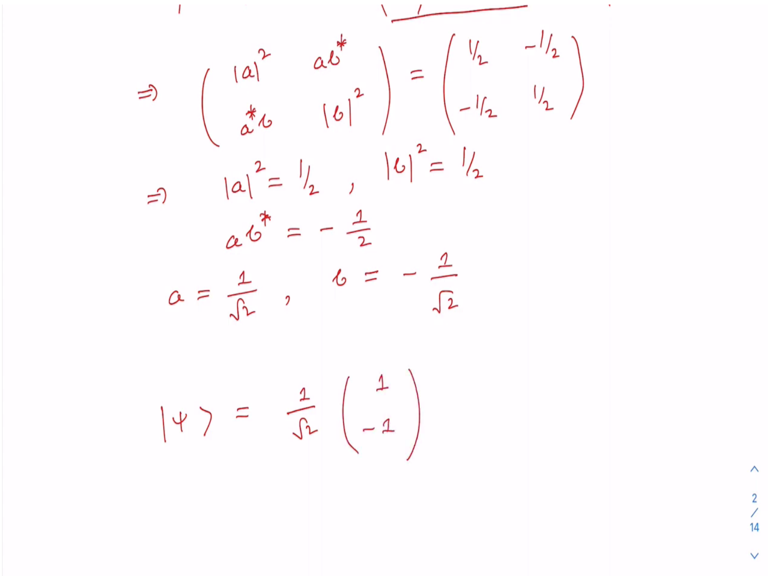

Let me show you one thing quickly.If I find out what is rho square. So, rho is half minus of minus half half. So, I have to take the multiplication of this matrix twice .If I do it as you can see, we will get here the first term would be half and here we will have minus half ,here we will have minus half, here we will have a half. So, rho square is equal to rho. This implies that rho represents a pure state.

We are asked to find out the normalized wave function as well. So, let us, state vector, let us work that out . Rho, we can write in terms of state vector as it is ket psi bra psi. Let us say the psi, ket psi is represented by column vector with elements a b and bra psi would be a rho vector with elements a star b star and if we do it but the way this we have to equate with rho that is your half minus half minus half half.

So, this implies that I have, from here we will have mod a square a b star a star b mod b square and this let me compare it element by element, first let me write it. So, clearly it implies that mod a square is equal to half mod b square is equal to half and a star b or a b star is equal to minus half. So, if I take a is equal to 1 by root 2 and my, I must have to take b as minus 1 by root 2.

So, therefore the state vector psi would be 1 by root 2 1 minus 1. So, this is the normalized state vector.

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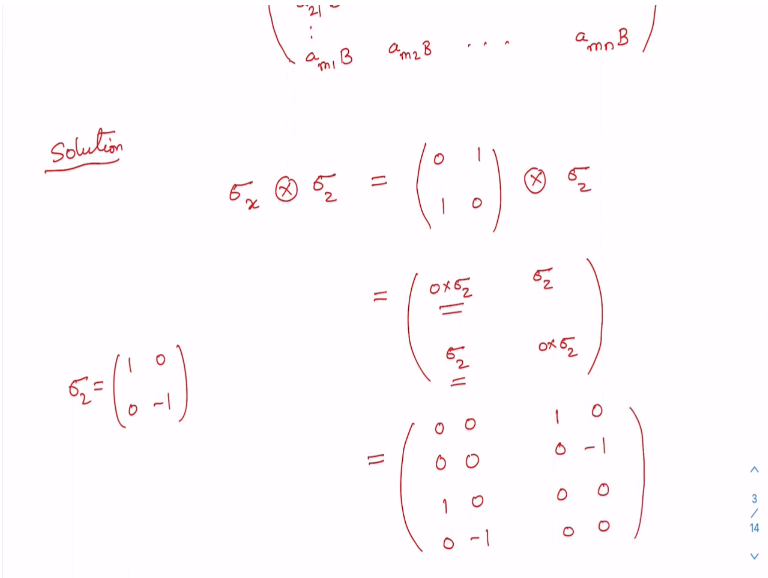

Now let us work out this particular problem .This is actually, you have to find out sigma x cross sigma z, this is basically tensor product between 2 matrices. Just let me quickly remind you about what is tensor product? Tensor product ,say, we have a matrix A ,is a, is an m by n matrix ,B is and say p by q matrix okay ,then tensor product of A and B is denoted as A cross B.This is the symbol for tensor product and that would be, so, we have the element for matrix A is a 11 a 12 and so on.

