Quantum Mechanics

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Lecture - 71
Tutorial 12

This is the end of our tutorial. We will wind up this tutorial by discussing few more problems on Clebsch-Gordan coefficient and Wigner-Eckart theorem which we did in the 11th tutorial also but there will be more problems on Wigner-Eckart theorem and this tutorial again I would emphasize that will give you how powerful, will give you the idea how powerful is the raising and lowering operator in determining the CG coefficient.

And how CG coefficient would help us to know the values of the observable using Wigner-Eckart theorem. So Wigner-Eckart theorem relates the irreducible and reducible representation which you might have seen in the lectures. So first problem, let us go with the first problem which says that it is deuteron problem, famous problem.

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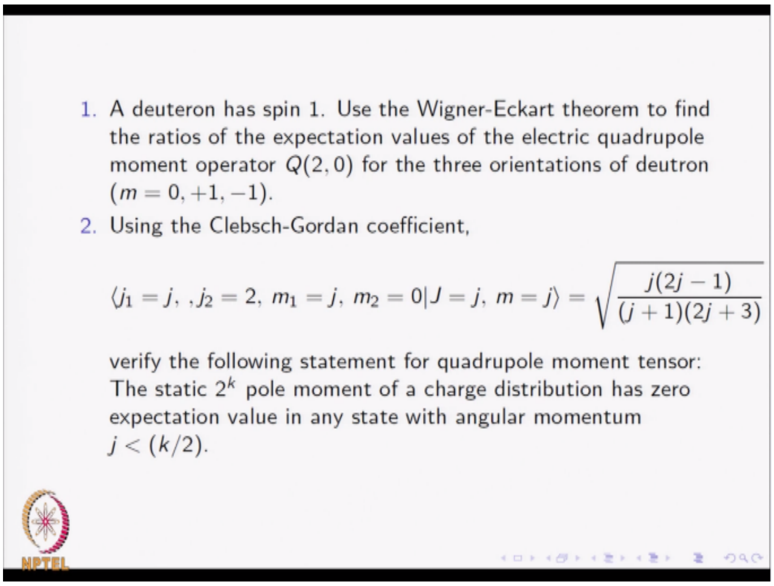

The deuteron has spin 1 and use Wigner-Eckart theorem to find the ratio of the expectation value of electrical quadrupole moment operator Q 2, 0 for the three orientations of deuteron. So for three orientation that means you need to find out the ratio or the expectation value of deuteron for Q 2, 1 Q 2, 0 and 1 -1 and 0. So the three orientation would be 1 0 and -1. So you have to find the expectation value for 1 0 state.

Using the 1 0 state the orientation for 1 1 state and 1 -1 state on Q 2, 0. So to start this problem, I will again be very sketchy especially for this tutorial because it is a kind of repetition of the previous tutorial but with more number of problems twisted and turned. So the first problem I will I thought of drawing a map for you all like a road map you have, so similarly a map wherein you will have like an algorithm actually.

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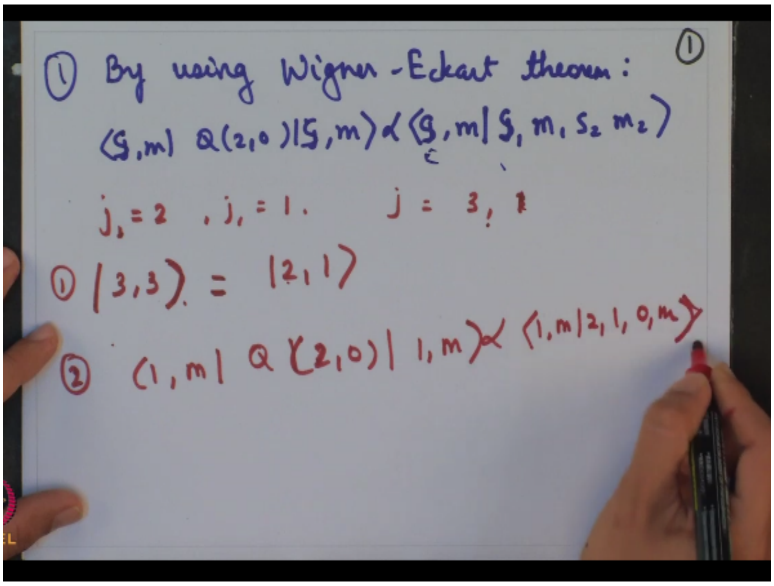

