Physics of Biological Systems
Prof. Mithun Mitra
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 60
Pattern formation in reaction diffusion system with stability

What we will do today's that will work out the conditions for this Pattern Formation in a Reaction Diffusion System.

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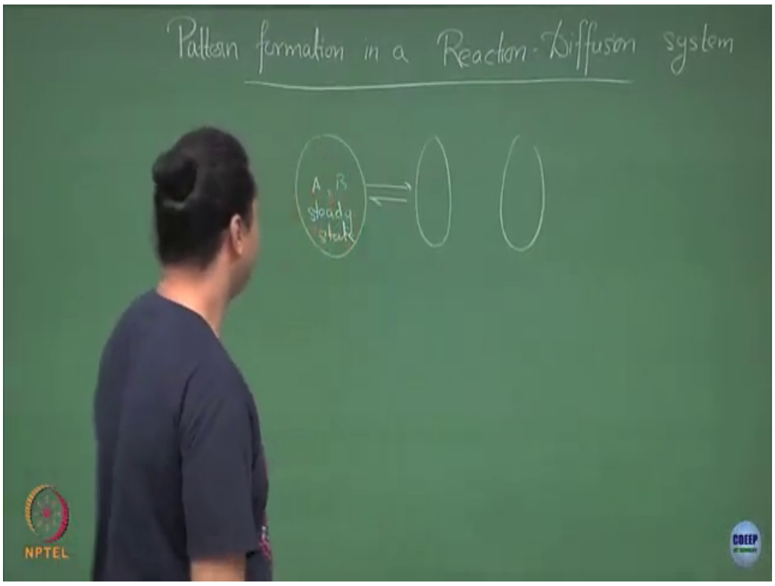

So, last class we saw examples of how you can generate patterns. The idea being that you have some cell, where you are undergoing some sort of chemical reactions between multiple species.

So, we saw one example with two species let us say A and B. And the idea being that the chemical reactions themselves can lead to some sort of a steady state, they can lead to some sort of a steady state. But if you now allow and if this reaction we consider it is taking place in multiple such cells the same reaction. If you now allow these cells to communicate with each other, specifically through diffusion.

Then what we showed was that in certain cases you can destabilize this steady state of the non-interacting system by introducing this sort of diffusive interactions between the different cells, right. And by destabilizing the steady state you can generate patterns which had you can generate spatial patterns, which are different from the steady state that we saw in this isolated model, ok.

So, what I will try to do today is that I will try to sort of formalize this, what are the conditions on this reaction system or this diffusion prop, what are the constraints on this reaction diffusion system that will allow us to see the formation of a pattern, ok. So, this is generic. It is true for any reaction diffusion system. And you can analyze any reaction diffusion system using this sort of a framework to say when will a pattern form or when will it not form, ok.

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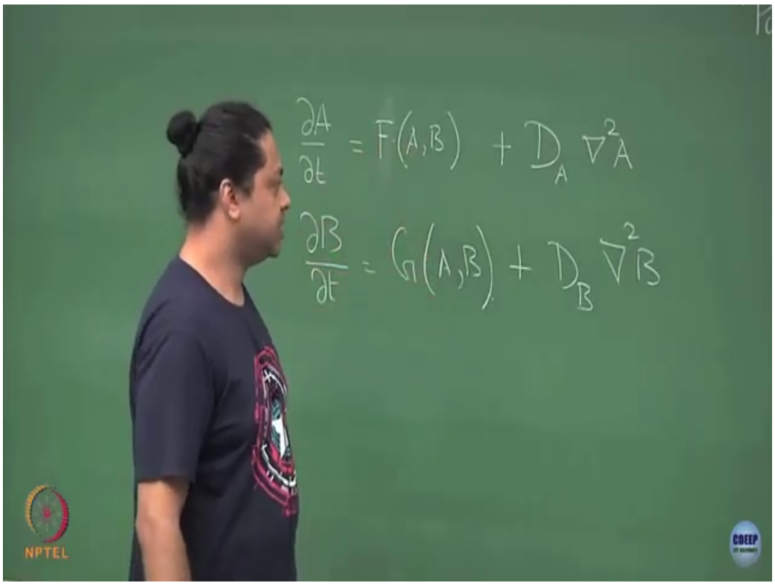

