Quantum Mechanics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology- Bombay

Lecture – 09
Conditions and Solutions for One Dimensional Bound States - I

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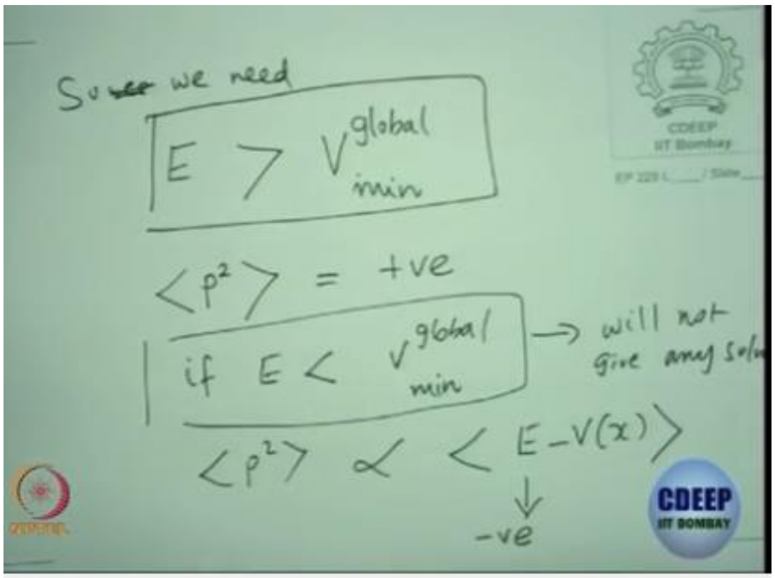

Okay, so we have seen that your energy must be greater than V global minimum otherwise, solution does not exist, right. How did I try to give you an indicator proof, if you try to find expectation value of p square, this should be positive but it E is; if E is < V global minimum, what happens; you can show this to be proportional to E - V(x), expectation value using the time independent Schrodinger equation and you can show that the this is, E will be less if I take, then this is negative, right, everybody is with me.

So, therefore this condition will not give any solution okay, so we need E to be always > V global minimum, locally there could be a minimum but it should not be the global minimum whatever it is, your energy values should be > the global minimum, is that clear.

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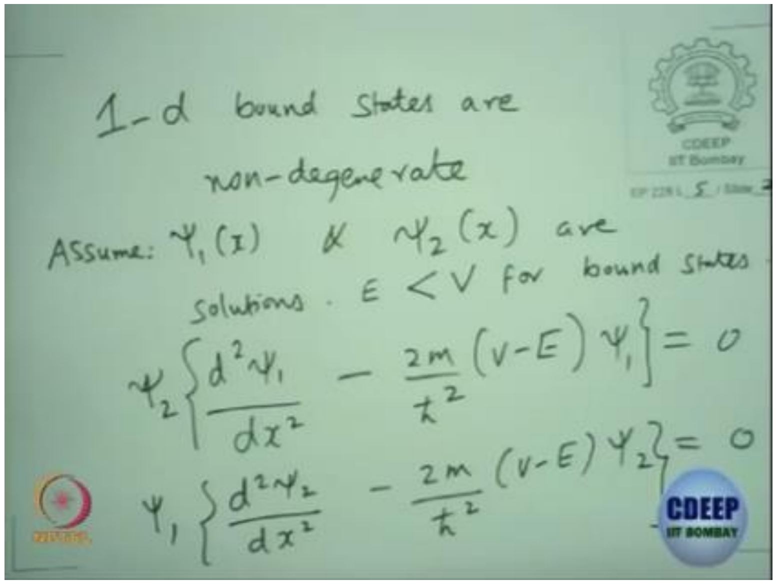

What about the other thing which I said, which is 1d bound states are non-degenerate, how do you prove this, you take the time independent Schrodinger equation okay, so you can take the time independent Schrodinger, assume that $\psi_1(x)$ and$\psi_2(x)$ are solutions okay, assume and proved by contradiction, so they are solutions, then $\psi_2 \left\{ \frac{d^2\psi_1}{dx^2} - \frac{2m}{\hbar^2}(V-E)\psi_1 \right\}$ =0, it is bound state.