So the algorithm will be that you start by writing so first step would be you know what is the Wigner-Eckart. So by using Wigner-Eckart theorem, you know actually that j, m Q 2, 0 j, m. So here j1=j2=1 and m1=m2 or you can write it as s, m. I am just writing it as j, m. So it is a spin state. So I will have s, m and this will be proportional to capital J and M or capital S and M and then I have s1 m1 s2 m2 or rather you know the Wigner-Eckart theorem.

We can write it in terms of so this proportionality constant U is a constant value which you cannot determine from the experiment but we can find out the ratio of the observables and then evaluate the CG coefficient and correspondingly you can find the ratios for s is=1, m is=0, s is=1, m is=-1, s is=1, m is=0 and so on. So you will find out the expectation value for 1 -1 1 0 and 1 1 okay.

It will be very simple but you need to also know how to evaluate the CG coefficient. I think you might have mastered by now. If you have done the previous tutorials, then you will be able to simply solve this roadmap which I am going to give you. So you will start by writing, so we know that j1 or j is=2 so j1 is=2 in this case and it is given to you okay, j is=2, j1 is=1. So here you can start actually by writing because your k is=2 okay. So your k is=2.

So you start by writing say state 3, 3 okay. So in this problem, you have a j1 is=1 or j is=1, j2 is=2 and you can start by writing j as 3, 2 state. So for 3, 2 state that is when j2 is 3 or when yeah j1 is 1 and j1 is 2 these two values you can take actually. So your capital J will be 3 and 1 but we are interested in taking J is=3. When J is=3 and M is=3 that is this is capital J and capital M okay.

So this state will correspond to the, this is the coupled state, this state will correspond to 2, 1 this is M1 M2. So when you draw this n x n matrix in this case you will have a matrix and this is the topmost corner of the matrix where you have the highest state will correspond to M1 is=M2=M okay. This is the highest state which I will calculate and then the next step would be.

So this will be point 1 and we will go to the roadmap but let me first draw some points. So 1, m this is what you are going to evaluate, 1, m this quantity is proportional to 1, m 2 1 0 m okay. So now let me draw the map for you if you are able to evaluate this quantity okay.

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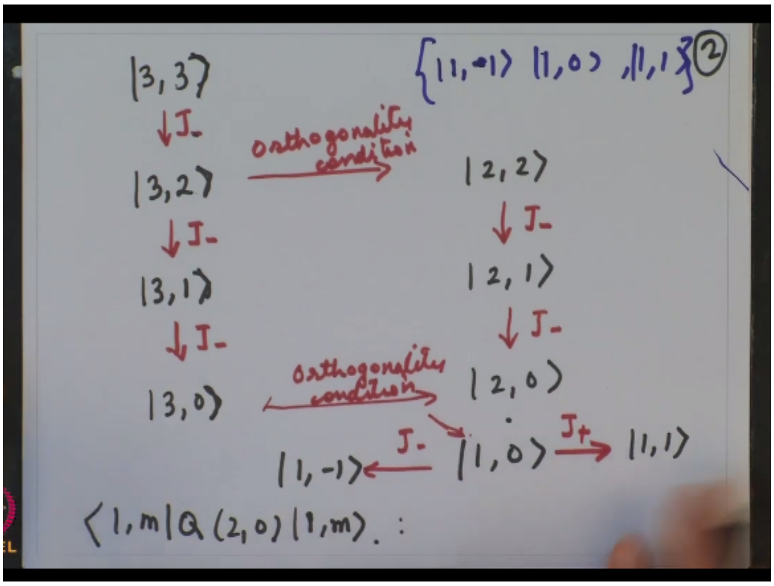