So, we have to this element would be multiplied by the matrix B and here you will have a 21 B a1 2B, this way we will have up to a 1n B and similarly if you go this side in the column will have a m1 B a m 2a m2 B and so on and finally here we will have a mn B. So, this is what we have. So, let us apply it for our problem . Sigma x tensor product with sigma z we know that sigma x is represented by the matrix 0 1 1 0. So, this is sigma z ,also we know let me go step by step.

So, here I have 0 into sigma z and then we have here 1 into sigma z so this is here sigma z 1 into sigma z here sigma z and 0 into sigma z. So, what I will have because sigma z we know that this is 1 0 0 -1. So, I will have here 0 0 0 0 from this and sigma z is 1 0 0 minus 1 from here I will have 1 0 0 minus 1 and from here I will have 0 0 0. So, you will see we will get a 4 by 4 matrix.

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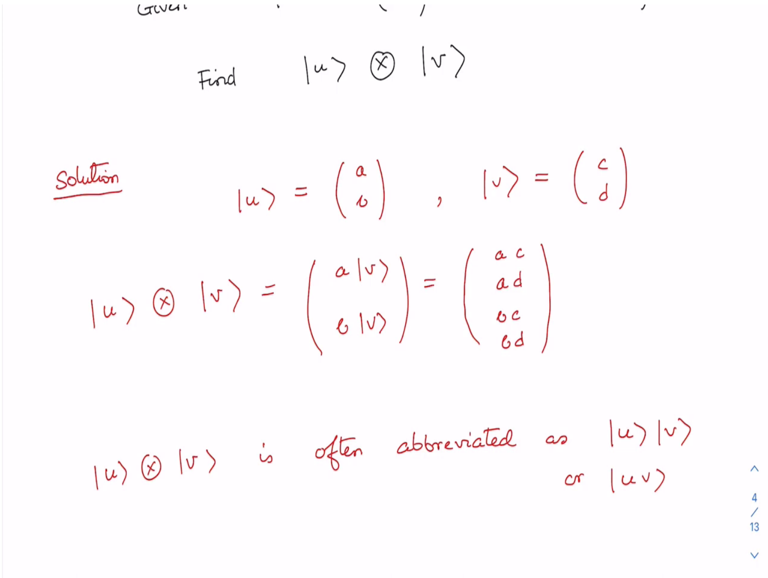

Let us now work out this problem .You are given 2 ket vectors and we have to find out the tensor product of this ket vectors. Let us do it. So, ket u is equal to a b column vector and ket v is equal to c d. So, tensor product of ket u and ket v would be, so, here we have the, let me first write down the elements of u a b and then we will have here the ket vector v. So, now if I write the k vector v as well here , then I will have a c a d b c b d. So, this is the tensor product.

In fact many times or often u tensor product v is often abbreviated as uv or even as uv .Okay.

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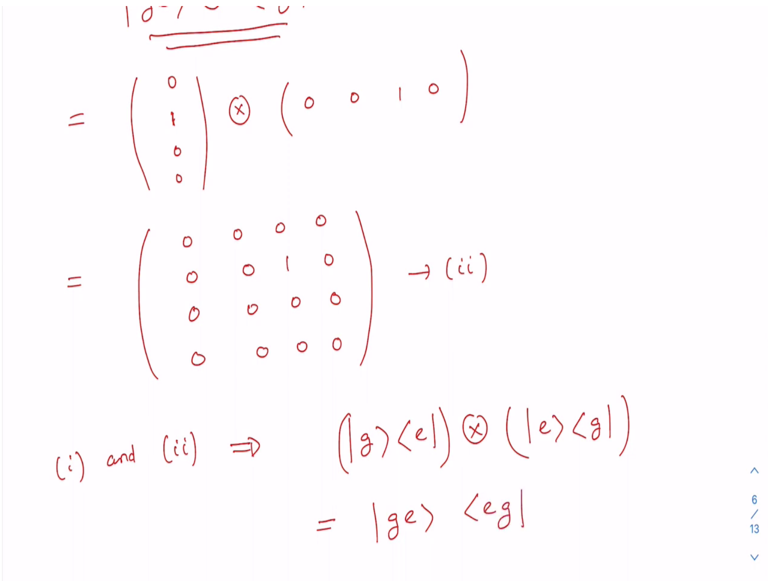