So, the basic I will stick with this two species model. So, the basic idea being that I have some sort of a chemical reaction between two species. So, I can write down the time evolution of the concentrations of the two species. So, let us say del A del t is some reaction term F of A comma B, that just to write capital F, F of A comma B and del B del t equal to some G of A comma B. So, this is my reaction part of the system. It encodes the chemical kinetics that is going on between these two species A and B, ok.

In addition now, I want to say that these species can diffuse in space. So, as A diffuses with some diffusion coefficient, B diffuses with some other diffusion coefficient. So, this is in some sense, the basic equation that I want to analyze, ok. The specifics of the equation we will come of course, in this F and the G terms, the specifics of the strength of the diffusion we will come in the D A and the D B.

Before I start analyzing I will just rewrite these. I will just rewrite these equations.

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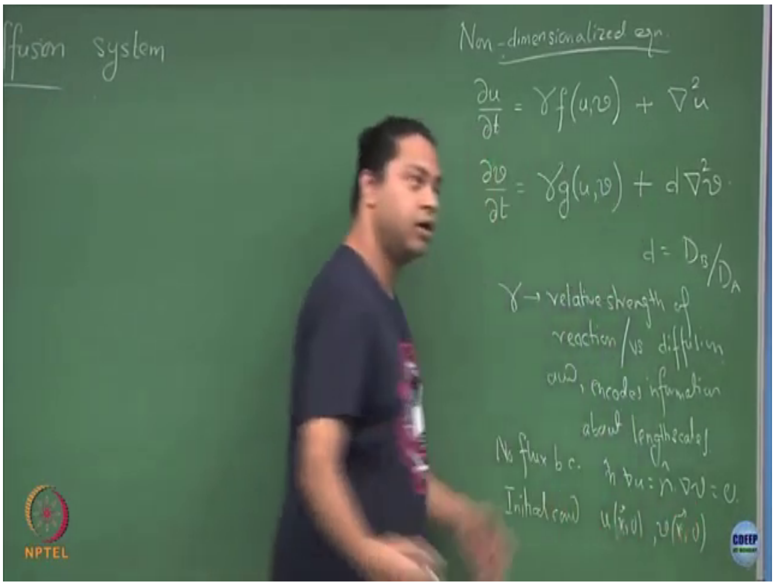

The first thing to do it is a non-dimensionalize this equation. So, let us say I have non-dimensionalize them and I write them now in terms of then this non-dimensional concentrations I write in terms of u and v, ok. What was A and B? I just non-dimensionalize and I call it u and v, and I will just write it in this form f u comma v plus.

So, it is essentially the same thing as here, except what have I done I have replace this D A by 1 and this D B by small d. So, you can think of this small d as the ratio of the two diffusion coefficients, ok. Similarly, this capital F and capital G after read after non-dimensionalizing, I have pulled out a factor gamma, quite generically.

What this gamma factor? I mean I could have called this some f prime, I just pull one turn out because I want to sort of encode. So, you can think of this gamma as sort of saying how strong is your reaction term in comparison to the diffusion term right or another way of interpreting gamma is that it contains information about the length scale of the system, ok. So, in a finite system for example, what is the size of the box that you are going to form this, am going to do this reaction since.

So, it encodes for example, the relative strength of reaction versus diffusion and also it has encodes information about length scales all right. So, this is non-dimensionalization. So, these are the non-dimensionalized equations that I want to stop. Quite generally again, not really change anything from here to here, it is a different way of writing.

Now, given that this is of course, a differential equation in order to solve it you need boundary conditions and you need initial conditions. So, let us specify some boundary conditions. Let us say no flux boundary conditions. So, let us say I have some boundary conditions, no flux boundary conditions which is n dot grad u is equal, n being the sort of surface normal n dot grad v is equal to 0. And let us say that I have specified for you the initial conditions, initial conditions something. So, u of x comma 0 and v of x comma 0.

