Quantum Mechanics
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Lecture - 70
Tensor Operators & Wigner-Eckart Theorem - III

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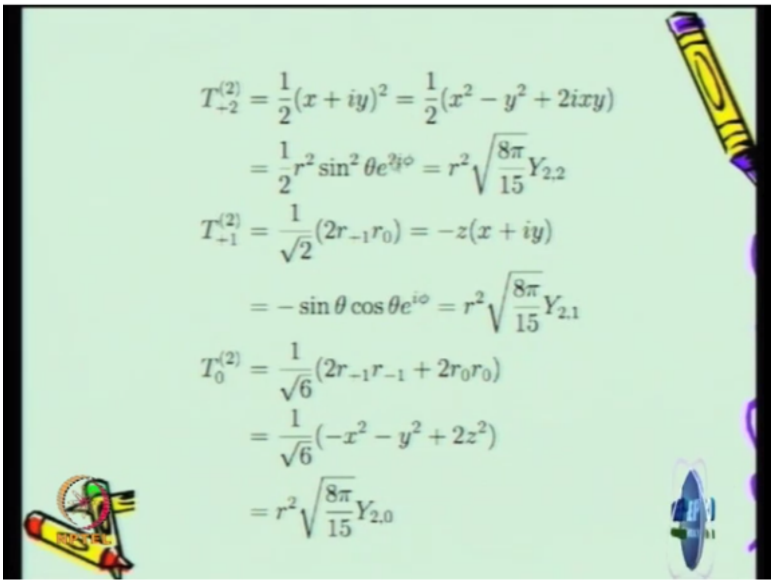

So let us do this on the operators which are similar to your spherical harmonics. Take a tensor product of spherical harmonics y where l is 1 y1 mi y1 m2. So that is nothing but this is a stretch state, this is like the stretch state. The stretch state will be just a square of the uncoupled. It is just the product of the uncouple states with the coefficient 1. Rewrite it and you will get x squared-y squared+2ixy and try to rewrite it in terms of theta and phi.

If you do that you show that up to a normalization it is Y 2, 2. So the tensor of rank 2, the component q will go from +2 to -2. You can construct out of tensor product of 2 irreducible tensor of rank 1 and we get this to be proportional to the spherical harmonic Y 2, 2. So there is a resemblance between this k, q and l, m. That is all I am trying to motivate you. You can do one more step here.

Do the ladder operation and go to the same rank two tensor the component 1 and you can show this? So please check some of these data but it is as simple as is just to show that there is a one-to-one correspondence of spherical harmonics in the position vector to any arbitrary tensors made out of two vectors. So there is a correspondence and you can explicitly verify for the position vector, it is indeed giving you Y 2, 2 up to normalization.

And by ladder operation you know how to do this. L- you know how to operate it here and you can directly say that this is the way the other components of the rank two tensors will be. What will happen to, I have written the 0, 1 and 2. What will happen to -1 and -2? They will just change your e to the i phi as e to the –i phi. This will become e to the –i phi when you make this to be -1.

Subscript is the q, superscript is the k and this one will become e to the -2i phi if the subscript becomes -2.

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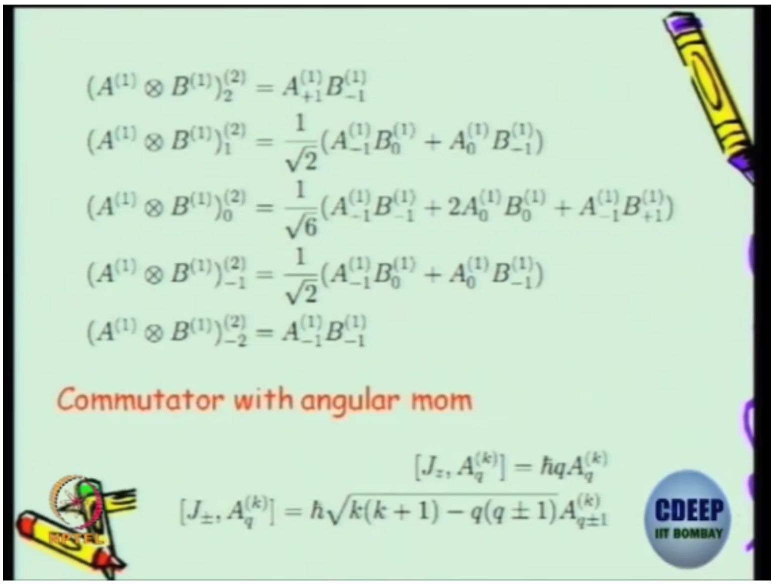

Elaborately, I have tried to put a step. Here take a tensor product of two vectors to irreducible tensors of rank 1 A and B. The 2 2 component is trivially taking the square is what we saw in the position. Here it is taking the stretch state components. Take the 1 1 component and the 1 1 component, just multiply them and that is what will give you the 2, 2 component. If you want to do the 1, 2 component, you have to apply the CG coefficient, you do it for the states.

