Nuclear and Particle Physics
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Module – 09
Electroweak Interactions
Lecture – 03
Electromagnetic Interactions

We will continue our discussion on the relativistic quantum mechanics. So, we talked about the Klein Gordon equation that in relativistic wave function a relativistic particle satisfy and we also mentioned the difficulties with the Klein Gordon equation. We will come back to the difficulties later, but let us continue our discussion now considering the particle interactions at first we will consider the electromagnetic interactions.

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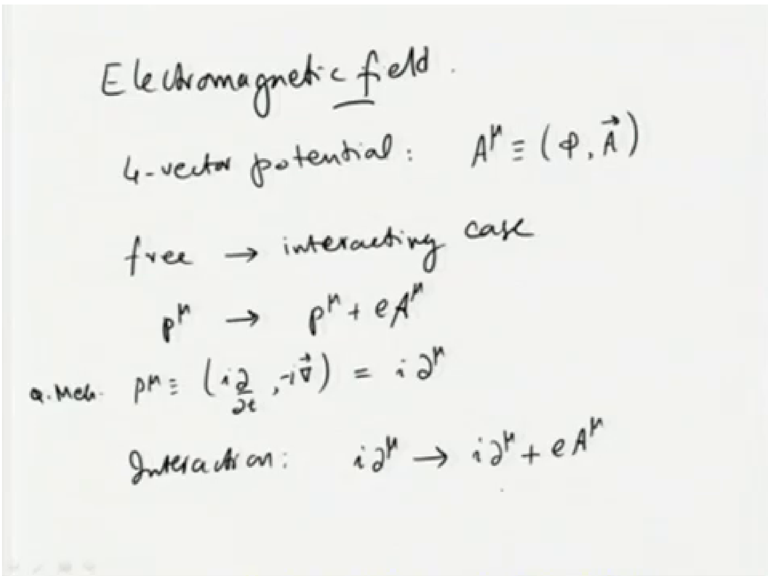

So, electromagnetic field can be described by potentials ok. We will denote this by a mu and the 0th component is the electrostatic potential phi and 1 2 3 components are essentially the electromagnetic. So, the magnetic vector potential.

So, now when we consider classical electrodynamics itself, we can consider this potential 4 vector potential in terms of phi and 3 vector a in nonrelativistic case and in the relativistic classical electrodynamics, we can consider them as components of 4 vector a mu.

And if you look at standard textbooks of electrodynamics which also incorporate includes the relativistic classical electrodynamics, then you will see that if you have a charged particle with momentum P ok.

And if you want to introduce electromagnetic interaction of this particle with the electromagnet with some electromagnetic field represented by a mu, then you can take the free case and consider or go to the interaction interacting case by taking the 4 momentum P mu to P mu plus e elect the charge of the particle times the potential A mu. So, here we are specifically considering a particle with electric charge e otherwise you will have to multiply it with the corresponding charge of the particle you know it.

So, since we have P mu represented by the 4 vector derivative dou by dou t and gradient we will have actually a minus s 1 minus i gradient, where we have taken you know h cross equal to 1.

So, from classical to quantum we will go to ok. So, this is the quantum mechanical conversion from classical to quantum mechanics what we had to do is to consider the momentum as the operator. So, p mu is basically i dou by dou t and minus i gradient and if we consider or this the compact notation, we can take it as i dou mu and in the interaction picture, we have to take i dou mu to i douw mu which corresponds to the momentum plus e times a mu a mu is at this moment the electric field.

Look with the vector potential in the quantum picture corresponding to the electromagnetic field considered. We will see that we will be able to quantize this later on we will see that we can interpret a mu as the particle or the quantum or representing the wave function of the quantum of the particle corresponding to the electromagnetic field which is photon.

