Quantum Mechanics
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Lecture - 68
Tensor Operators & Wigner-Eckart Theorem - I

So today I am just going to continue. I gave you a brief flavor of how to construct tensors familiar ones are your scalars and vectors to you, but you can also start constructing higher indexed object by taking products of these vectors is what I kind of gave you a flavor, right. So today what I am going to do is, I am going to try and formalize and tell you about tensor operators, okay.

Even though it looks abstract, if you remember scalar and vector and how I constructed moment of inertia tensor for you, you will know how though higher tensors can be constructed. Is that clear? Quadrupole moment is called quadrupole moment tensor, right. You have done the multipole expansion. Dipole moment is a vector, then quadrupole moment, then you have these higher moments, these are all tensors, okay, just to give you some more examples.

So what I want to do is we have talked about selection rules, which transition is possible under parity and so on. So this is formally for rotationally invariant systems. You can put it in the form of a theorem, which is called Wigner-Eckart theorem, okay. So these 2 are the themes for today's lecture and let us see how much I can progress okay. Okay, just to recap what we did for the states.

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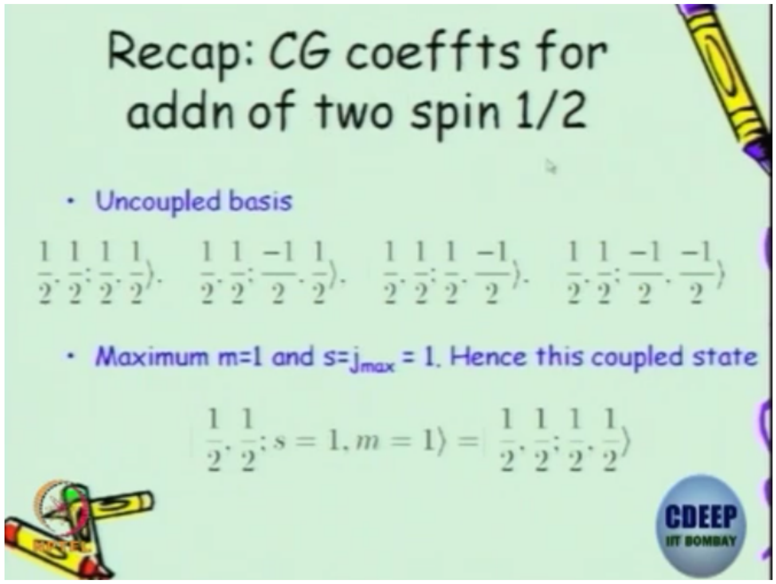

In the context of states, if you take 2 spin half particles, you can either look at it as an uncoupled basis. This is j1, j2, m1, m2 the notation. Now I am going to blindly follow this notation and there is a couple state, which is equivalent to again in j1, j2, but you can call it as j or s. Because I am looking at spins, I put it as s=1, m=1 and this is what is this stretch state.

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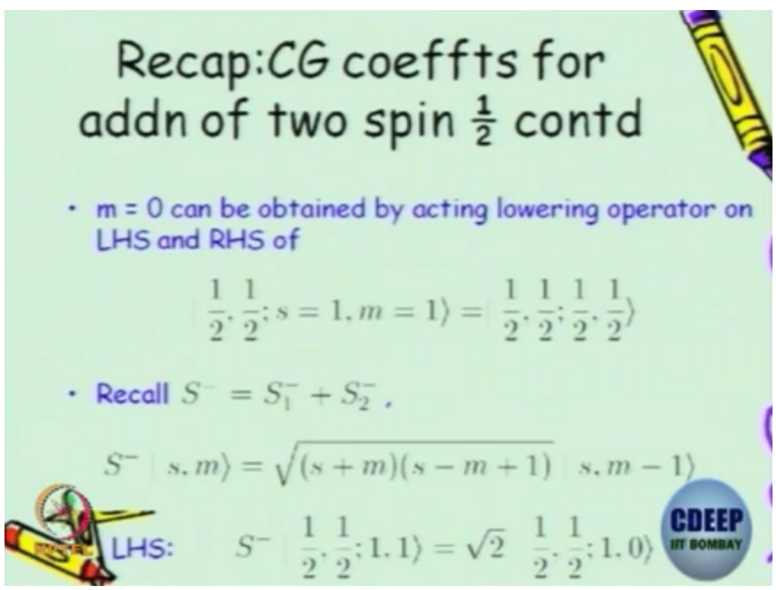

And then on the stretch state, on the stretch state, we operated ladder operators. On the left hand side is the couple state, right hand side is an uncoupled state and then we did this s minus operator on the couple state. It only looks like these 2 and the right hand side s1 minus will operate on the m1 and reduce it to m1-1 and s2 minus will operate on m2 and reduce it to m2-1, right. So we can do this and we did this elaborately and we found that the couple stated.