So, we need to take E to be < V, okay let us look at this constant barrier potentials, take the energy to be less than the potential energy of the barrier. Similarly, you can write a similar equation for ψ 2 = 0 both are allowed solutions is what we are assuming which means, for E < V, both have to satisfy the time independent Schrodinger equation, so multiply the first equation by $\psi_2(x)$ similarly, multiply the second equation by $\psi_1(x)$.

And then you try to subtract one from the other, what happens when you subtract, which term cancels, the second term cancels.

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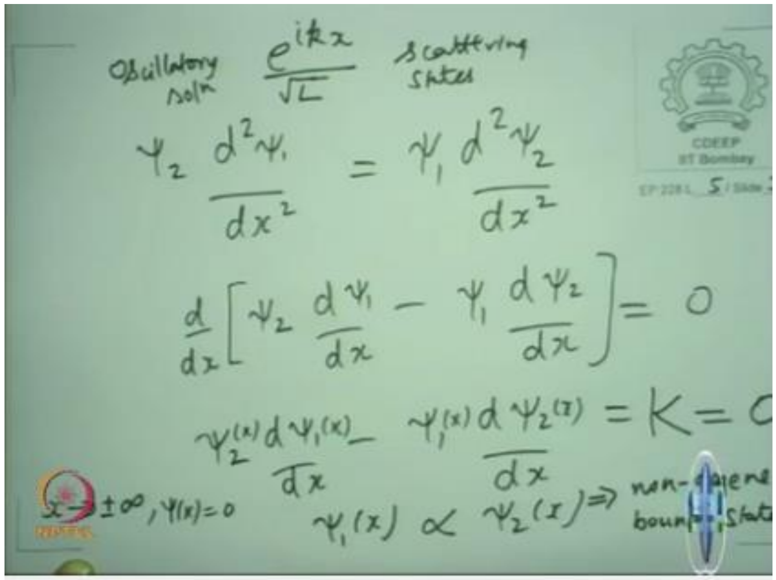

What you will be left with is; what will you be left with $\psi_2 \frac{d^2\psi_1}{dx^2} = \psi_1 \frac{d^2\psi_2}{dx^2}$, this can be further simplified as $\frac{d}{dx}\left[\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx}\right] = 0$. What is this tell us; the square bracket can be put to be some constant, right you can put it to be a constant, $\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx}$ is a constant okay, it is a function of x and it should be valid at all x.

For x tending to + or – ∞, we are looking at a bound state solution where it should be exponentially damπng, you expect so what do you want; x tending to + or – ∞, what happens; wave function itself will be 0, which means this constant; it is the constant, it should be valid even at x tending to + or – ∞, so for bound state, we can definitely put it to be 0, we cannot do this for a scattering state.

Because it will be oscillatory solution with a box normalization as I said, if you had a solution for an oscillatory state as E to the ix, this is oscillatory solution, we do a box normalization putting a root L and then after we do all the expectation value, we will put L tending to ∞ and nothing happens, everything is fine, so there are all not the states we are considering for bound states, bound states are the ones where the wave function vanishes.

So, these are scattering states, this is not the; this is not what we are looking at in the specific proof, we want to look at bound states which vanishes at + or - ∞ and we are assuming that it is degenerate and we are getting a condition like this, so can we further simplify this condition since this is 0, this first term = the second term, so right, excellent. So, what does that tell us, yes, so which means that you get ψ_1(x) to be proportional to ψ_2(x).

This k = 0 will automatically show that the 2 wave functions are not independent, they are proportional to each other which means our starting point that it is degenerate is not valid, degenerate means those 2 should be linearly independent but we find that systematically substituting that they are degenerate and putting it into the Schrodinger equation and imposing that the wave function vanishes at + or – ∞, forces k to be 0 for the bound state.

And this condition will automatically show that $\psi_1(x)$ is proportional to $\psi_2(x)$, which implies non-degenerate bound state, this is only applicable in one dimension okay, so one dimensional problems bound states are always non degenerate that is what you saw in the harmonic oscillator; one dimensional harmonic oscillator you saw that all the solutions in the one dimensional particle in a box are all non-degenerate because of this process okay.