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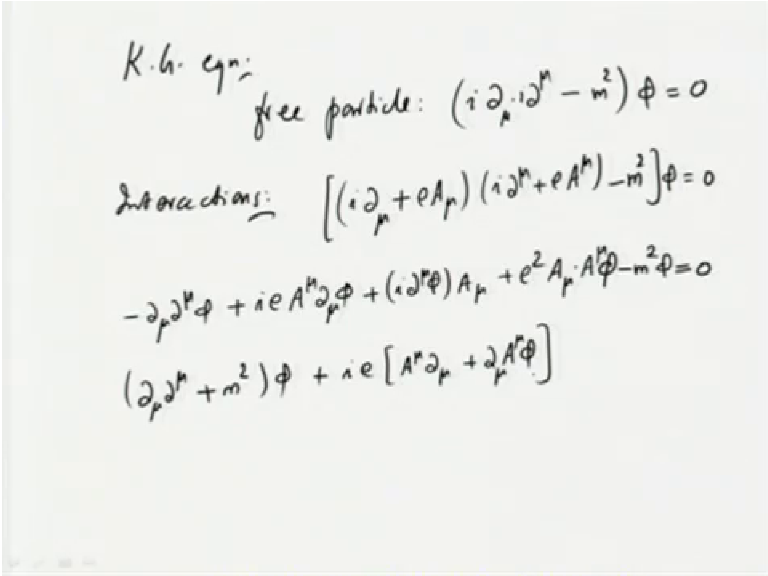

Now, with this let us look at the Klein Gordon equation. Klein Gordon equation for an equation for a free particle we had dou mu or P minus sorry i dou mu i dou mu minus m square phi equal to 0 this gives P square plus m square phi equal to 0.

Now interacting case when we switch on the interactions we will i dou mu plus e A mu i dou mu plus e A mu instead of i dou mu and then m square as usual and acting on phi, expanding this will give you minus dou mu dou mu phi plus i e A mu dou mu phi plus, i dou mu phi A mu plus, e square A mu A mu minus A m mu A m mu phi minus m square phi equal to 0.

With this is basically clubbing or the non-interacting the free case and then the interacting case separately multiplying. Firstly, with a minus sign I get dou mu dou mu plus m square acting on phi which is basically the free part of the equation plus i e A mu dou mu plus dou mu A mu phi.

Sorry here there is a mistake that we made the earlier expression it should have been dou mu i dou mu a mu phi ok.

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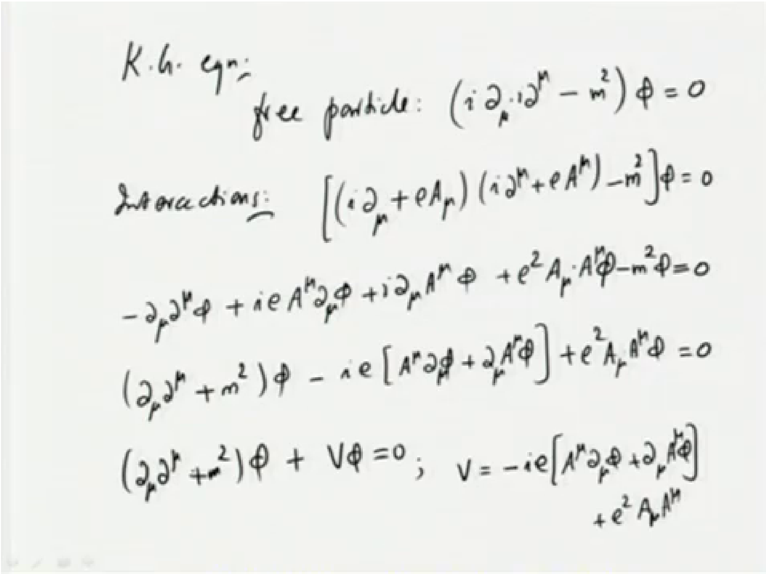

So, this plus e square a mu a mu phi equal to 0. Now this has 2 terms which should correspond now to the interaction of the electromagnetic field with the charge particle phi phi was missing there. So, I just added that call this the vect that the potential. So, let me rewrite this equation as dou mu dou mu plus m square, which is the free part acting on phi plus V phi equal to 0.

So, this is the usual way of writing the kinetic energy term mass term and the potential. So, potential here is essentially minus at let me say the minus sign also here there will be more careful. So, minus ie A mu dou mu phi plus dou mu A mu phi plus A square A mu A mu all right. So, this is again the potential operator. So, there is no phi as such there A mu A mu.