So 3, 3 this is the highest state which will correspond to 2, 1. This we have just now seen. Now if I operate a ladder operator J- what do I obtain here is 3, 2. So when you apply the ladder operator J- operator, you obtain the next lower state. You again go to the next lower state by applying. So repeatedly I am applying the ladder operator or do the ladder operation. This will give me 3, 0.

So these states you can obtain, these are all left hand side terms. Similarly, the corresponding right hand side terms you will obtain okay. You have seen in the calculation of CG coefficient and now for this particular state, so this 3, 2 will be a combination of some terms which will on operation of orthogonality condition if I use you can obtain or you can find out the coefficients of state 2, 2.

Now you go down in the ladder by down in the state 2, 1 by applying the ladder operator J-. Again, I obtain 2, 0. In this step again, you can apply orthogonality condition okay and go to the next state which will give me. This can be obtained from here also or you can apply orthogonality condition and this on further application of orthogonality relation, you obtain 1, 1 state okay.

This will give you 1, 1 state and here you can do two things. You can obtain 1, 1 because basically our idea is to obtain three states. What are the three states? 1, -1; 1, 0 and 1 1. So these are the three states we need to determine. That is why we have to go through these steps. The other way would be to start from 0, 0 and go up to 1, 1 state. So this I can go in two ways.

I apply a J+ or I apply a J-. So when I apply a J+ now obtain 1, 1 state. If I go backwards now obtain 1, -1 state. So once you obtain 1, -1 state and 1, 0 and 1, 1 state, you have to find out this quantity 1, m Q n 2, 0 and m 1, m okay. These you can obtain for different values of m.

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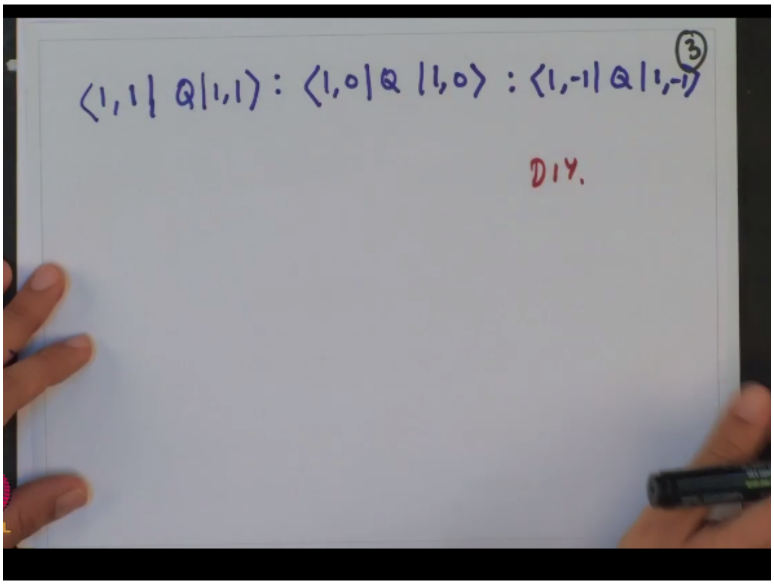

So this ratio will be in the next step you will have I will obtain the ratios that is 1, 1 Q. So let me just go through the roadmap once again. This is the road map, 3, 2 state you go to the lower state, you apply J- multiple times to obtain 3, 0 state and using orthogonality condition you can switch to this state 2, 2 and again series of ladder operation would result in 1, 0 state by orthogonality condition and J-.