So, the next thing is; once I have tried to take this assumption that the energy of the particle should be greater than the global minimum and the 1d bound states are always non degenerate, we can look at symmetric potentials and proved that the solutions have to be either even solutions or odd solutions, so that is the next thing.

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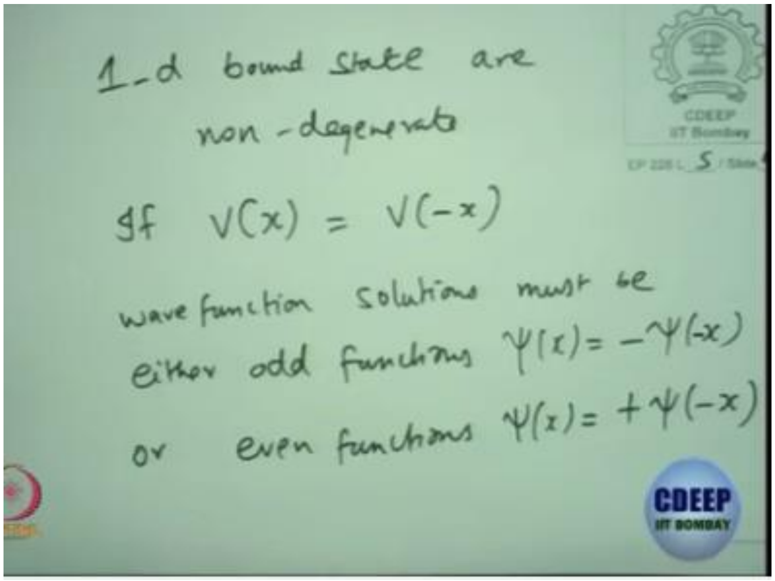

So, 1d bound state are non-degenerate, if V(x) = V(–x), wave function solutions; solutions must be either odd functions, what is the odd function; $\psi(x) = -\psi(-x)$, or even functions; $\psi(x) = +\psi(-x)$, this is what we want to prove again, how do you prove this? Take the Schrodinger equation again.

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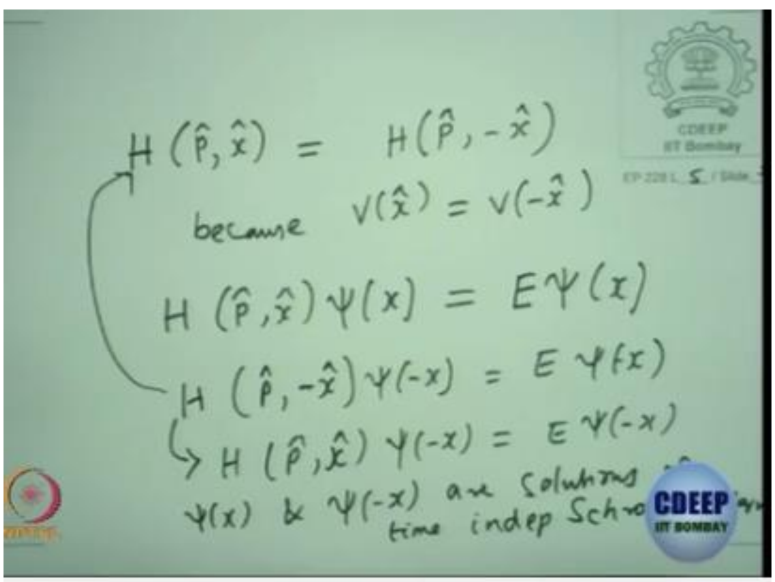

If you take the Schrodinger equation, Hamiltonian, which is a function of p and x; $\hat{p}$ *and* $\hat{x}$ to be very precise, we find that it is same as H($\hat{p}$, $\hat{x}$) because V($\hat{x}$) is same as V (–$\hat{x}$), we are looking at this class of potentials, one of the example is the harmonic oscillator, other example is a symmetric box, which I indicated in the last lecture and the third example is your double delta potential which we are going to solve today in the remaining half of the lecture okay.