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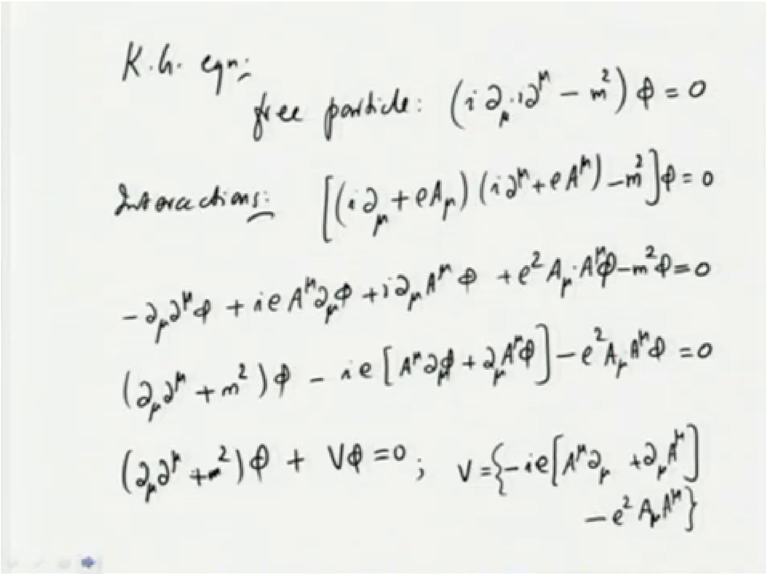

And the whole thing is a vector is the potential which acts on phi .

Now, let us consider the transition amplitude T f i in this case.

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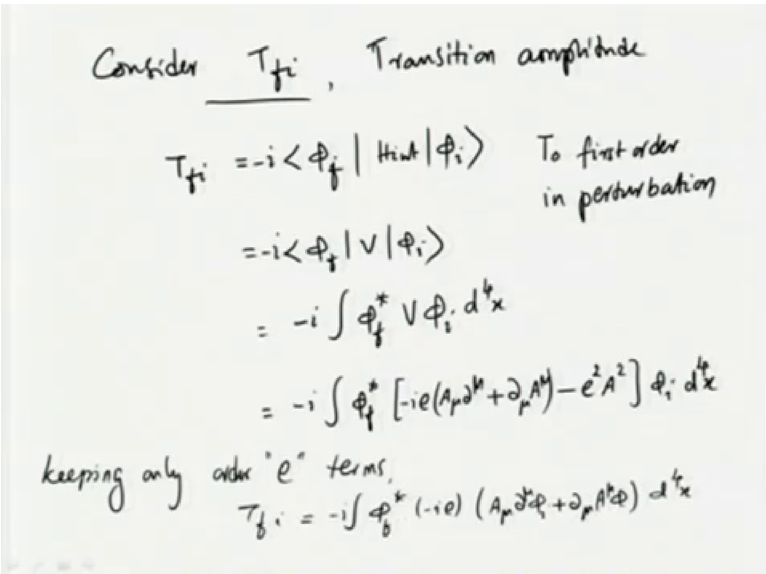

Let us say in the interaction case what we interest we are interested in is if you have an initial particle after interaction, what happens to that particle is one interesting question . So, this transition amplitude we had written earlier in the case of non-relativistic quantum mechanics as inish the final particle wave function h interacting and initial particle this I will remind you this was done to first order in perturbation.

So, if you consider the perturbation theory, the first order correction or further to first order the transformation transition amplitude is given in this fashion, you know, but the case when we actually have interaction written as in terms of the potential it is basically V phi i and for us this is minus i interaction right there were the I am using if you look at it carefully.

So, I will take the position representation and write it as phi f star V phi i d 4 x. For the potential that we just now considered we can write this as minus i interacting integral phi f star and the potential which is minus ie A mu dou mu plus dou A mu plus actually A minus because this is minus e square A square phi i d 4 x.

So, another mistake here with the sign so, we should have taken it as ok. This is the transition amplitude we wanted we were the first order term. Now when you consider the second order perturbation, you will have as essentially the square of this thing coming in here because there will be other things changes, but essentially this will be square of the potential that will come in.

When you look at the square of the potential even how actually the first term square will have an e square times, whatever else with that and then there will be an eq which is a cross product between cross term between these 2 terms in the potential and there is an e power 4. Important thing is to notice the e square term appearing in the second order current. So, we will take the approach that instead of taking the first order second order in expansion of V, we will look at what is the order in e square e, what is the power of V that appears in this thing in the expression.

If you do that you will see that if you include the e square higher order terms there would not be any more term which is linear in e, which will have a which is proportional to e the charge. When you consider this next order that is e square we will see that there is already a term property square in the first order perturbation, but then there will be other e square terms coming from the second order.

So, if you want to take all the particle all the terms to the e square order then we will I to go to second order in that this thing.