So, this is the time t equal to 0, what were these concentrations? u and v, ok. x or whatever you know, generally r you can write if it is in 3 d, ok. So, these are also given. So, given the boundary conditions and the initial conditions together with this differential equation, I now have everything in principle that I need to solve this problem. So, that is what we will try.

So, the first thing is of course, to find out remember the initial idea is to say that well there is a steady state in the absence of diffusion, and there is a stable steady state in the absence of diffusion and we want to see where diffusion can be stabilized the steady state, ok. So, first then we will consider this problem the absence of diffusion and then find out what is the requirement on these reaction terms in order to have a stable steady state. So, let us say I have a steady state.

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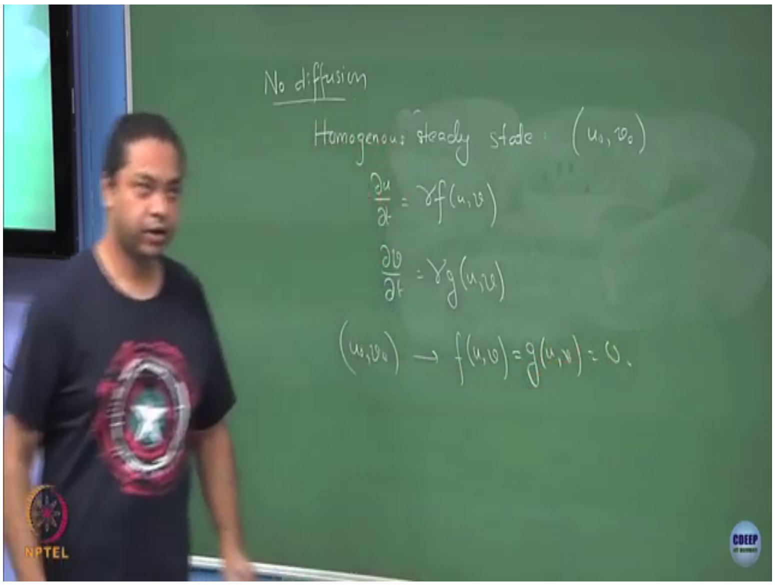

So, in the absence of in the absence of diffusion, so no diffusion, no diffusion let us say I have a homogeneous steady state which I call some u naught v naught, ok.

So, these are the concentrations of the steady state, if there was no crosstalk between the cells. And so, these are my equations now del u del t is equal to gamma times f u comma v and del v del t is gamma times g of u comma v. So, how do I find the steady state of this equation? I just say the del u del t is 0, del v del t is 0. So, this u naught v naught are solutions of; so, u naught comma v naught are solutions of f u v equal to g, u v equal to 0, right. So, this is the homogeneous steady state in the absence of diffusion.

Now, I want, so whatever it is. So, I as long as I do not specify f and g you cannot find the particular solution, but in general given an f and g you can equate them to 0 you can find out what is the stable steady state, sorry. You can find out the steady state. Now, we must look at the stability of the steady state, right. So, how do I look at the stability of the steady state? So, what I can do I will do? A linear stability analysis. So, I will define perturbations away from this steady state.

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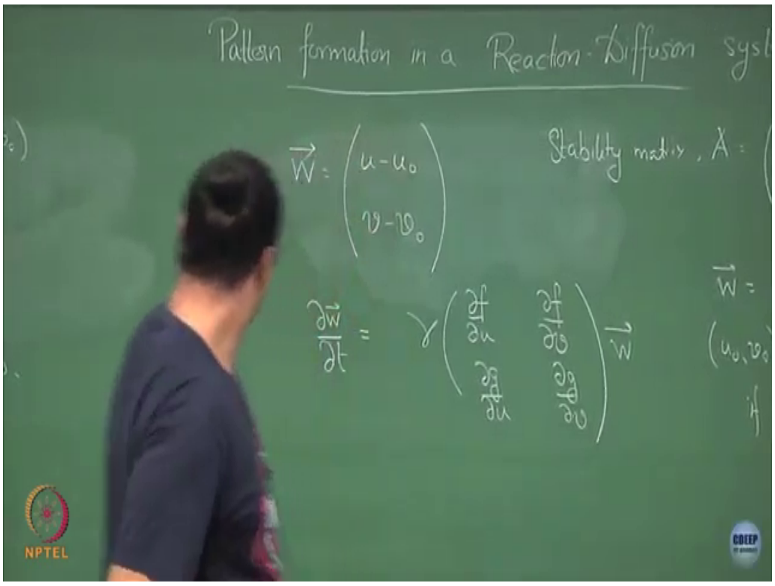

