Quantum Mechanics
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Lecture - 69
Tensor Operators & Wigner-Eckart Theorem - II

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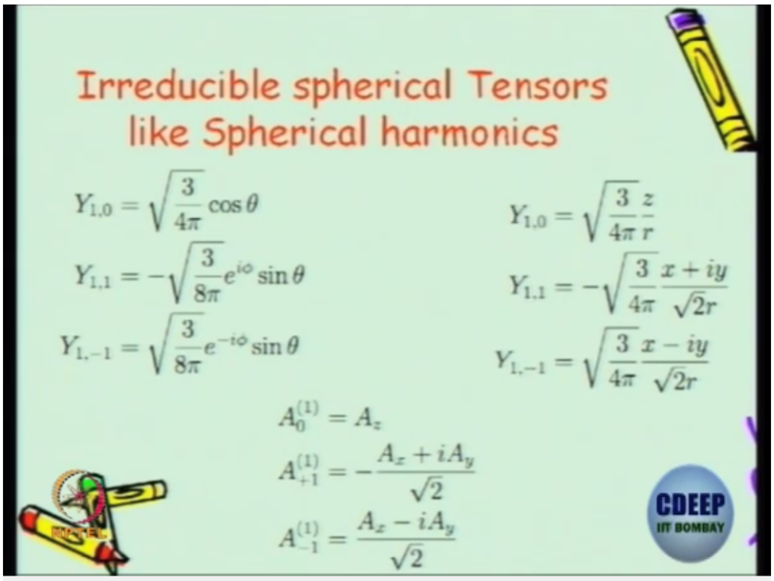

So this is for this completeness I thought let me put the data for you. Why this iy and x ix+y all these things comes in –x –iy. So this is because if you see the spherical harmonics, Y 1, 0 up to this normalization is cos theta. Cos theta is nothing but the z-component, proportional to the z-component. Y 1, 1 has an e to the i phi which you can write it as cos phi+i sin phi right and there is a sin theta.

To resolve them into components what you get here is – of x+iy and if you look at Y 1, -1 it is x-iy okay, clear. Remember these things, so this is why this is for the position components, you can write for any vector components in the same way saying that it is a rank 1 the superscript is denoting my rank 1 here. You should have put in k, q I put a superscript for the rank and subscript is for the q values. It is just a different notation.

Some books follow this notation, so you should know what exactly it is. So just seeing this for a position vector which is proportional to z, you can call this to be the z-component. Similarly, you can call the magnetic quantum number, the q=+1 component to be exactly similar to this which is what we have written here and the last one. There is some subtlety of normalization and the convention followed in the normalization is this.

Probably, I had it in that slide but when I wrote it on this I did not write it. Is that right? I had it on the yeah there is a square root 2 here which should also be incorporated. Is this clear? Actually, the rank 0 tensor is a scalar should be clear to you; rank 1 tensor is a vector and the vector keeping your knowledge on spherical harmonics which is applicable for position operator. You can try and write any vector operator in that notation.

Instead of writing x-component, y-component, z-component, you will like to write it as k, q as 1, 0; 1, 1 and 1, -1 or you can put a superscript under subscript where superscript corresponds to k and subscript corresponds to q. Is that clear? So this is for a tensor of rank 1.