What we will do here is since we are not going to consider the second order perturbation theory and to be consistent keeping only the same keeping the correct the interactions at the same order of e, we will keep only the first order term and then discard the second order term. This can be done also because e square is a very small quantity compared to e all right. And e square itself is of the order of 10 power minus 2 and e is going to be therefore, order of one and then there is an order of magnitude difference between these 2.

So, we can actually neglect it ok. So, essentially keeping only or the e T f i is equal to minus A integral phi star minus ie A mu dou mu phi plus dou mu A mu phi d 4 x. So, the second fight first phi is the final 5 second one is the initial phi.

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$$T_{fi} = -i \int \phi_f^* \, (-ie) \left(\partial_\mu A^\mu + A_\mu \partial^\mu\right) \phi_i \, d^4x$$

$$= (-i) \int (-ie) \left[\phi_f^* \, \partial_\mu A^\mu \phi_i + (\phi^* \partial^\mu \phi_i) A_\mu\right] d^4x$$

$$= (-i)(-ie) \int \phi_f^* (\partial_\mu A^\mu) .$$

Now let us take this T f i and look at it and closely look at it closely phi star minus ie dou mu A mu plus A mu dou mu phi i before x equal to minus i integral, minus i e phi star dou mu A mu phi i plus phi star dou mu phi i a mu.

First term I will write as minus i e phi star phi f star dou mu A mu we will actually write it as a total derivative right.

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$$T_{fi} = -i\int \phi_f^* (-ie)(\partial_\mu A^\mu + A_\mu \partial^\mu)\phi_i \, d^4x$$

$$= (-i)\int (-ie)\left[\phi_f^* \partial_\mu A^\mu \phi_i + (\phi^* \partial^\mu \phi_i) A_\mu\right] d^4x$$

$$= (-i)(-ie)\left\{\int \partial_\mu(\phi_f^* A^\mu \phi_i)\, d^4x - \int(\partial_\mu \phi_f^*) A^\mu \phi_i \, d^4x + \int (\phi^* \partial_\mu \phi) A^\mu \, d^4x\right\}$$

So, we will write it as a total derivative acting on the plotter all of those times phi f star mu phi i d 4 x. So, I split this. So, from this term to get the first term in the earlier line we had to subtract dou mu phi star a mu phi i f d 4 x and then the remaining term integral phi star dou mu phi a mu difference.

Now, look at the first term the first term is derivative term integrated over volume for volume. So, that is going to give me. So, let me look at that.

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$$\int \partial_\mu(\phi_f^* A^\mu \phi_i)\, d^4x = (\phi_f^* A^\mu \phi_i)\Big|_{\text{at the boundaries.}}$$

$$\int_0^1 dx$$

So, integral dou mu phi star A mu phi d 4 x is equal to phi star A mu phi the subscripts this is A mu phi i at the boundaries what do i mean by that?

Look at this integral d x 0 to 1, what is this right it is not integral integral d by dx of f dx 0 to 1.

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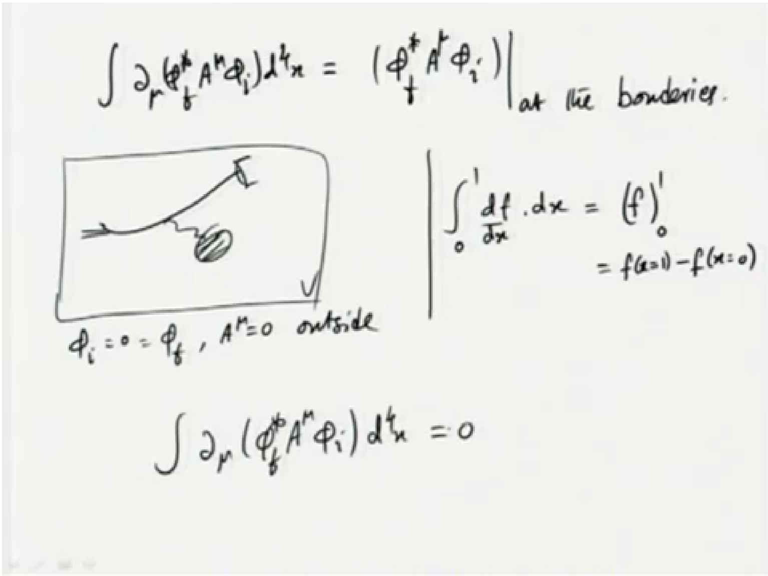

What is this this is equal to value of f at boundaries and 0 and 1, which is equal to ef x equal to 1 minus f x equal to 0. So, this is a 1 dimension case and the boundaries are 2 points line integral boundaries are 2 points, the ends of the line segment where over which you are integrating it and that will give you value of the function at the 2 boundaries or the difference in that is.