So, let me say let me define a quantity W. So, I will just write in vector form. So, which is perturbations away from the steady state, so W is u minus u naught and v minus v naught, right. So, I can now substitute this W into this equation, right. So, what will I get? I will get a del W del t. So, by introducing W instead of having to write two equations for u and v, I am just writing it in vector form. So, I will just write it to them.

So, del W del t is going to be what? So, if I do an Taylor series expansion of this f and the g terms around the steady state, right, so the first term will be, sorry I do not how to write. So, let me just write del. So, if I do a Taylor series expansion the first term will be f at u naught comma v naught, right, plus delta into del f del u delta into del f del v and so on, right.

The f at u naught comma v naught is of course, 0 because I am expanding about the steady state which means that if I just write it in the matrix form. What will I get? I will get a gamma times del f del u, del f del v del g del u, del g del v, right times this W itself, right. So, because I am doing a linear stability analysis, I do not keep the higher order terms, I just keep the first order x, ok.

So, this matrix, let me call it as my stability matrix.

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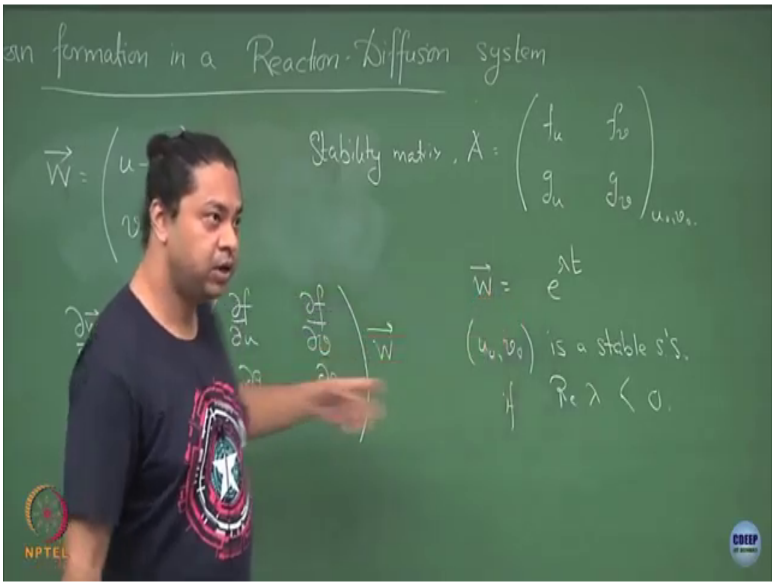

So, let me define it as my stability matrix, let me define it as a stability matrix A let us say and in short form I will just write f u, f v, g u, g v. These being the partial these partial derivatives, f u as del f, del u and so on, ok. Evaluated at this, evaluated around at this steady state, evaluated at u naught comma v naught, ok.

So, next in order to say, whether this steady state is stable or unstable. We can look for solutions of this type. We can look for solutions of the form W is equal to some e to the power of lambda times t, right. And then I will say that this steady state u naught v naught is stable if these lambdas are negative, if the real parts of these lambdas are negative, right. So, u naught v naught is a stable steady state, is a stable steady state if the real parts of the lambda is less than 0, right, that is my constraint. Because if this is less, if the real part is less than 0 then as time goes to infinity these perturbations will die out, you will come back to the u naught comma v naught state, ok.