That is also another exercise for two spin 1 you do the state CG coefficient. The same thing will be applicable here and you have to make sure that this is total m is 1, this is the total q, q1+q2 should add up to give you q that is important. This is exactly what is happening. So you please verify and get your hands on to feel that this is the way tensor product of tensor operators irreducible tensors if you take product you can reduce it.

If you take the products to this side which I have written is like an uncouple state of the reducible components and the reducible components can be rewritten in terms of the irreducible components and each of these irreducible components has some linear combination of the reducible components. These are your raw Cij's, this side is the composed one which will be some linear combinations of those Cij's which is what I wrote.

The Aij is a linear combination of Cij –Cji. Sij is a linear combination of Cij+Cji-. So these are things which I wrote and this will fall into the place the way I have given it here. Once you do this, you can actually, this is something which I already said T k, q. We did this for k=1 but you can look at these tensor products and argue that this is the way the commutator algebra of tensors of rank k any qth component it will always satisfy the result.

Very nicely fitting you know many things which look like completely disjoint but you can try and make contact with what you know by using the CG coefficients okay.

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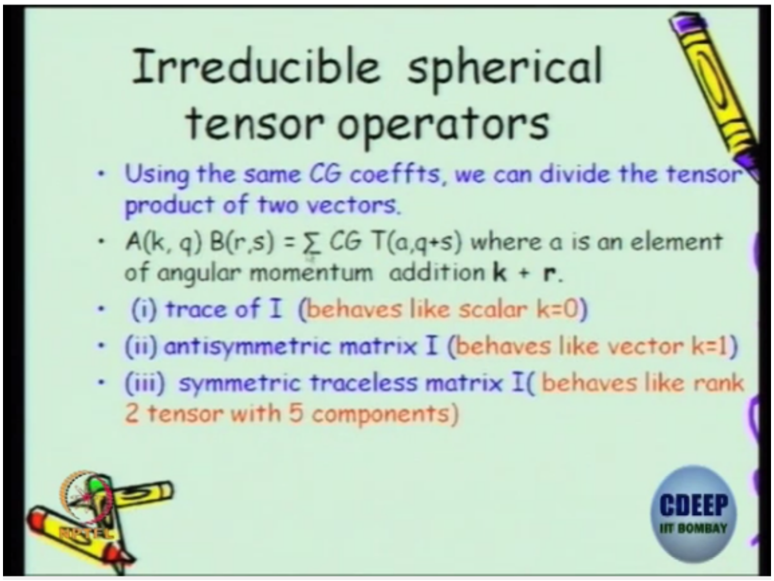

So this is just to summarize what we did. You can take a product of two vectors or two tensors, two irreducible tensors and you can try to resolve. Then, this product left hand side is formally a reducible tensor but you can resolve them with the CG's with q+s is what I have written okay. Then, this is what I elaborated for k=1 and r=1 in the earlier slide okay.

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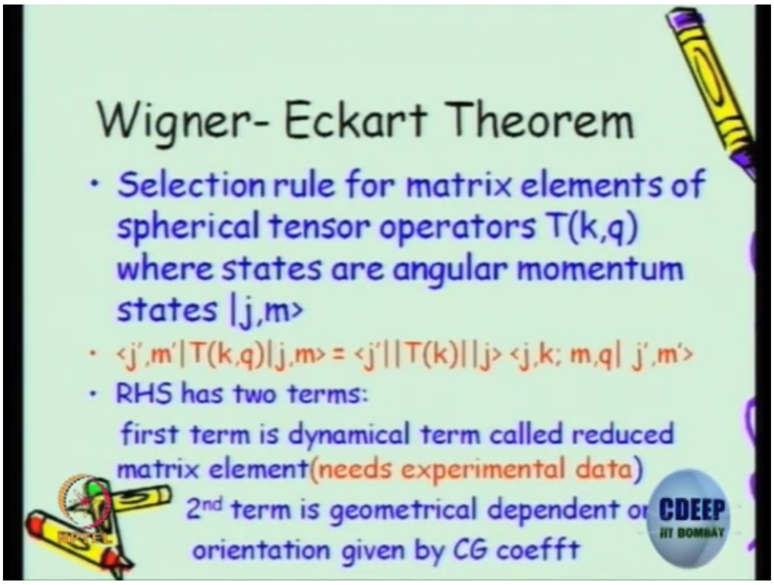

So this is the selection rule. One can actually prove this theorem. This is the matrix element of an irreducible tensor of rank k and q between two angular momentum states j, m and j prime, m prime okay. You can formally prove this. This depends on the CG coefficient. What does it mean? If the CG coefficient m+q so this side is an uncouple state, this side is a couple state okay. So if m+q is not m prime, it is going to be CG coefficient is 0.