Similarly, here this is the boundary of this there is a 4 volume that we are considering integrating. So, at the boundary meaning it is the boundary of that 4 volume, which means essentially the surface of the 4 dimensional spaces. Now see what we are usually doing is we are interested in is a particular kind of a something happening in a in a particular region let us say electron interacting with some nucleus or something.

So, if you take the larger of volume then whole thing is happening inside this. So, there will be a beam generator and then it is going in and there will be a detector that you can put here to detect this etcetera. In any case essentially you can think about everything happening in a large volume V and integrate over that and outside there is no field there is no sorry there is no wave function there is no particle or there is no potential.

So, the electromagnetic potential exists only in a small local local space or in the volume V and outside the volume V this is equal to 0. Similarly the wave functions initial and final wave functions also we will consider to be present within the volume.

In that picture this values of phi f phi i mu all of this will vanish at the boundaries phi i star equal to phi is equal to 0 at the boundaries. So, is phi f and A mu all are equal to 0 outside so, on the boundary also.

So, therefore, this integral therefore, vanishes phi star A mu phi i d 4 x is equal to 0 coming back to our expression for T fi the first term vanishes second term is this.

So, let us write that down.

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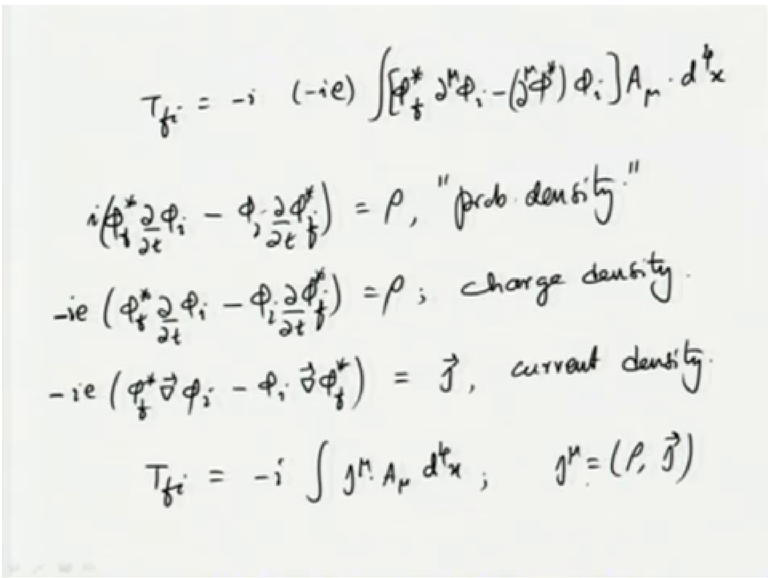

So, we have a minus i T f i is equal to minus i minus i e integral phi star f dou mu phi i ok, this is essentially the last term here. And the second term is minus the minus sign there dou mu phi star phi star phi i acting on sorry multiplied by mu d for yes. Now what is this term here. So, we have phi f star i phi f star dou by dou t of phi i minus phi dou by dou t of phi star earlier we had denoted by rho probability density.

Now, we have here minus i or e. So, we have actually i e phie f star dou by dou t phi i minus phi dou by dou t phi star this is what we call rho ok. And we call this the new rho the charge density we have multiplied the probability density though well let me put this under course the probability density for the way we had defined earlier and then put this as a probability density under course.

Because that is not exactly we are going to interpret it now. So, this with minus e we will call it the charge density. Similarly we have the spatial component here, which is minus i e phi star grad phi 3 phi minus phi i grad phi f star this is fine time 5 f is essentially what we call the current density.

So, the probability density in the probability current density we multiplied by the electric electric charge, we call it charge density and the current density respectively I think this is with A with A minus sign already check that out.

So, now with this I can write T f i as minus i integral j mu A mu before x where j mu is rho j the 4 vector current density. The charge density and the current density together considered as a 4 vector, which is the charge current for vector current density for this.