And then you can obtain the plus and minus ladder can be used to obtain these two states. So this is your final end product that is required. So the ratio that we want to calculate is 1, 1 is to 1, 0. I am dropping this Q 2, 0 term because of space issue, 1, -1 Q here it is 1, -1. So this ratio you can evaluate. So all the terms which are constant term in the Wigner-Eckart theorem you have seen this proportionality.

So if you take the ratio, the proportionality constant will be as it is dropped and then it will be very easier to evaluate. So this again DIY for you, do it yourself. So this hint is sufficient for you to go about and solve this problem.

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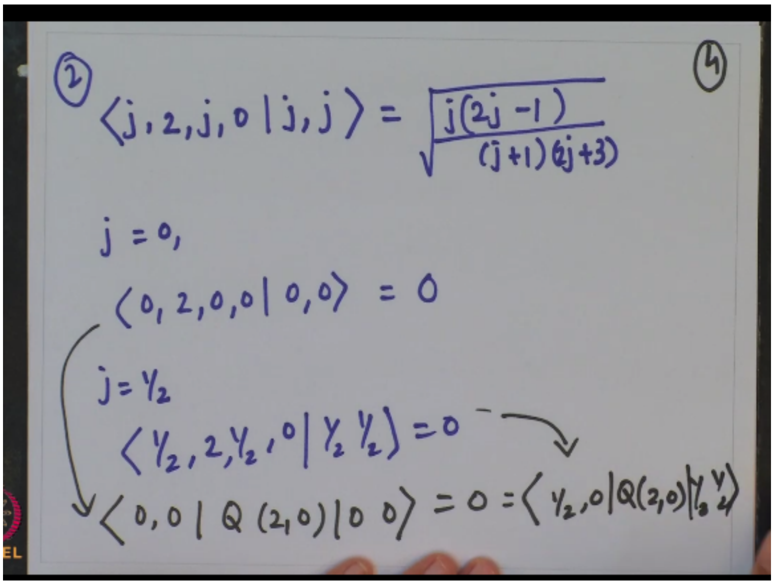

In the second problem, simple steps, you can do the second problem just by inspection. So in the second problem, you have j, 2 j, 0 j j this is given to you okay and this is square root of 2j j*2j-1/j+1*2j+3. This is given to you. This quantity is or rather this quantity is 0. When it is 0, let us check. When j is=0, when you take j is=0 this term that is 0, 2, 0, 0 0, 0 will be 0 okay and when j is=1/2.

So when j is=1/2, this term will be 0 that is to say that 1/2, 2 1/2 and 0 is 0. So by inspection, you can see that these terms are 0 and in the question you are asked to verify that verify the statement that the quadrupole movement of the static 2 raise to k pole movement of the charge distribution is 0 and the expectation value of this state with angular momentum. So from here you can see that these quantities are 0.

And when you write in terms of the expectation value, how will I write it in terms of expectation value? I will have 0, 0 state okay and I have Q 2, 0 I have a 0, 0 correct. This term is 0 okay and another term is 0 which is this is the first term and this will correspond to this that is I will have here 1/2, 0 Q 2, 0 1/2, 1/2 okay.

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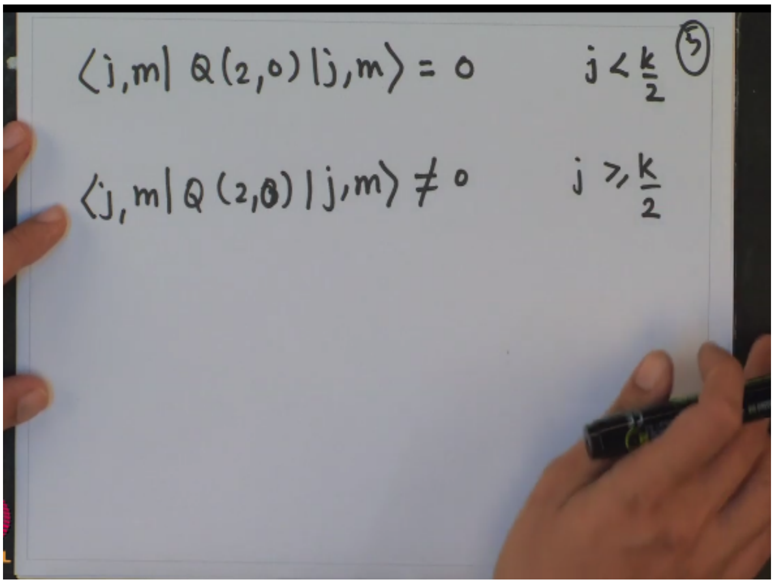