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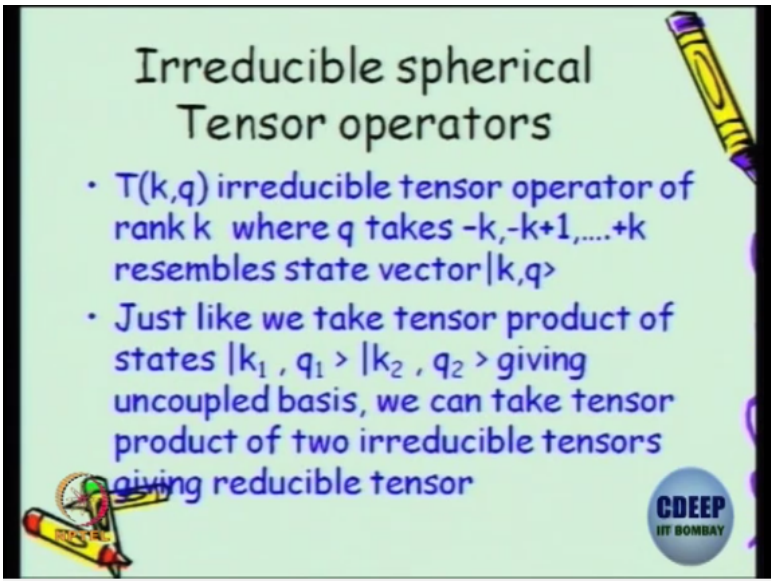

So this I have already said. There is a lot of resemblance between spherical harmonics and the k, q and similarly formally for any other vectors other than position vectors you can have a similar behavior of how k, q behaves under rotation. That is the same way to operate as well behave in the similarity transmission. So then we have to look at you did the CG coefficients by taking two spin half particles or 2 particles with angular momentum j1 and j2.

Why cannot it compose two tensors and see what happens right. This is what you will ask. Suppose I have a tensor, I take another tensor, if I combine these two tensors that is like an uncoupled state which what we call it as a reducible tensor. Why reducible? It is the raw thing where I will reduce it and make it like a trace, anti-symmetric. This is what we did and made it into irreducible tensors okay.

So I am bringing in you over the concept of reducible and irreducible just like uncoupled and coupled states right. Uncoupled states are the ones which you have just taken a tensor product and written but then you also said the coupled state has some properties like when you took the two spin half particles, the spin j=1 all the states there you can go by ladder operation of j=1 only and all those states was symmetric right.

And then the spin j=0 state, we found by orthogonality with 1, 0 state and then we found that it is anti-symmetric. So there is some kind of a separation and each of those states is what we call it as you know separate module. It is a separate module. So let me try and say formally what we mean by that and that is what we are going to do by composing tells us. So let me take few minutes and try and tell you some of the jargons if you read books you will get to see them.

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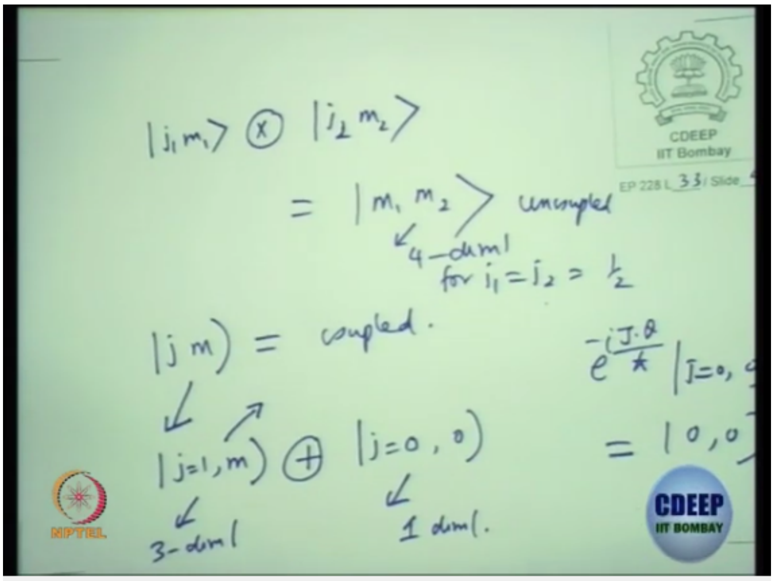

So this is what we did j1, m1 tensor product of j2 and m2 which we called it as a coupled state. Let me suppress the j1 j2. So this is uncoupled okay and then we had another one which was jm which I for convenience I tried to put this bracket which I called it as coupled. This is also four-dimensional for j1=j2=1/2 and this one can be written as j=1, m and a sum with j=0 and 0 right. This is what you saw and this one was three-dimensional and this one was one-dimensional.