So, what we have done is we had considered interacting picture in which the transition amplitude can be written in the first order perturbation in this fashion and by considering the or interpreting them in terms of charge density and current density, we could write T fi as an interaction of the electromagnetic field represented by a mu with the 4 vector current density j mu ok.

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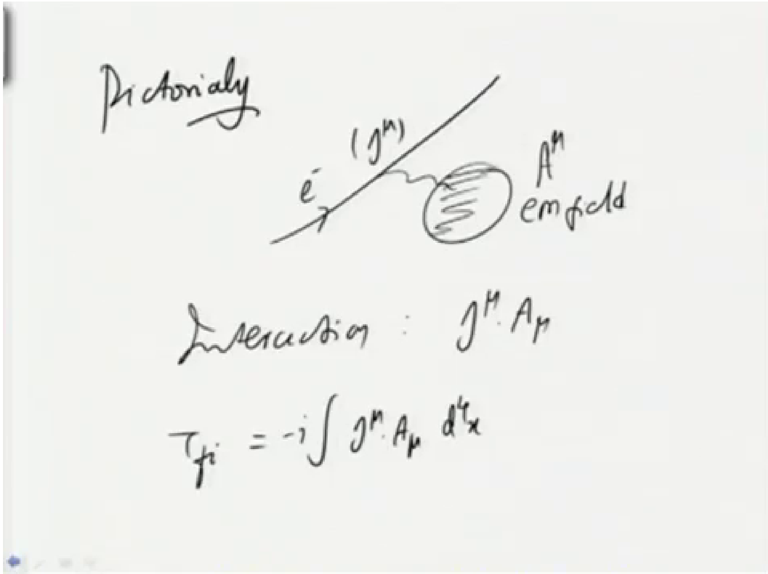

So, pictorially we can consider an electric current say due to electron interacting with the electromagnetic field of something, whatever it is represented by the electromagnetic potential e mu. So, this is your current j mu and this is A mu.

So, the interaction is basically between the current and the electromagnetic potential. And the transition amplitude when you have an initial and final state wave functions included in the current or interaction amplitude, the transition amplitude is given by minus i j mu A mu over the volume d 4 x we will consider more aspects of this as we go.

Now let us look at this electrostatic electromagnetic potential a mu a little more. So, we had first considered the free particle wave functions the equation corresponding to that the dynamical equation called the Klein Gordon equation corresponding to that, incidentally the problem with the negative probability density is almost taken away.

You remember in the last discussion we had said that rho is equal to twice e the normalization wave function, when we consider the plane wave's free particle can be considered as plane waves or superposition of plane waves. And then if there is any problem with any particular solution then that is a difficulty that may linger in the more realistic solution, which is superposition over the plane wave solutions.

So, plane wave solution had a problem that energy is going to be negative in some cases which means that negative energy solutions excess we have to do something about it.

And then we saw that the charge the probability density is equal to or it is proportional to the energy and therefore, energy was 2 negative values then this probability density goes to negative values and then therefore, we cannot interpret it physically as probability density.

But here if we interpret it as charge density by multiplying the probability density does whatever we had earlier called the probability density, by the electric charge of the particle concerned we could charge density and charge the density because the electric charge can be either positive or negative can be negative so, no problem sir.

The problem still remains of the negative energy and then we will come back to that later. And then we consider the interaction picture here and then interaction picture we had introduced everything and then we finally, said that we could consider the transition amplitude in an interaction. As product of the charged in current density and the electromagnetic potential and integrated over the volume, before going into more details of this interactions come back to the Electromagnetic field in relativistic quantum mechanics.

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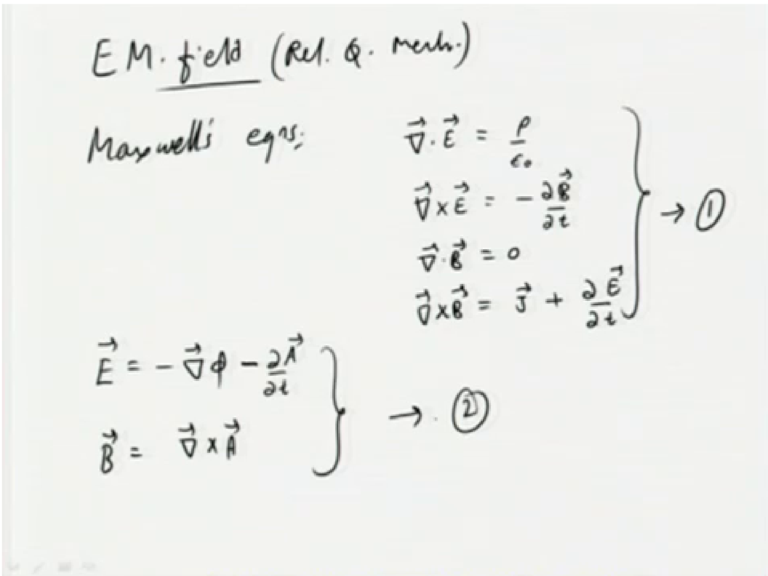