So in general, I can write this as j, m Q 2, 0 j, m is 0 when j is<k/2. So k is 2, so when j is<1/2 that is so this is 2 so when j is <1. So when j is 0 or 1/2, this quantity is 0 while j, m Q 2, 3 sorry 2, 0 j, m is !=0 when j is >= k/2. So if j is =1, 3/2, 2, 2 etc., then this quantity is nonzero. So we have shown here that the pole momentum of the charge distribution has zero expectation value in any state where the angular momentum or j here in this case is<k/2. So it is just by inspection, you can understand and solve this problem.

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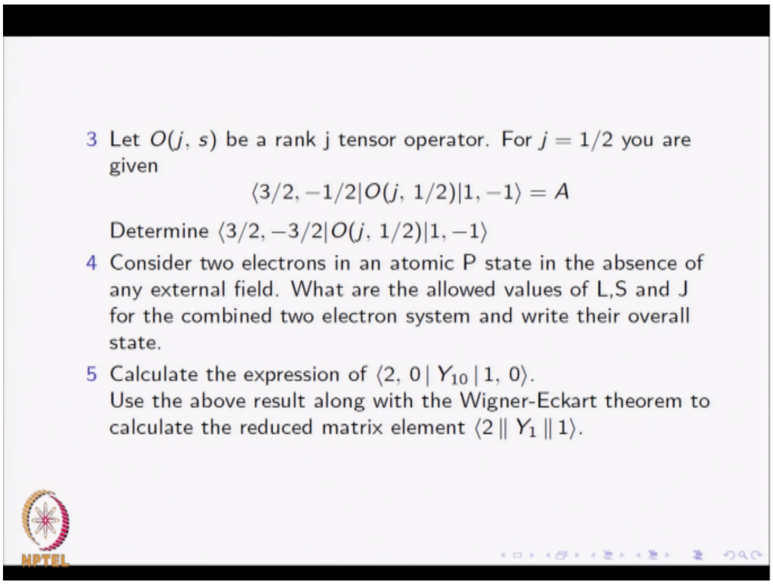

Now let us go to the third problem. Third problem is again recollect the problem we did in that tutorial the 11th tutorial we had done one problem wherein the question was of similar kind. So you have to evaluate this matrix or the jth rank tensor and for j is=1/2, you are given some states. So here also you will use Wigner-Eckart theorem.

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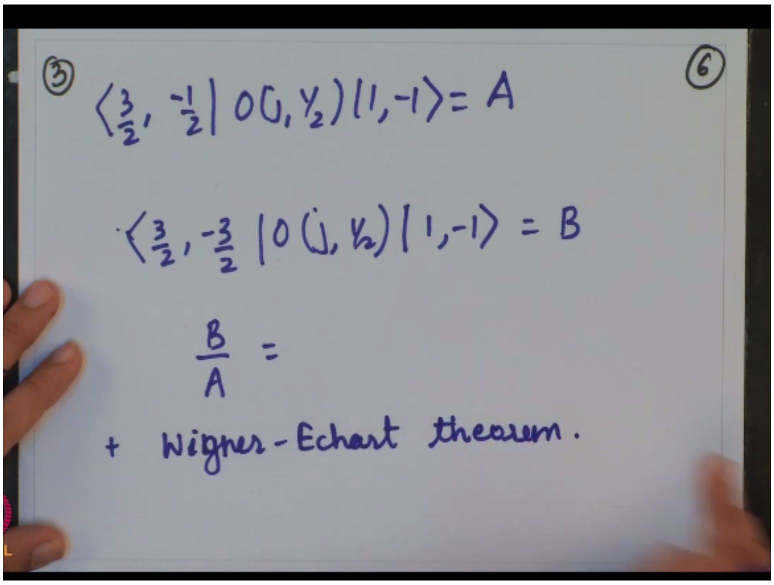