So, these are the 3 examples, always try to; when you try to go over a proof, recall some examples and keep it in that mind okay, then you will never forget some of these statements, you will be able to correlate with an example and you will remember it properly okay, so that is why I am trying to give you those 3 simple examples, so that you remember these things clearly. So, what do we do usually?

When we write the Hamiltonian operating on ψ(x) right, Hamiltonian will be in general a differential operator will be time independent Schrodinger equation is some energy eigenvalue E times ψ(x). So, I could also write $H(\hat{p}, -\hat{x})$ ψ(–x), I am just changing x to – x, nothing more okay, so allowed because if you write the explicit operator, it will be in the position representation, we find that the potential has this property.

So we can write it in this fashion where the energy eigenvalues independent, it does not care about which position you are looking, so what is this; so this equation is nothing but because of this property, this is nothing but the same Hamiltonian operator on ψ (- x) is E of ψ (–x), okay, you all agree. So, this equation by using this property, I can try to write it as this, so what does this tell; ψ(x) is a solution, ψ(–x) is also a solution, okay ψ(x) and ψ(–x) are solutions of so Schrodinger equation; time independent Schrodinger equation.

But they are all we are looking at bound state problems which means, ψ(x) and ψ(- x) cannot be linearly independent right, so ψ of their solutions.

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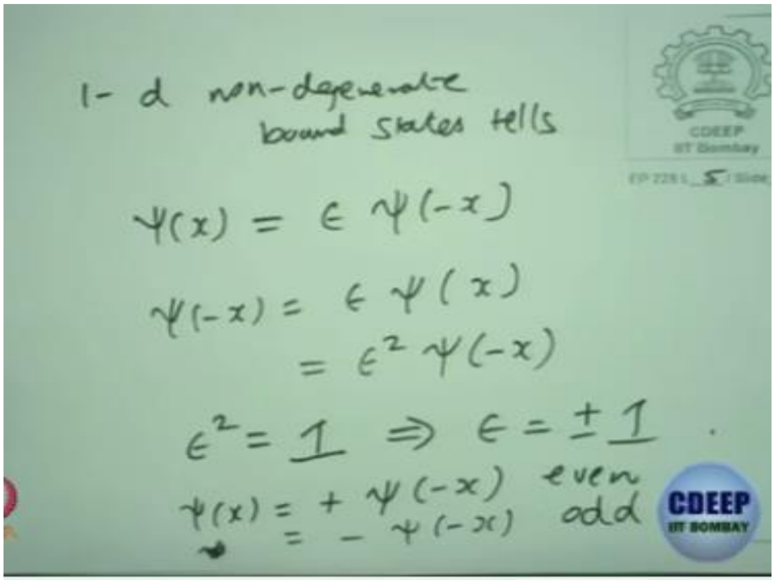

But 1d bound state tells us that 1d non degenerate bound states tells ψ(x) should be some ε times ψ (– x), they cannot be independent, x is a dummy variable, you can call x as – x, if you do that ψ(- x) is ε times ψ(+ x), so you can substitute twice instead of ψ(x), you can substitute the above equation, it becomes ε square ψ(– x), so ε square has to be 1, so which implies ε has to be + or – 1.

So, the ε is + or – 1, the + 1 will correspond to is even, ψ(x) and the - 1 will correspond to as odd, so I did not do anything, I am only assuming that we are looking at bound state solutions for symmetric potentials, looking at the time independent Schrodinger equation, you find both are allowed solutions, ψ(x) is an allowed solution, ψ(–x) is an allowed solution but 1d bound states have to be non-degenerate which means, there should be proportional.

And then the proportionality constant ε squared has to satisfy constraint which is 1, so the only solution possible are odd solutions and even solutions, so when we did this elaborate exercise of solving the time independent harmonic oscillator, we did not do use these proofs but the solutions which we found where either odd functions or even functions, right and also the particle in a box when you put it symmetric, you first ground state solution will become a cos function, right.