So, what I need to then find is the eigenvalues of this matrix.

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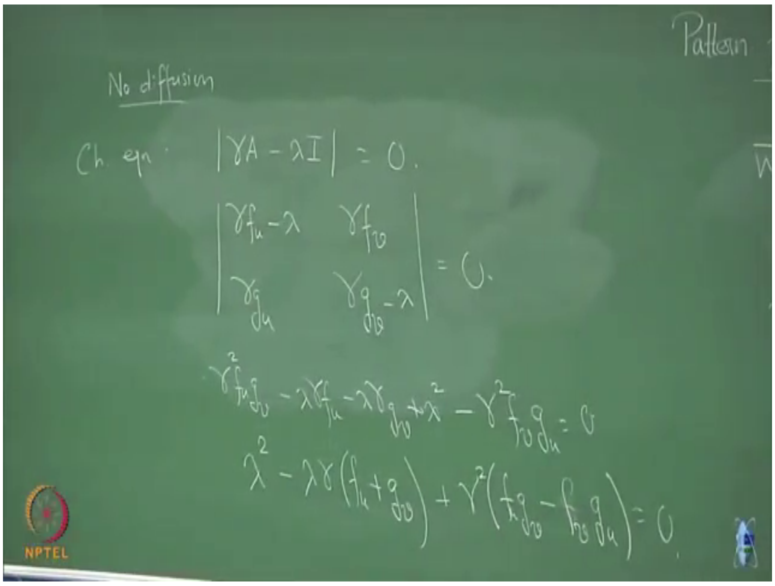

So, I need to solve this characteristic equation, I need to solve this characteristic equation. This is gamma times the matrix A. So, I need to solve this gamma times this matrix A; minus lambda times identity matrix is equal to 0. So, this is my characteristic equation, ok.

What does this mean? This means basically gamma f u minus lambda, gamma f v, gamma g u, gamma g v minus lambda is equal to 0 right, ok. So, what does this give me? If I just write it out, so gamma square f u, g v minus lambda gamma f u minus lambda gamma g v plus lambda square minus gamma square f v g u is equal to 0, ok. So, in order to find out these eigenvalues lambda I need to solve this equation which is just a quadratic equation. So, this is lambda square minus lambda into gamma into f u plus g v, right and then plus gamma square f u, g v minus f v g u. So, here is my quadratic equation that I need to solve. So, this is of course I know what is the solution.

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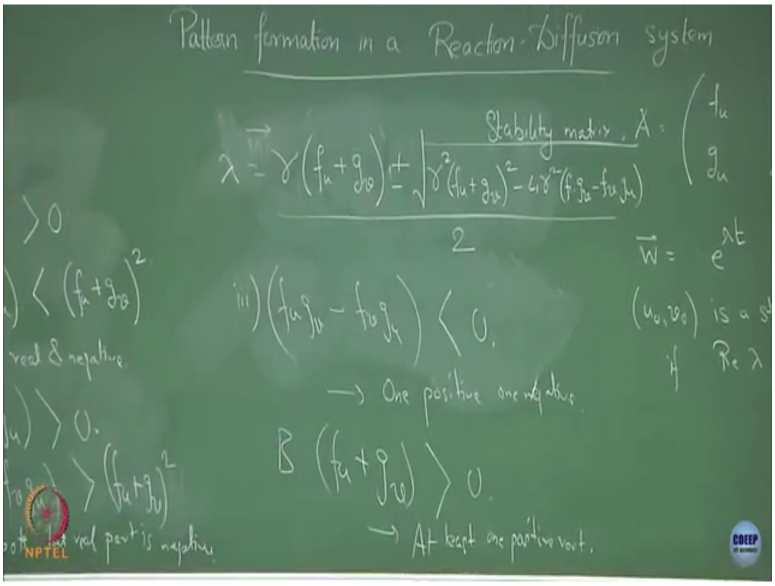

So, lambda is equal to lambda equal to minus v. So, gamma into f u plus g v minus v plus minus square root of v square. So, gamma square into f u plus g v whole square minus 4 a c minus 4 gamma square into f u g v minus f v g u. This is under square root divided by 2 a, ok. So, these are my eigenvalues, lambda times f u plus g v plus minus this quantity divided by 2.