Let us consider the Maxwell's equation, in the nonrelativistic form classically we can write it as del dot e is equal to rho over epsilon 0 these rho is exactly the charge density we have the same sometimes the same notation representing different things, but in depending on the context I believe you will be able to without confusion understand what is what?

So, curl of e is equal to minus dou B by dou t the electrostatic potential is called free, but electric potential due to changing magnetic field can be. So, diverse of B is always 0 curl of B is basically the current, but if you have changing electric field then again you can have minority field or the curl of magnetic field then is proportional to change in the rate of change of electric field these are the Maxwell's equations.

So, now what is e in this picture of potential we have e equal to minus grad phi, which should have been the case when you have electrostatic potential, but the curl is not 0 here it could be due to the change in the magnetic field. So, we can actually have dou by dou t A mu were non gradient term. So, the curl of this need not be 0 and B is equal to curl of the third comment the special component of the potential l.

Et me call this set of these equations Maxwell's equation the well-known Maxwell's equations 1 and electric field and magnetic field expressed in terms of A and phi plus equation 2.

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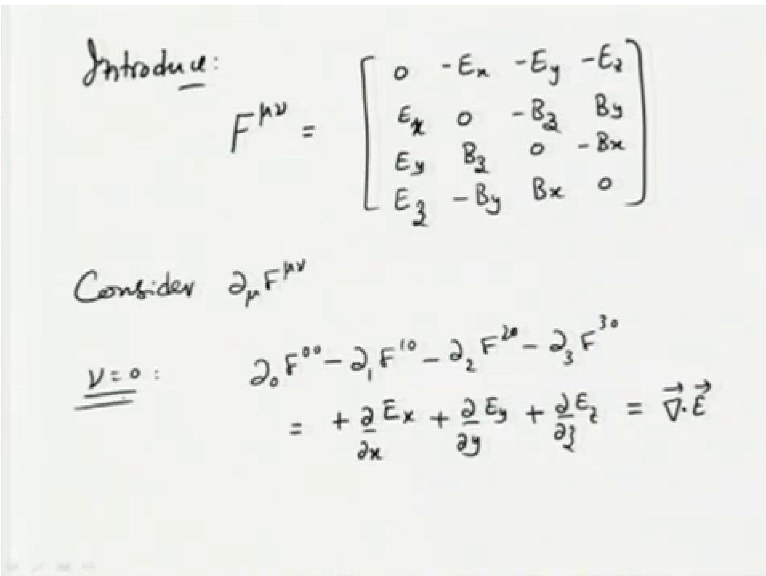

Let us introduce a an object f mu nu with 2 indices conveniently written in a matrix form has minus E x, minus E y minus E z as the first row plus E 1 0 minus B z B y as the second row E 2 B z 0 minus B x as the third row.

And E 3 minus B y B x 0 as it is clear from the explicit form F mu nu is an anti-symmetric matrix in this fashion in this way of writing. It is actually an object which is anti-symmetric under the exchange of mu and nu first and second indices consider dou mu F mu nu let us take nu equal to mu is summed over. So, nu is free, but take nu equal to 0 case that will give you 0 F 0 0 minus 1 F 1 0 minus 2 F 2 0 minus 3 F 3 0 equal to equal to first term is 0 I thing is there.

Second time dou mu is with a minus sign. So, it is minus dou by dou x, but there is already a minus sign there this is a covariant vector. So, there is a minus sign associated with this the result is plus here therefore, if 1 0 1 0 e is E x ok. So, I made a mistake here in the notation it should have been written as E x E y E z not E 1 E 2 E 3, because that is where the notation B is. So, second 1 is E y dou by dou x of dou by dou y of v y plus dou by dou z of E z or this is essentially divergence of E that is when A nu equal to 0.