Now let us go to the next problem.You have to show that the tensor product of 2 combination of 2 ket vector, one is ket vector ,okay this is your ket vector, this is your bra, this is your ket ,this is bra. So, you have to basically show that this would be simply equal to this one and you are asked to take ket g as one 0 and ket e 01. So, let us do it. So, first let me work out ket g bra e.

So, this would be the this is the outer product. So, ket g is 1 0 bra e would be 0 1. So, if I take the matrix multiplication I will have 0 1 0 0 and then outer product of ket e bra g that would be ket e is 0 1 and bra g is 10. So, therefore I will have 0 0 1 0. Now let us take the tensor product or direct product ket g bra e tensor product with ket e bra g that would be this multiplication of this matrices.

So, let me write here 0 1 0 0 0 0 1 0 .Of course this is tensor product. So, therefore, what we are, we have here is 0 0 0 0 0 0 0 0 then here we will have 0 0 1 0 and 0 0 0 0 I think it is easy to see. let me say this is equation number one. Now we have ket g ket e actually ket g ket e ,let me get this ket g ket e would be ket g is 1 0 ket e is 0 1. So, its tensor product would be 0 0 0 one 0 0 and ket e ket g, the tensor product again it is ket e 01 ket g is 1 0. So, that would be 0 0 1 0.

So, therefore actually this I can write as abbreviation ket ge and here I can write it ket eg. Now the tensor product of ket ge and bra eg that would be ket ge we have worked out and that is 0 1 0 0 and this is would be the rho vector, eg, ket eg we know,so bra eg would be 0 0 1 0. So, if I do the maths I will get here 0 0 0 0 0 0 1 0 because 1 is sitting there and here I have 0 0 0 0 0 0 0. So, let me say this is my equation number two. So, if I compare one and 2 this is one and then this is my 2.

If you compare both sides ,both the expression 1 and 2 implies that ket g bra e tensor product with ket e bra g, we can write it as ge eg. This is actually a general result and this result is important and it will be utilized in the next problem.

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$$+ \; |e\rangle\langle e| \otimes |0\rangle\langle 0| \big)$$

$$\rho' = Tr_1(\rho)$$

$$= \frac{1}{2}\Big\{ \underline{\underline{\langle g|g\rangle}}\, |e\rangle\langle e| - \underbrace{\langle g|e\rangle}_{0} |e\rangle\langle g| - \underbrace{\langle e|g\rangle}_{=0} |g\rangle\langle e| + \underline{\underline{\langle e|e\rangle}}\, |g\rangle\langle g| \Big\}$$

$$= \frac{1}{2}\big(\underbrace{|e\rangle\langle e| + |g\rangle\langle g|} \big)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let us do this problem .You are given a state vector ,a state ket and you are asked to find out the corresponding density matrix. Then partial trace it over the first Hilbert space to find the density matrix of the second system. So, as you can see from this expression for the state vector here that this is actually a composite system, system 1 and system 2 is there and we have to find out the density matrix for the second system by taking partial trace method. So, let us do it.

So, we have ket psi is equal to 1 by root 2 ket g ket e, of course this is tensor product, right, ket g tensor product ket e minus ket e tensor product g .Density matrix would be ket psi bra psi and if we do that 1 by 1 by root 2 will appear twice. So, we will have half here, let me write here ket g ket e minus ket e ket g tensor product with the bra psi that would be bra gi bra e minus bra e bra z.