Equivalently this matrix element can be put to 0 or you can tell the experimentalist for quadrupole moment tensor of rank 2 or a vector that has a matrix element such a transition you do not even try to look at, it will be 0. This much we can tell them okay. Just from the angular momentum algebra and the CG coefficient requirement, we can tell them which matrix elements or which transition; this is the operator which triggers the transition from an initial state to a final state.

We can tell them such a transition for triggered by some operator like dipole moment or quadrupole moment do not even you would not be able to see because that CG coefficient is 0. So for scalar for example, if suppose this is a scalar operator, what can we say m has to be m dash. Only diagonal elements are possible in those matrices that is it, m has to be m dash and similarly for vector operator, suppose I put m dash is m+2 if suppose this was an irreducible tensor of rank 1, q will be + or -1 and 0.

When you add an m to it, I need to get m prime right. If I put m prime to be let say m+3, it is going to be trivially 0. So some of these matrix elements, some of these transition elements you can argue from this Wigner-Eckart theorem. I am not proving it here in this course but it can be done by composing vectors with the states sorry operators with the states which I have not done now.

But if you do it systematically, you can prove this theorem. This matrix element I have no information, which is what we call it as some reduced matrix element, which has information about what operator I am looking at and looks at only the total angular momentum of the initial state and the final state. This information I do not have but definitely the CG coefficient I have the information.

So that kind of tells me that I can tell which element will be 0, which elements will be nonzero from the CG coefficients but if I ask you to tell me what is the exact coefficient, exact matrix element, you need to give me. Suppose I give you an answer for T of 1, suppose I tell you that the experimentalist tells me for m dash and m T of 1, 0 this is the element. This is what I measure.

I can put that measurement here as a instead of this unknown constant and then I can do the other components. This is what is the power of this Wigner-Eckart theorem okay. So let me give you some examples.

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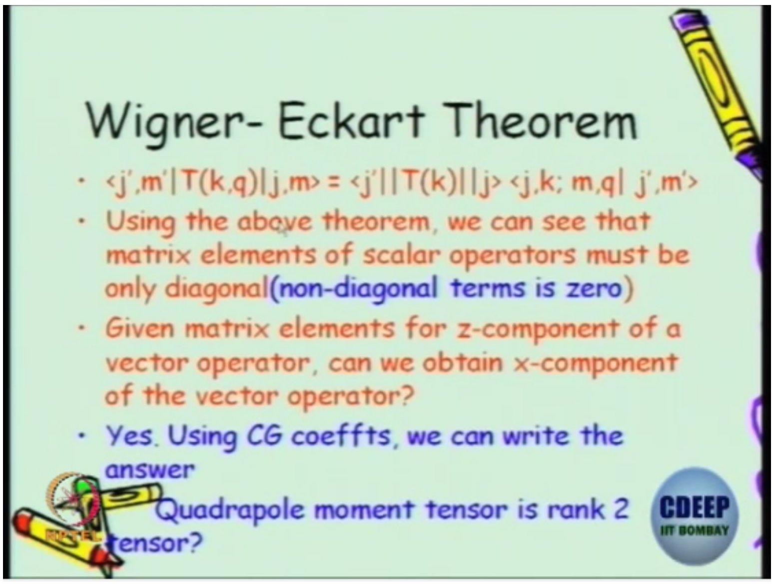

So using the above theorem, we can see that the matrix element of scalar operator must only be diagonals, this I have already argued for you. Non-diagonal elements are 0. Suppose I give you a matrix element for z-component of the vector operator. I tell you what is answer. Can you determine the x-component of the same vector operator? There you can use this Wigner-Eckart theorem where the ratios depend only on the ratios of the CG coefficients.

So I will leave it to you. Suppose I give you this answer, so let us state for dipole moment. I give you the dipole moment k=1q=0. I give you an answer as some value. Let say some value, some constant k or something. Then, I ask you can you find T of 1, +1 q=+1. Can you do it? Given that data okay. So that is the question for you.