What I want is that the real parts of these lambdas are negative, then I know that this the solution that I have written down the steady state solution u naught comma v naught that is linearly stable. So, it is a stable steady state, ok. So, what can I say? So, what are the conditions for this lambdas to be negative? So, let me see. Is this clear? So, that is my lambda. So, let me say that if f u plus g v this quantity, if this was less than 0, ok.

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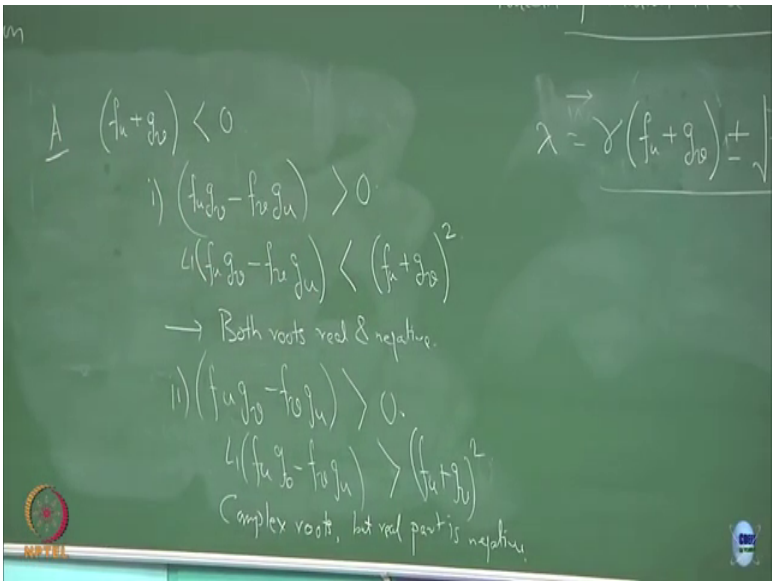

So, if this quantity was less than 0, and so, let say this case A, and then let me say that f u g v minus f v g u this was greater than 0, ok. Let me say this is greater than 0. And what else?

And say that is less than f u plus g v whole square. I think, I have specified everything. So, if this was the condition, right, that this term is negative, this quantity in the bracket is positive, and is less than this quantity. Then other, what are the roots? Are the root positive or negative under these conditions? Anyone?

Student: Negative.

Negative. Are they real? So, yes, I should have included the factor 4 somewhere, ok. Then, then these are real and negative, right. So, in this case under this condition both roots are actually real and negative, are real and negative, right. Because anyway when this sign, because this is minus when this sign is minus anyway it is minus, even when this is plus this is less than this, so therefore, its it will not be greater than that f u plus u, ok.

What would happen if I said that, this is still positive f u g v minus f v g u is still positive its greater than 0, but this is [FL]. So, f u g v minus f v g u 4 times is greater than f u plus g v whole square. Then, then the real. So, then the roots would become complex, but the real part would still be negative, right. So, this would be complex roots, these would be complex roots, but real part is negative. What if I now relaxed? So, what if I now say that f u g v minus f v g u is less than 0. So, if this is a negative quantity, then what happens?

Student: (Refer Time: 20:59).

So, one would be positive then and another would be negative, which means you would have this blowing up, right; this state would no longer be stable. So, so one positive, one negative, ok.

Now, what would I say if I said that f u plus g v, which I had taken to be negative if this was positive, then can you ever have both roots the real paths are negative? No, right. They will always be at least one which is positive. So, here you will always have at least regardless of whatever on the other condition at least one positive (Refer Time: 21:53), at least one positive.

So, quite generally then having written down a steady state u naught comma v naught, the condition for that steady state to be a stable steady state of this non-diffusive, non-interacting system is that f u plus g v must be negative. And what is f u plus g v? f u plus g v is the trace of this matrix, that is the stability matrix that I have written down. And this other quantity f u g v minus f v g u which is the determinant that must be positive, right; because in both these cases this was positive and I had real parts were negative, ok.