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$$\partial_\mu F^{\mu 0} = \frac{\rho}{\epsilon_0} = \rho \rightarrow (3) \qquad \Bigg| \; \begin{array}{l} \epsilon_0 = 1 = \mu_0 \\ c = \hbar = 1 \end{array}$$

$$\overset{\nu=1}{=} \partial_0 F^{01} - \partial_1 F^{11} - \partial_2 F^{21} - \partial_3 F^{31}$$

$$= \frac{\partial(-E_x)}{\partial t} + \frac{\partial B_3}{\partial x} + \frac{\partial(-B_y)}{\partial 3}$$

$$= -\frac{\partial E_x}{\partial t} + \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_y}{\partial z}\right)$$

$$(\vec{\nabla} \times \vec{B})_x - \frac{\partial E_x}{\partial t} = J_x$$

So, we can say that dou mu F mu with mu equal to 0 is equal to rho by epsilon 0. If we consider the Lorenz heavy side units epsilon 0 equal to 1 and mu 0 is equal to 1 term similarly this is why C s we have taken C is equal to 1 h cross is equal to 1. So, in that unit it is equal to rho now nu equal to 1 will give you. So, this is 1 we already denoted 2 equations as 1 and 2.

So, here this will be 3 . So, when nu equal to 1 we have dou 0 F 0 1 minus dou 0 F dou 1 F 1 1 minus dou 2 F 2 1 minus dou 3 F 3 1, this is equal to dou 0 F 0 1 F 0 1 0 1 is minus

E x dou 0 is dou by dou t minus E x F 1 1 is 0, but then minus the dou by dou 2 t is one over sorry dou by dou x with a minus sign because it is a covariant derivative term.

Then F 2 1 F 2 1 is be z with a positive sign so, of so dou z F 3 1 is minus B y right. Now this is equal to minus dou by dou t of E x plus dou by dou x of B z minus dou by dou z of the y. The term in the simple brackets is the x component of B curl of B. So, what we have here is curl of B x component of this minus time derivative of the x component of E. And as per the Maxwell's equation the last one it is equal to x component of the current density J.

And you can consider other components nu equal to 2 and nu equal to 3 to get the other parts of this equation.

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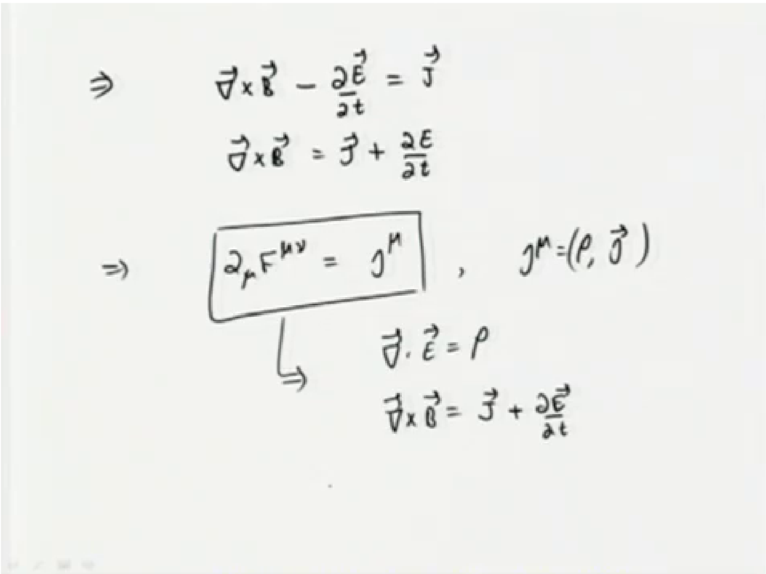

And finally, we will have curl of B minus dou E by dou t equal to J or curl of B equal to J plus dou E by dou t, which is essentially given by considering dou mu F mu 1 F mu nu F mu 2 F mu 3 etcetera.

So, all of these things together ok. So, we will give you dou mu F mu nu 0th component gave us rho and the other components gave us j mu.

So, this with j mu equal to rho and j is essentially a compact way of writing Maxwell's equations divergence of E equal to rho, which is essentially given by nu equal to 0 and the other 3 values of mu will give you curl of B equal to J plus dou E by dou t.

So, these 2 equations Maxwell's equations can be represented in a compact way using F mu nu dou mu F mu nu equal to J mu.

We will see how the other 2 equations in the Maxwell's equation set of Maxwell's equations can be represented in a similar fashion that we will do in the next discussion.