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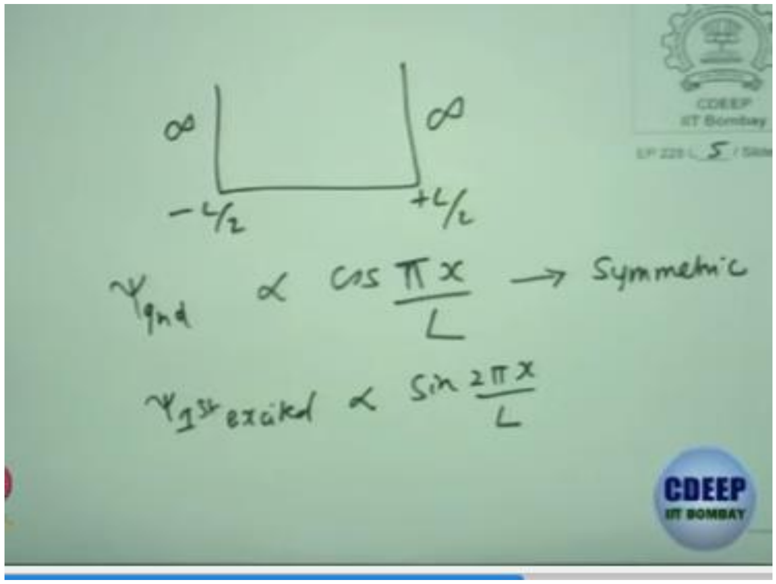

For a symmetric, if you remember that if you took a – L/2 to + L/2 ∞ potential, the ground state was proportional to cos (π x/L), similarly the first excited state is proportional to sin (2π x/L), this is what you will see in a; particle in a symmetric box and I also said that the ground state will always be symmetric, okay what we will do now is; we will take the double delta function potential, how many of you have already tried, if any of you tried the solution for the double delta, okay.

So, we will just go over it, so that you can recall if there is any mistake in my tyπng, you should point it out and basically, you will find that because for the double delta, you have the symmetric condition, a solutions can be assumed using these proofs that you will have an odd solution or an even solution and you will find that the even solution as a lower energy in comparison to the odd solution, okay.

So, we will go over it and both graphically as well as yeah, so you could take it but then when you take x_2 – x, ε squared is what you can do, $e^{i\pi}$is only -1, if you do that it is $e^{2i\pi}$ that has to be set to 1, so you are stuck, whatever you do the nature knows when you do the Schrodinger equation, it picks out hermite polynomials which are odd functions or even functions, it is very beautiful, one is just rigorous mathematics.

The other one is by this kind of simple proofs that you can show that your symmetric potential should allow only for odd solutions or even solutions in one dimensions, no, no, no so what he was saying is let me put that sheet again, I do not know whether it is misspelled, let me see, hope it will be, so what he was saying is that instead of this ε which is to be + or -1, why cannot it be a complex phase factor.

But what I was trying to tell him is that you need your ε squared to be 1, if you take a complex phase factor is $e^{2\,i\,\Phi}$ and that is not 1, for an arbitrary Φ, unless Φ is taken to be π, the phase factor, if you take Φ to be π, then it is only + or - 1 okay, so with this warm up, let us get to the double delta function potential, I thought just to make it clear rather than putting too many complicated equations.

I thought let me put it on the power point or a PDF file, so just take a look at it and then is this clear the fonts and all.

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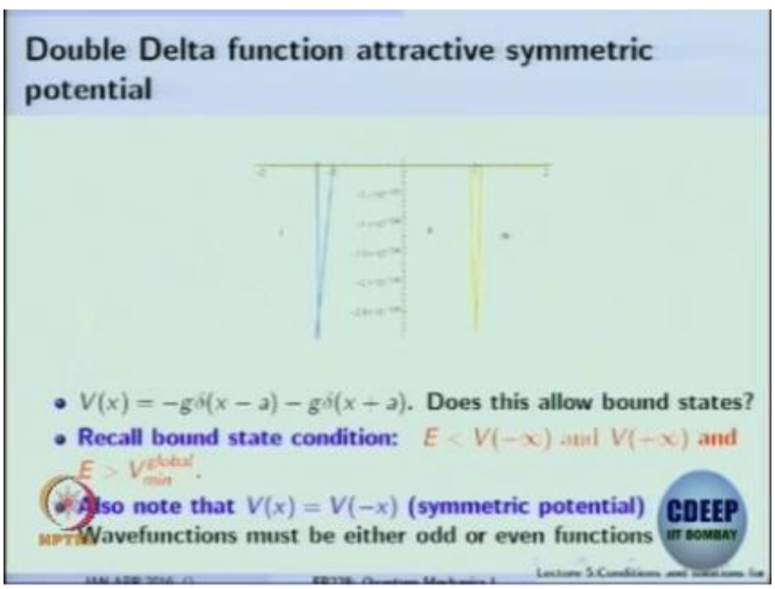