So, the condition for the steady state. So, then let me rub out somewhere.

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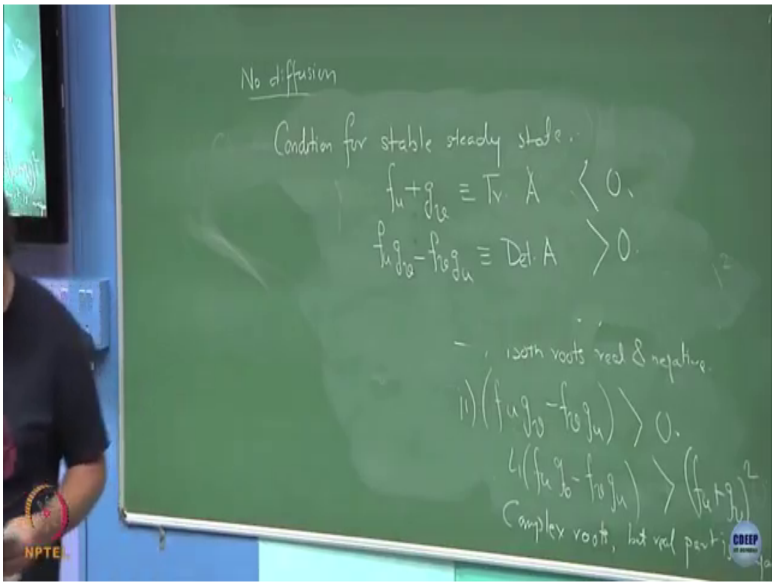

So, if I solve this characteristic equation what it tells me is that the condition for the steady state for a stable steady state. Condition for a stable steady state is that f u plus g v which is nothing but trace of this stability matrix A that must be negative. And f u g v minus f v g u, which is nothing but determinant of this matrix A; that must be positive.

So, if I satisfy these two conditions then these are sufficient for me to guarantee that this steady state that I have written down u naught comma v naught is a stable steady state, right. So, this is as far as this non-diffusing particles, where I am just talking about the reaction parts of this equation. I have not yet brought in diffusion. Actually, I should have written it here, so that I can rub that out. So, let me just, if you allow me. Let me just rub this out.

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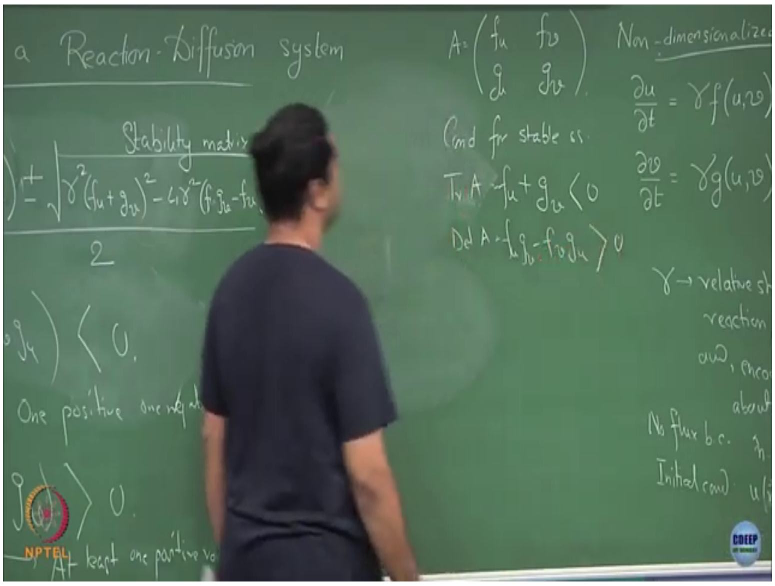

So, my stability matrix A, let me just right is f u, f v, g u, g v and condition for stable steady state is that trace of A equal to f u plus g v is negative and determinant of A is f u, g v minus f v g u that is positive, ok. So, these are my two conditions.