Let me now open it up and have I can use the previous problem's result here. So, this one tensor product with this one is going to give me ket g bra g tensor product with ket bra e .Then this one with this one will give me ket g bra e tensor product with ket e bra g ,then let me take this one with this one ,so I will have minus ket e bra g tensor product with ket g bra e. And finally let me take tensor product with this one ,with this one and this will give me plus ket e bra e tensor product with ket g bra g.

So, this is what we have that would be our density operator. Now to get the system ,second system ,we have to trace out first system. So, the second, density operator for the second system would be, I have to trace out system one. So, that is simple we know it how to do that. So, that would be here half, if I take the trace here then as you know that this is my first system, this is my second system.

So, if I take the first system ,here it would become simply this direct product would be become. Now scalar product like this and here I have second system would be ket e bra e. Similarly for the second term that would be I will have scalar product g e and here ket e bra g and the third term will give me scalar product eg and here I have ket g bra e and the last term will give me scalar product ee and here I will have ket g bra g and I know that these are ket g and ket e are orthogonal to each other.

So, this is normalized ,this is normalized but this is equal to 0 ,this is orthogonal to each other, scalar product would give you 0. So, I will end up with half ket e bra e plus ket g bra g, in fact this is nothing but identity matrix, this part is identity matrix 1 0 0 1.

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$$= \frac{1}{2}\Big[\, |0\rangle\langle 0| \otimes \ldots$$
$$|1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \Big\}$$

$$\rho_A = Tr_B\, \rho_{AB} = \frac{1}{2}\Big[\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{\mathbb{1}}\Big]$$

Sly, $\rho_B = Tr_A\, \rho_{AB} = \frac{1}{2}\Big[\underbrace{|1\rangle\langle 1| + |0\rangle\langle 0|}_{\mathbb{I}}\Big]$

Thus, $\boxed{\rho_A = \rho_B = \frac{1}{2}\mathbb{1}}$

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Let us now work out this problem.This is similar to the previous one for a system of 2 qubits in a single state is given in terms of density operator ,this is the density operator for the composite system .Here the first one this ket 0 belongs to system A and ket 1, this is actually tensor product, it belongs to system B. Similarly here it belongs to A and this belongs to B ,here it belongs to A ,it belongs to B, this belongs to A, this belongs to B.

Let us work it out .You are asked to find out the reduced density matrices for each qubit separately. So first of all ,let me open up this density operator rho AB .Let me write first it is half ket 01 + ket 10 tensor product to it bra 01 + 10. If I open it up term by term. So, let me take the tensor product with this one with this one. So, I will have ket 0 bra 0 tensor product with ket 1 bra 1.

So, you see this is system A , this is system B belonging to and similarly we have, now you take this one with this one then we will have ket 0 bra 1 ket 1 bra 0. And now let me take direct product with these 2 then I will have ket 1 bra 0 direct product with ket 0 bra one plus ,now let me take this one with this one I will have ket 1 bra one tensor product with ket 0 bra 0. So, then finding out the reduced density matrix for a would be easy, I have to just trace out B from rho A B.

Then I will get rho A and this is system A, this is system B, similarly for all this all these other terms. So, if I take trace out B ,we know we will get ,because ket 0 and ket 1 are orthogonal to each other. So, if I take trace out B. So, this is going to be the scalar product of 1 1. So, that would be one and here I will have ket 0 bra 0 this term would go to 0 because the scalar product of 1 0 would be 0 and I will be left out with, I need this term from this term I will have ket 1 bra 1.

Similarly you will get rho B, I have to trace out A from rho AB and here I will get half ket 1 bra 1 plus ket 0 bra 0. So in fact as you can see I have rho A is equal to rho B and that is equal to half and as you can see these are identity ,this is identity matrix, this one also. So, this is what I have .Okay.