So this is like a scalar state or like a scalar operator or analogy with a scalar operator or a singlet state and the singlet state remains as a singlet state in the rotation okay because even if you do the e to the j+ operator j- operator it is nothing happens. Exponential of that will be like an identity operation right, e to the power of ij dot theta by h cross on j=0 0 will give you what, the same state.

This is behaving like an identity operator. Singlet state is unaffected by rotations. This is similar to your scalar operator which remains the same under rotations and this one will transform under rotation right. The j=this k, this will transform under rotation like a spin 1 state and the same thing will happen for operators which are vector operators but in the operators, you have to do a similarity transformation.

State vectors transformed by U operator, operators will transform by U dagger operator U. This is all is the modification, so you can see that some kind of a ringing that it tells you that a total four-dimensional representation in the uncoupled basis, you can resolve it having some specific properties under rotation, a state which is invariant under rotation, another state which is behaving like a vector under rotations, three components.

So this resolution is what we dealt by the CG coefficients. Remember, we did the CG coefficient where we did this resolution. Now I would like to do the same thing for operators okay. So this is what I was indicating in the last lecture that if you take two vectors, all the operators will have only integer j's so that is why I am calling k and q.

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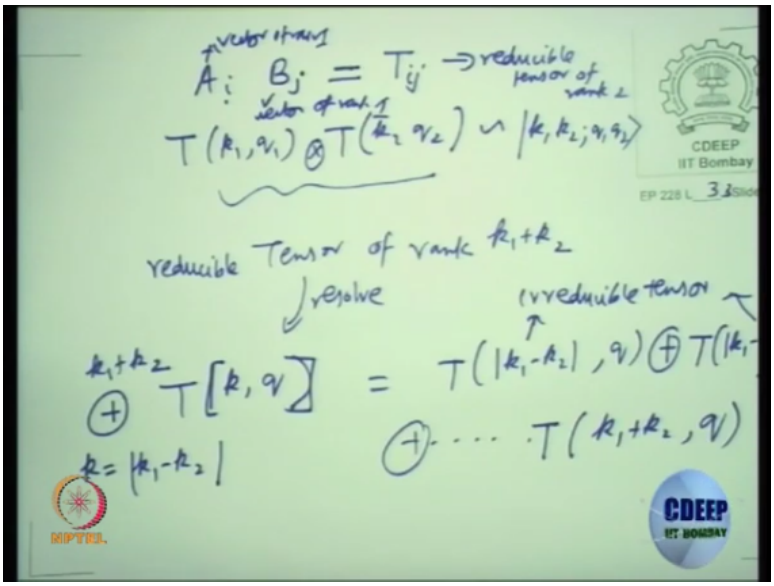

So if you take T of k1, q1 and take a tensor product of T of k2, q2 okay. This one is formally similar to there is an analogy to k1, k2 q1, q2 the similarity but you should know that these are operators, these are states okay that similarity tells you that this dimensionality of this. This will give you a tensor of what rank, so k1+k2 formally right but it will be a reducible tensor. What do I mean by reducible?

It is a raw tensor where you have combined them but now I have to resolve with respect to the rotation properties. Resolve them, how will you resolve? Again using the same CG coefficients and break it into a k, q or correspondingly I could say it is T of k, q. So it is a linear sum going from k=k1-k2 k1+k2, it is a linear sum. So by this linear sum I mean T of mod of k1-k2, the corresponding q, take a sum of T of mod k1-k2+1, q dot.

Finally, the last one will be k1+k2, q. This sum which I am writing is what we call it as some kind of a direct sum like this is the direct product. This is what we call it as a direct sum. Each of these are irreducible tensors okay all of the separate okay. So this piece when I just take a product like an uncoupled state, let us consider to be a reducible tensor. This is exactly what we did.