So, I tried to draw it to the limit as possible, I am not very good at drawing but I hope you understand that it is a double delta function potential for graphically, just I have taken A to be + 1 and so you have a delta function potential at A = +1 and A = -1 but you can keep this to be an arbitrary, so I tried to plot in Mathematical unfortunately, one side I did plots, the other side it does not plots I was struggling for some time but then I drew it manually this piece.

But then I have not done a very good job okay but you get the point, so I have just try to give you a graphical potentials, so this is the potential versus x, where you can see that there is the symmetry it is not very obvious in the figure but at least from this expression, you can see that it is an attractive potential, if you take x to –x, this term goes into this term and this term goes into this term, is that clear.

So, does this allow bound states, the first thing you have to see is what is the global minimum; global minimum is negative ∞, so your energy can be less than the local minimum at + or – ∞, so you can have a E negative and your; we do get bound states, okay, so recall bound state condition and E should be greater than global minimum and also note that the potential is a symmetric potential.

So, I have already proved for you that for symmetric potentials and 1d bound states, now we are looking at bound state, the wave function has to be non-degenerate and it should be either odd functions or even functions, we are going to use this property, when we write the solutions now, we are not going to write blindly solutions, now we are going to incorporate this back okay.

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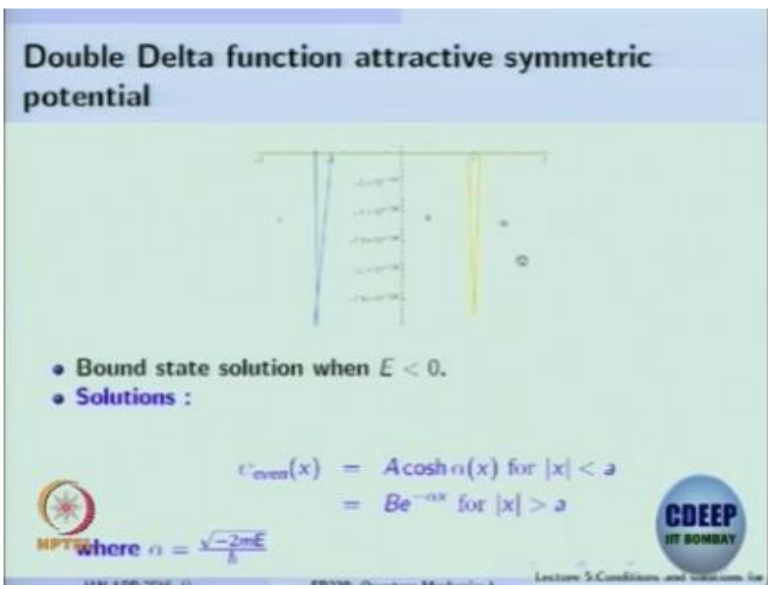

So, the bound state condition will be energy should be < 0 because at + or – ∞, the energy is the potential is 0, so you can have an even solution, you can split this into a regime; region 1, region

2 and region 3 and region 2; you can have this cos hyperbolic not a linear combination of e to the α x and e to the α x, because this should be a symmetric even function and similarly, for region 1 and region 3, mod x is > a, you will have a symmetric function.

The same B will appear in region 1 and region 2, so you do not need to put a $Be^{-\alpha x}$ and $Ce^{-\alpha x}$, this is all I am trying to say that is what you would have done now, you can simplify it and say that the symmetric potentials will allow this, okay, α; of course you all know it will be E is negative, so α is positive and this is the definition of α.