So 3/2 and -1/2 is j, 1/2 1, 1 is A. This you will write in terms of Wigner-Eckart theorem, you will have the j1 j2 term with the operator j okay and another term where you will have a Clebsch-Gordan coefficient and now you have mastered how to calculate Clebsch-Gordan coefficient. You can evaluate that and then let you can call this quantity 3/2, -3/2 j, 1/2. The observable is the same similar one okay, j, 1/2 and j, 1/2.

So this will be 1, -1, you can call this as B. Take the ratio of B/A okay and then evaluate this quantity. You can easily determine this quantity. You have to use, this you will do plus Wigner-Eckart theorem. I think this would be very clear, it is nothing to doubt on but you can go back always to the lectures in the 11th tutorial also and just find out if you can do it. If you find it difficult then we can discuss this later okay.

Now let us go to the fourth problem. In the fourth problem actually, you have to consider two electrons in an atomic P state. Now so first question in this would be what is a P state or a P wave in the absence of electric field.

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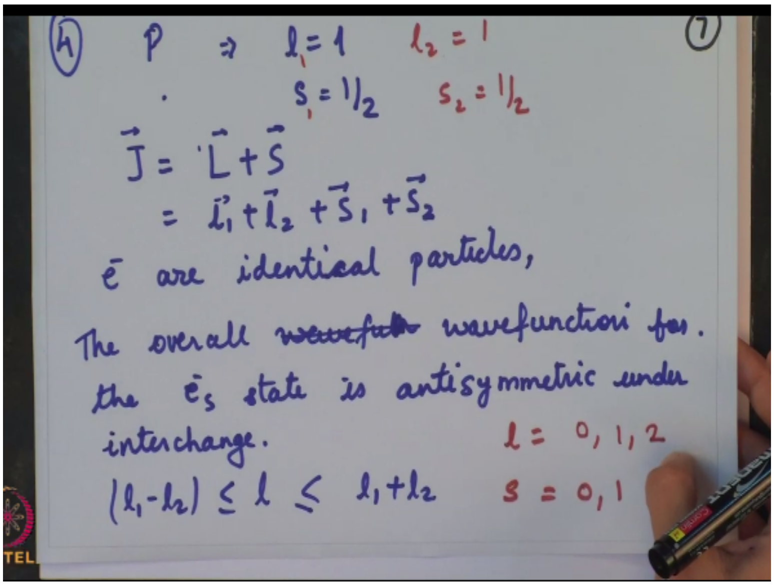

So P wave or the P state is when would imply that L is=1. S state we have seen L is=0, P state L is=1. S is=1/2. So when we are doing the addition of the angular momentum and spin, the total spin or the total state is orbital angular momentum+spin will give me the total angular momentum, so L+S where L is l1+l2+s1+s2. When I am doing all this, I will do the addition J is this okay.

First for no values of Z we can add l1+l2 s1+s2 and evaluate. So electrons are identical particle and so here they have same radial component; however, the angular momentum, orbital angular momentum part will change in sign. That is, it is anti-symmetric. So the overall wave function for the two electrons or the electron state has to be is anti-symmetric under interchange or exchange of particles, under interchange okay.