We took an Ai under Bj, this was rank 1 a vector Ai and a vector Bj the components and then I wrote formally this as some tensor of rank 2. This is a reducible tensor. This is vector of rank 1. You understand what I am saying. This is also a vector of rank 1. You had two vectors of rank 1, you took a tensor product of two vectors and you got a raw tensor of rank 2 that is called reducible tensor okay.

And that reducible tensor you can write it as components or resolve it into irreducible tensors and each of those components will have a specific behavior under rotation. The raw one will not have any specific behavior. You can break it into subcomponents so that a trace of that matrix is nothing but A dot B that is invariant under rotation. One which is invariant under rotation is like a singlet state which did not change under rotation operator.

So that scalar is an irreducible tensor pulled out of the tensor product of two vectors and how will you pull it out, you can use the CG coefficients same methodology and pull out the irreducible tensors from the reducible tensor which you have just got by raw multiplying the two separate tensors. When we did vectors, you take vector A and a vector B, when you just compose them, you are not doing dot product or cross product. You just formally taking Ai with Bj where i could be anything one of the 1, 2, 3 or j could be 1, 2, 3.

How many components are there? 9, 3 cross 3 which is 9 right. So the 9 can be written as the 3 x 3 matrix. I can call it as a rank 2 tensor or I can formally write it as Tij but this rank 2 tensor is a reducible tensor. I want to resolve it and write it as components which has specific behavior under rotations okay. To do that what I will do is I will try to resolve it this way. If I do it this way, this is resolution.

You remember that when you did this uncoupled state to go to the coupled state, this resolution gave you the CG coefficient. The same CG coefficient will help me to do, this is not uncoupled state, this is called reducible tensor. I can break it into irreducible component. Is a spirit clear? You should go back and look up shift now okay. It is very well said in symmetries of quantum mechanics, there is a section where these tensors are very well related.

Now if you go and read it, you will understand okay. You can also do mix and match; you can have a tensor operator under state also. Nobody tells you that you should only look at because now you can go back and forth; you do not need to only work with tensor operators multiplied with another tensor operator. You can also have a tensor operator operating on a angular momentum state.

So those compositions also can be done okay. So please go back and read shift and then we will try to appreciate what I am trying to say okay but at least you should be clear that this is a reducible tensor and you will use the CG coefficient just like we use the CG coefficient to break this into mod k1-k2 in steps of 1 to k1+k2. The same thing you have to do it for the tensors also.

So T k, q is an irreducible tensor where q takes values. This is just again I am trying to repeat, similar to a state vector k, q just like we take tensor products of two vectors, you could take tensor product of two, the two tensors which we take are trivially this T k, q is a irreducible tensors, so product of two irreducible tensor will give you a reducible tensor okay in general. So whenever we take tensor product of two T k, q's each one is an irreducible tensor.

But when we take a product, we will get a new tensor with the rank looking like k1+k2 but that is not a irreducible tensor.

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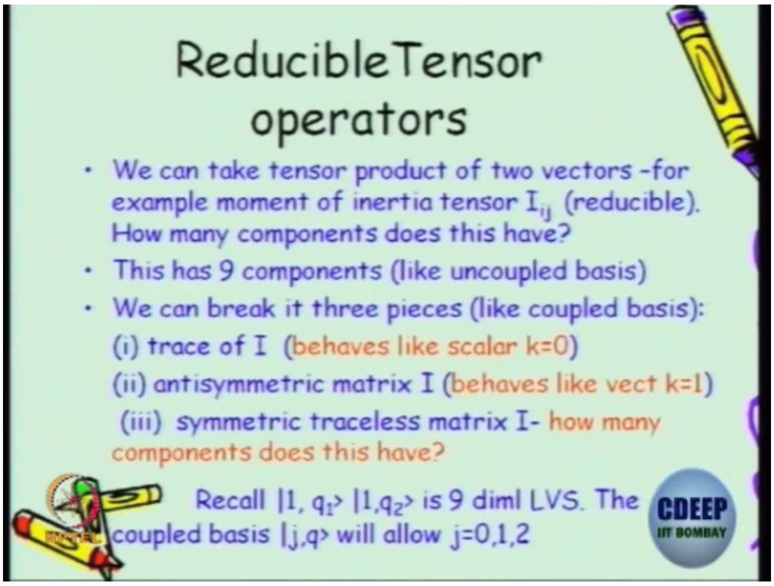