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$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}} \left(\quad \right)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar m\omega}} \left(m\omega\hat{x} - i\hat{p} \right)$$

Verify the following four facts :

(i) $\left[\hat{a}, \hat{a}^{+} \right] = \mathbb{1}$

(ii) $\left[\hat{a}, \hat{a}^{+}\hat{a} \right] = a$

(iii) $\left[\hat{a}^{+}, \hat{a}^{+}\hat{a} \right] = -\hat{a}^{+}$

(iv) $\hat{H} = \frac{\hbar\omega}{2} \left(\hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} \right) = \hbar\omega \left(\hat{a}^{+}\hat{a} + \frac{1}{2} \right)$

Now let us work out this simple problem on harmonic oscillator. Consider the quantized simple harmonic oscillator, the Hamiltonian is given and the annihilation operator and the creation operator are also given .Using this you are asked to verify these four facts that a a dagger is equal to 1. And that is actually identity and these relations you have to prove and finally you have to show that the Hamiltonian can be written in this form using this annihilation and creation operator and this is equivalent to this form well known form .Let us do this.

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(iv) $$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$= -\frac{\hbar\omega}{4} \left(\hat{a}^{+} - \hat{a} \right)^2 + \frac{\hbar\omega}{4} \left(\hat{a} + \hat{a}^{+} \right)^2$$

$$= \frac{\hbar\omega}{2} \left(\underline{a a^{+} + a^{+} a} \right)$$

$$\left[\hat{a}, \hat{a}^{+} \right] = 1$$

$$\hat{H} = \hbar\omega \left(\hat{a}^{+}\hat{a} + \frac{1}{2} \right),$$

So, we have annihilation operator a is 1 by square root of twice h cross m omega m omega x cap + i p and a dagger is equal to 1 by square root of twice h cross m omega m omega x cap - ip. So, from these 2 relations we can express what is x cap and what is p cap, x cap would be we actually know it that it would be square root of h cross by twice m omega that is the zero point fluctuation then a plus a dagger and p would be equal to i square root of m h cross omega by 2 a dagger - a.

Now first relation a a dagger ,so if I put the expression for a in a dagger then I have here a is equal to m omega x k + i p, then the factor of 1 by h cross m omega, let me take it outside, here I have 1 by twice h cross m omega outside and from a dagger I have m omega x cap - ip and again the factor of twice h cross m omega let me take it out. So, therefore I will have 1 divided by twice h cross m omega m omega x cap + i p m omega x cap - i p.

Now if I take the commutation I know that x cap and x cap this will get commute ,they commute only x cap, I have to just bother about the cross terms only ,these cross terms. So, therefore what I will have here is 1 by twice h cross m omega ,let me write it this way, I have m omega minus i x cap p okay. So, I am taking this term and this term .Then I have from this term and this term I have i m omega p cap x cap .We know the commutation relation between x cap p cap that is equal to i h cross.

So, therefore I have one by twice s cross m omega minus i m omega i h cross and here I have i m omega i h cross. So, you can immediately see that this would be 1 by twice h cross m omega and this one and this one would be twice h cross m omega and this is equal to 1 .This is the first part we have solved. Second one ,second relation we have to prove a a dagger a, these are very trivial but only for those who may not be familiar I am doing this problem is for them only.

So, you have a a dagger a, I can write it this way and we know that a a dagger is equal to 1. So, this will give me simply a .And then the third relation we have to prove a dagger a dagger a and here I have ,I can write it as minus a dagger a a dagger ,this I can write as minus a dagger a a dagger and I know that a a dagger is equal to 1. So, I have here minus a dagger. And finally in the last part of the problem, the Hamiltonian we have to express in terms of creation and annihilation operator .Hamiltonian is given as p square by twice m plus half m omega square x square.

If I put the form of x and p which , from here x and p ,if I put it here I will get the first term would turn out to be minus h cross omega by 4, you can check it .You'll have a dagger minus a whole square plus h cross omega by 4 a + a dagger whole square. So, if you do the maths you will get h cross omega by 2 a a dagger + a dagger a and again utilizing the relation a a dagger is equal to 1 ,you can also show that x is equal to h cross omega from here ,you can show that this would be a dagger a plus half.

So in the next problem solving session we will actually try to solve problems from coherent and squeezed states as well as some problems related to quantization of electromagnetic fields.