So you have to be very careful and see which part of the, which states give you the symmetric and which part will give you the anti-symmetric wave function. So l value, l can take value from l1-l2 to l1+l2 okay. So l can take value from l1-l2 to l1+l2. So l will have what values? l will have l1 so l1 is 1, l2 is also 1. So I have l1 l2 as 1 1 and s1 s2 will be. So there are two electrons, two electron state will be 0, 1 and 2.

And s can take values 0 and 1 because you have 1/2 as the electron spin. So this would give us the possibilities. Now with this possibilities you have to make combinations only which are allowed. Which combination will be allowed? The combination which will give me the symmetric state. The symmetric state will be forbidden sorry and the combination which gives me an anti-symmetric state will be allowed state.

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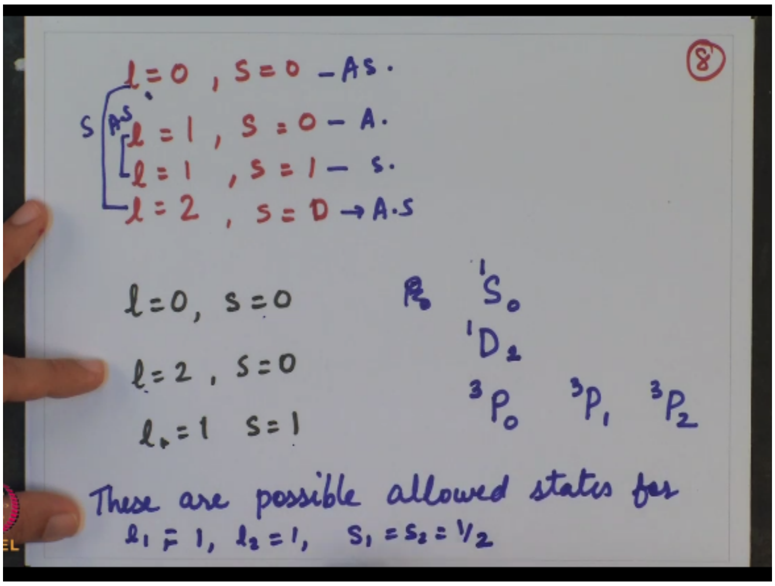

So this will give me, I will have a combination like I can take l is=0, s is=0. I can take l is=1, l is=2 these combination. Then, s is= 0 state then l is=1, s is=1 state, l is=2, s is=1 state. So I have a 0 state okay. So out of these combinations, l is=1 and l is=0 and l is=2. These two combination are symmetric combination and when l is=1 it is anti-symmetric combination okay, l is=1, this is symmetry, this is anti-symmetry.

Similarly, s is=0 will give me an anti-symmetric combination. This will be a symmetric combination, this will be an anti-symmetric combination, this will be an anti-symmetric combination. So from these you have to choose the combinations which are allowed. So l is=0, s is=0 will be allowed; l is=2, s is=0 will be allowed and l is=1, s is=1 will be allowed. So in the spectral notation, the corresponding states can be written as.

I can write them as P0 and l is 0 so I will sorry I will have here S because l is=0 so S state. S state s is=0 so I have here just 1 and j for this combination is 0 okay. Then, you come to l is=1, s is=1. So there will be a P state and here you will have s is 1, so this will be 3. The combination for j can be 0, 1 or 2. So I will write here 0 3P1 3P2. Similarly, the combination for this will be you can have only one possibility.

So here you will have s is 0 1 and here you will have 2 j is=2. So in terms of spectral notation, these are the possible values or the allowed state for the combination. So these are possible allowed states for l1=1, l2=1, s1=s2 is=1/2 okay. So these are the possible states which are allowed okay. The final problem, the last problem will again take you back to Wigner-Eckart theorem and Wigner-Eckart theorem by now you know that you can write it in terms of reducible and irreducible representation.

You can link or write the reducible and irreducible representation by a constant of proportionality.