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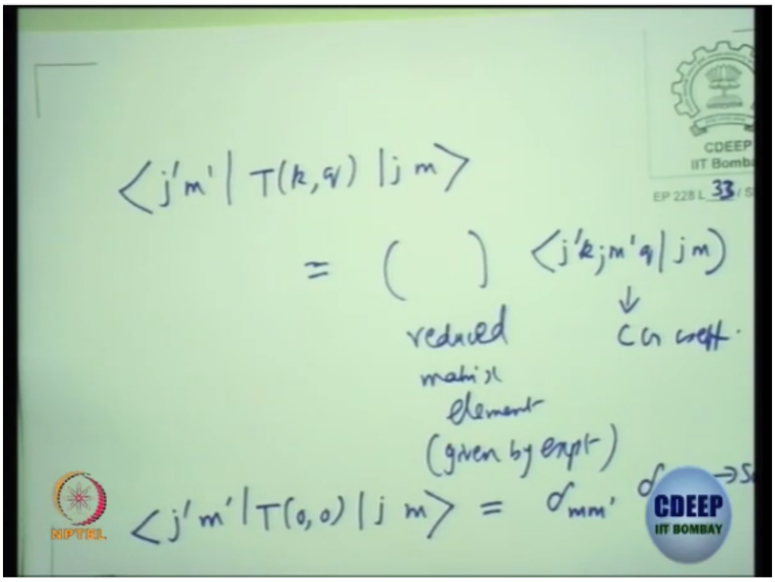

Let me write this. So Wigner-Eckart theorem tells you that if you want to find, this is the piece which is dependent which I call it as a reducible reduced matrix element. This is given by experiment but then you also have the CG coefficient j prime k;: m prime. Did I do it wrong? It is okay but it does not matter whichever way we do it. So you can write it this way okay. This we know by the computation of CG coefficients.

This we do not need an experimentalist. From here, we can say that j prime m prime T of 0, 0 j m will be delta mm prime delta jj prime okay. It is like an identity operator, the states are orthogonal, so let better be that, so only diagonal elements will contribute for scalar operator okay scalars. This is a scalar which is a rank 0 tensor. Is this clear? So now let us do it for the vectors.

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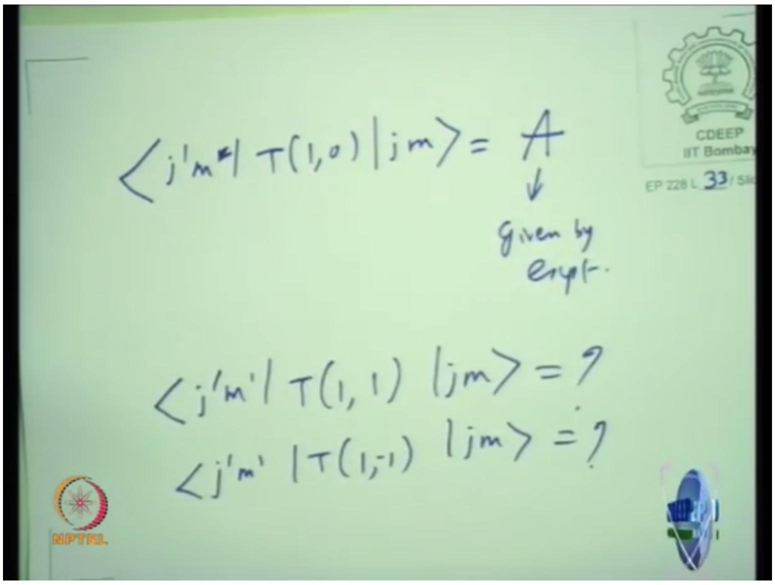

If I do it for the vectors suppose I give you some number here, so let me call it as some capital A. This is given by experiment. Suppose I give you this answer, then the experimentalist is asking me can you tell me what is 1, 1 jm. He says I can measure the z-component of the dipole moment but I do not know how to measure the x component or y component but this is not x component. What is this?

It is - of x+iy after you measure you can also do, similarly you can also find out what is this. If you do find out both, you can take a linear combination and determine what is the x component of the dipole moment; you can take the y component. So my question is use the Wigner-Eckart theorem definition and determining terms of A what is this okay. I leave it you to do this.

We will have some exercise problems. Similarly, do this. These two will not be in general same and then from there I extract out the x component of the dipole moment operator and the y component of the dipole moment operator. You can even take these m prime to be some specific values and do it also. So here it is actually the same m, otherwise it will be 0. So this is something which I want you to put your hands on and check.