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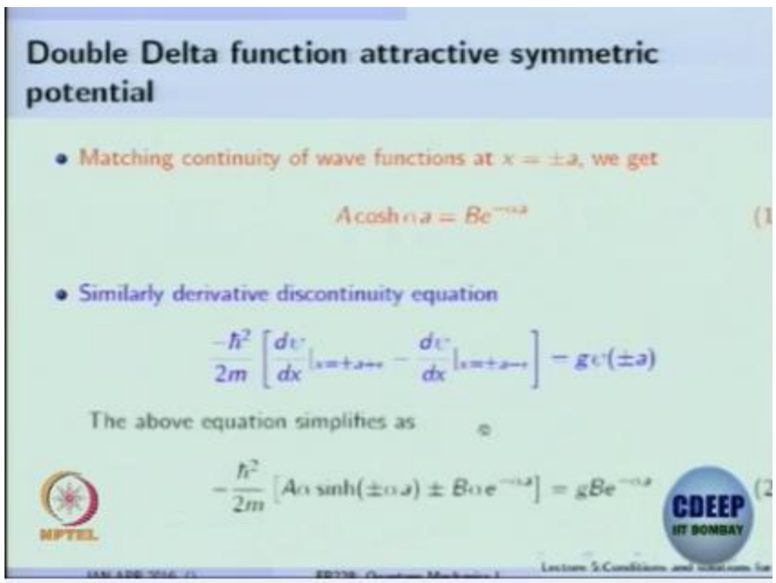

So, what is the next thing, wave function must be continuous at x = + or – a, those are the 2 boundary regions and you can put in that condition now, very easily we can do this at + a or –a, okay, I have taken one of the regions but you have this delta function potential requires that there is a definite derivative discontinuity, we have seen this, so substitute that the definite derivative discontinuity and you will get an equation where it is + ε, the derivative at + or – a,

And similarly, the derivative at -ε shift at + or - a boundary, is that okay, so you can simplify this, in the even solution which I have given, so you get a condition from the derivative discontinuity and equation, what is our aim; our aim is to determine energy for the bound state or α; α is the function of that energy, α is positive and energy is negative, so what next; you divide 1/2 right.

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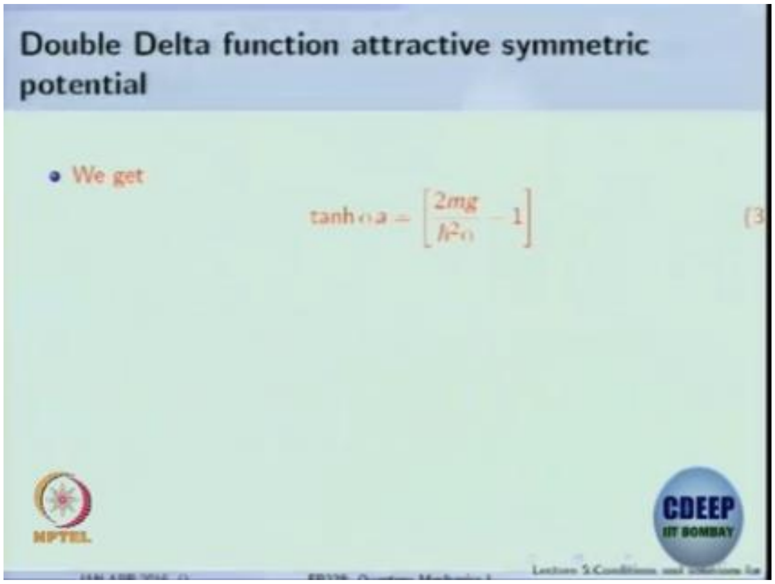

Then you get a simplified form from which is it easy to determine α, you have done some similar problems in your assignment sheet, where you do not really get an explicit analytic solution for α, right from this expression but you can start getting the trend, take α in comparison to the width a, the position of the delta function whether it is small, whether it is big and we can get some kind of a trend.

And then we could also go into trying to plot them in a graph and see whether there is a solution and wherever is the solution, you can pick out what is the energy.