So this we did elaborately last time. I am again repeating it here. We can take tensor product of two vectors and we can write Iij. This is in our notation, now it is a reducible tensor correct. Ai was an irreducible tensor of rank 1, another if you take the same vector Ri and Rj each one is an irreducible tensor of rank 1 but the product which gives you a moment of inertia tensor is a reducible tensor.

And you can break that reducible tensor into pieces. This is what I was saying but I did not do the CG coefficients here but you can do this also okay. So if you do the CG coefficients, it will give you pieces one is A dot B which is like a trace, another one is A cross B which is like an anti-symmetric piece of this Iij tensor. The last one is the remaining out of the 9, you already have one component here.

A vector has 3, 3+1 is 4, how many are left? Five, a symmetric traceless tensor will have 5 independent components. With trace, it will be 6 but without trace if we want to reduce the trace to be 0, then one more component is constrained right. So it is 5 components. So once I get 5 components, you can see a resemblance. When I took a tensor product of two spin one states, which was nine-dimensional, allowed j values are j1+j2 to j1=j2.

So it is 0, 1 and 2, so this is what you would have written for the states. This is exactly what I have tried to resolve here and rewrite your states, rewrite your operators, tensor product of two operators as a resolved irreducible components okay. So let me just put it for you in more complete thing.

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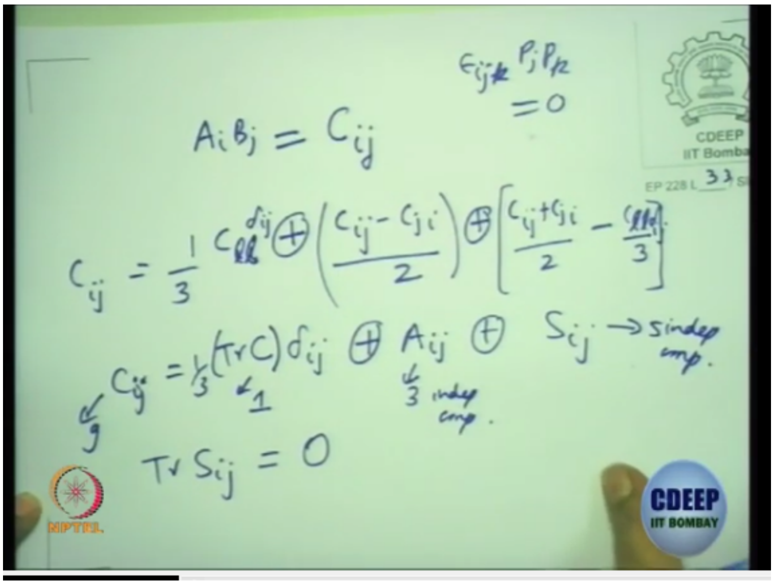

So I am going to take Ai, Bj, let me call this as some this T if you are getting confused let me put it as Cij okay. This Cij I can write Cii+Cij-Cji/2. Is this correct? I have not done anything. I have written the Cij in a little easy looking fashion, nothing more I did. This I can write it as 1/3 trace of C times delta ij. This I will write it as Aij, loot at this bracket, this I will write it as Sij. Can someone check trace of Sij is 0 or not?