Because most of the other courses you will be straightaway asked to work out whether such as transition is possible from an initial state to a final state in the presence of a quadrupole moment operator or other operators you know and you need to know what is the irreducible tensor of what rank and then you can determine. At least, you can say whether it is 0 or nonzero.

The exact value cannot determine because the reduced matrix element an experimentalist has to give okay but you can find the ratios of the components. Why you can find the ratios of the components? This reduced matrix element is independent of the components, independent of q or independent of magnetic quantum numbers. All those magnetic quantum numbers and q dependence is only in the CG coefficient.

So once I give an answer for one component for a specific ranking, other components are only dependent on this constant times a CG coefficient, so it is dependent only on that constant times difference CG coefficient. It is very powerful. If one component answer, the experimentalist gives you for a rank 2 tensor you have 5 components. If he gives you one component answer, one of the components let us take rank 2 q=2 he gives you an answer.

You can actually determine all the remaining 4 using this Wigner-Eckart theorem because this constant gets fixed from that experimental data and this is independent of the components of the tensor or the magnetic quantum numbers of the state. All those informations are only in the CG coefficient. You can actually determine all of them and this is the power of the Wigner-Eckart theorem where we do the composition of tensor operator on the states.

And that tensor operator on the states again can be resolved in terms of CG coefficient is really beautiful. That is the way to prove it but right now let me state it as a theorem for you.

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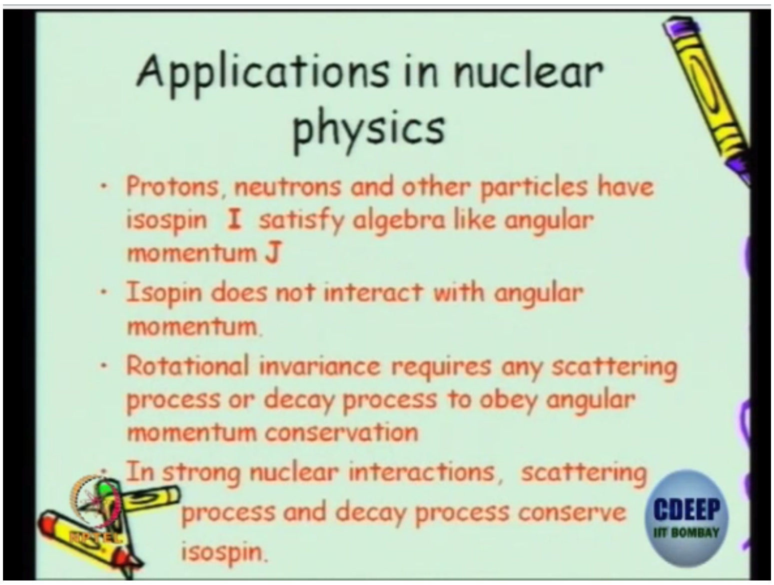

So quadrupole tensor is rank 2 and there are a lot of applications which you will start learning when you do nuclear physics.

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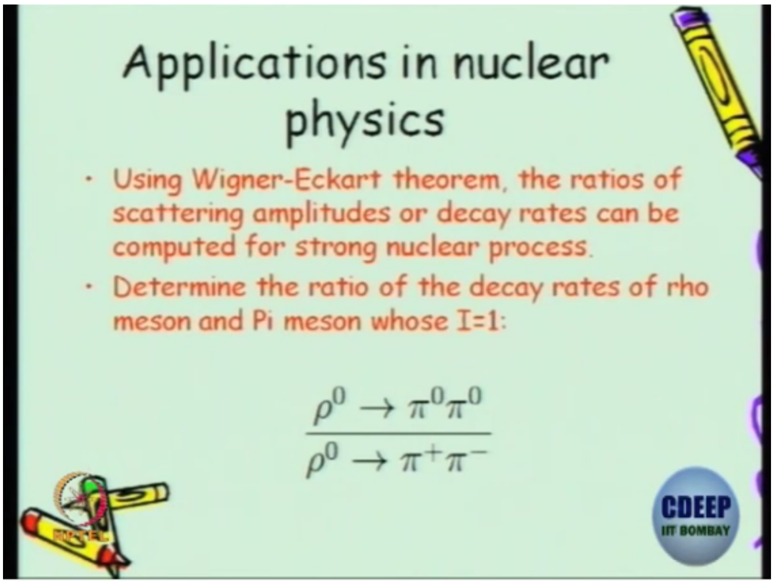

And particle physics where you can use Wigner-Eckart theorem to say whether a process A going to B+C, whether that process is allowed, if it is allowed what is the ratios of those scattering cross sections and so on. Lot of these ideas will come into picture. So let me stop here.