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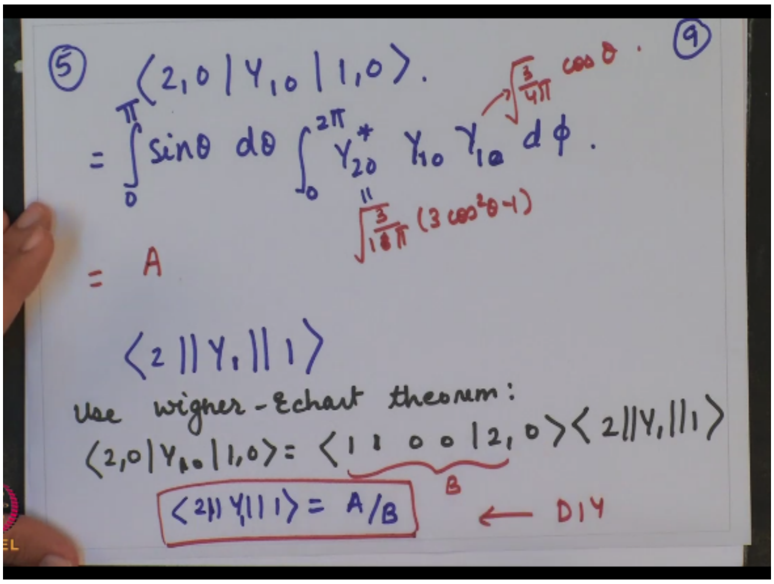

So this problem a part that is you have to calculate the expression for 2, 0 Y1 0 1, 0, this is what you have to calculate. So you use Wigner-Eckart theorem to calculate the reduced matrix element okay. So the reduced matrix element is written as 2 Y this is another way of representing the matrix element, reduced matrix element. So here you have 2 1 Y is 1 0. If I want to write this in terms of integral operator or integration, this will be 0 to pi sin theta d theta integral 0 to 2pi.

This will be you have Y1 0 here you have Y1 0 and Y2 star Y2 0 star. So this I can write as Y2 0 star, so this problem actually we have put to give you the flavor that you can write these kets in terms of the harmonic operators. So the spherical harmonics and these states how they are related, that you can see here and you know these values Y2 0 theta phi can be written as in terms of the coordinate, angular coordinate is 3 by I do not remember.

I think 3/16 pi or something you just check this, 3 cos square theta-1, this is 1 0 and here this is I think 3/4 pi cos theta. You can evaluate this, just put these angular terms, this 0 to 2pi and this phi goes from 0 to 2pi, so you will have a 2pi factor and the remaining you can just work out by putting cos theta is some t or something like that and just evaluate. You will get some answer; let me call it as A, some constant you will get okay.

And that constant you have now. Now you can write use Wigner-Eckart theorem. So you use Wigner-Eckart theorem and using Wigner-Eckart theorem you can rewrite this as 2 1 1 0 1 0, you can write this as I am directly writing the terms relevant term okay, 2, so here we have 1 0, so this is 2, 0. This is the term, it has got flipped, just do not bother about it, 1 0 1 0 so 1 1 0 0 2 0.

I have just flipped it; it does not matter if you flip the terms. It will not change the result anyways, 2 Y1 1. So this is the term which you need to evaluate. So this term will be this you know is A, you calculate the CG coefficient of this. Now you have to I mean just by looking at the terms you should be able to evaluate the CG coefficient. So let me call this value as B. So in the final expression, you will have 2 Y1 1 is this is A/B.

So this you will DIY and check okay. So do it yourself and check whether you get this ratio. This will be a number; this will also be a number, B part will also be a number, some square root of something. Here also you will have some number and just take the ratio of these two and you are done. It is not at all difficult. So in these tutorials, you have to handle the CG coefficient if you have mastered.

Then, you will be able to do problems on Wigner-Eckart theorem very easily. So with this, I wind up on the tutorial and this is the last. This was the final tutorial which I have discussed now and hope you all enjoyed the tutorial and also worked out each and every tutorial by yourself and wish you all the best for your assignments.