Can you check? So what did I do? I just rewrote the same element which was on the left hand side is a reducible element. I resolved the same element into 3 pieces, delta ij yeah I put it as trace C times delta ij so that the ij elements I have taken. So I have just written any component as formally as adding pieces as is pointing it out, I can write this explicitly as this way so that the indices are all visible yeah.

So you can use Cll and then put a delta ij, is that what you are saying? So Cll means it is lambda okay. So I have just tried to rewrite it in a fashion where can an anti-symmetric tensor under rotation become symmetric tensor? That such kind of chaos happens then you will have you know two del r's when you have which is symmetric and if you have an epsilon ijk you put it to be 0 because the product are on two different spaces.

I am sure you would have played around with (()) (22:46). Suppose I give you an epsilon ijk and I give you a Pj Pk what is this? This is symmetric, this is antisymmetric. So always 0, so the symmetric product and anti-symmetric products, anti-symmetric tensors and symmetric tensors are two different spaces always okay. So only thing is that the trace term is invariant under rotation or it is behaving like a singlet.

And then you have an anti-symmetric tensor which is actually an axial vector kind but it behaves like a vector under rotations and then you have the remaining which can be written as a symmetric traceless tensors. This has 5 independent components and this has 3 independent components again and this is just 1. So this which has 9 components and the 9 has come out of 3 x 3, one vector has 3 components, another vector has 3 components.

And this 9 components can be resolved as exactly the way we do the CG coefficients, you can resolve it and write it out and you will find that the rank 0+rank 1 + rank 2. So what have I tried to show now is that at least with the vectors I have tried to show this given you an indication.

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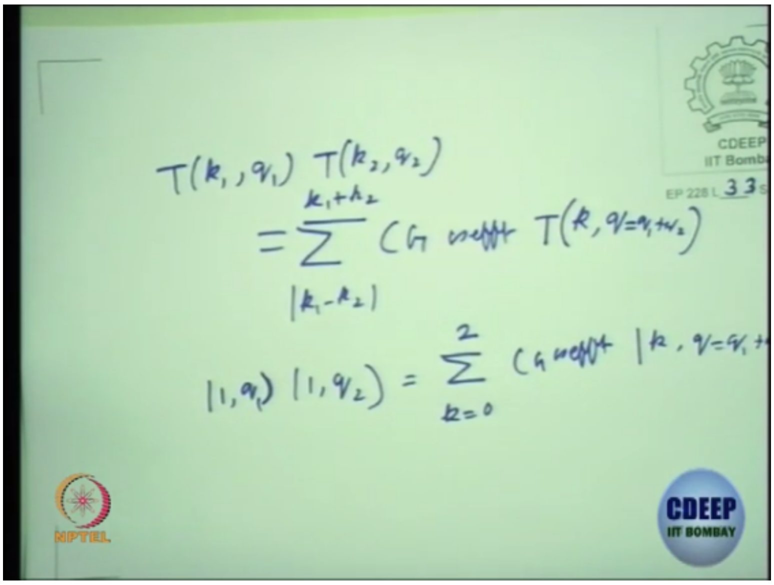

So what I am trying to say is if you take k1 q1, take k2 q2. You can write it as in terms of the CG coefficients with the k going from mod k1-k2 to k1+k2 CG coefficient I can write that also explicitly but you understand what I mean okay and this q is restricted, q has to be for a specific element. So whenever we take 1, q and 1, q1 and q2, you will get this as some linear combination of k going from 0 to 2 CG coefficient of this is what I am saying okay.

We did this elaborately for spin half, for spin one also you can do this elaborately and this is exactly the way this will also have and we can verify and I will give you an example and show this example clearly. This I have not done it now but we will see it now okay. Is the spirit clear how things are going?

So this is the way I resolved seeing how the rotation properties are and each one is an irreducible tensor and you can see that there is a lot of resemblance by taking tensor product of states which will give you an uncoupled state which is nine-dimensional and you can resolve it in the coupled basis which will allow one-dimensional, three-dimensional and five-dimensional, 2j+1 is the dimension of j, clear?