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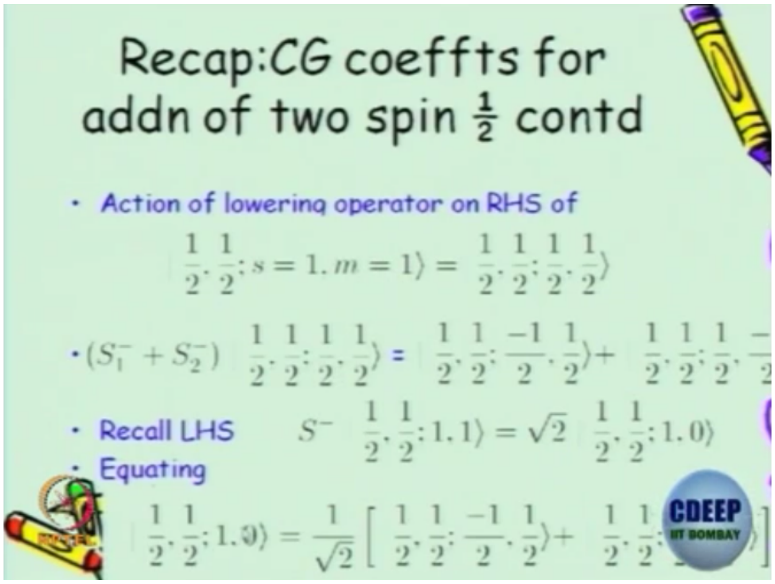

This is the stretch state and then we get the coupled state with one lower of this magnetic quantum number, which is 1 0 in terms of the uncoupled basis states and these coefficients are what we call it as a Clebsch–Gordan coefficients and we could write the Clebsch–Gordan matrix. I would like you to do the spin 1 and spin 1/2 the Clebsch–Gordan coefficients okay. Take one of the particle to be spin 1, other the particle to be spin 1/2, j=1 and j=1/2.

Compose, write the stretch state, do the ladder operation, find all the CG coefficient and write the matrix completely.

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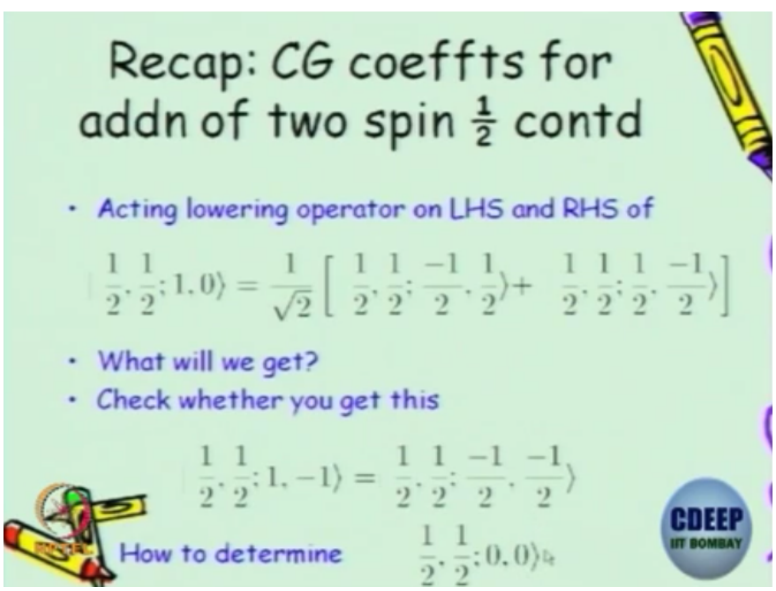

Yeah and then we also went on to say that how do we find there is also one more state, which we have to worry. It is 1-1, again you can get by the ladder operation, lowering operation, you can get this and you can show that it is exactly equal to this with coefficient 1 and to determine the coupled state with total angular momentum 0 and identity quantum number 0, how do we do that? This state has to be orthogonal to the state and using that you can fix that there is this coefficient with a relative sign and that is what we did.

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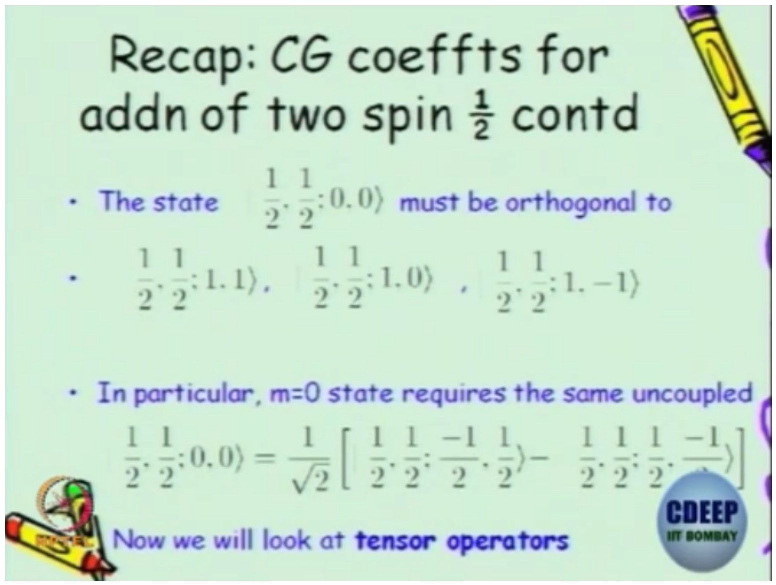

So these 3 states correspond to spin j in the coupled state and then m=0 is the only state and that state is a linear combination of these 2 with the negatives. So redoing this, this is the very simplest example, but I want you to do it for taking this to be 1 and this to be 1/2. I want you to do that okay. So now we will take a look at tensor operators, where there is a lot of correspondence between the states and the tensor operators okay.

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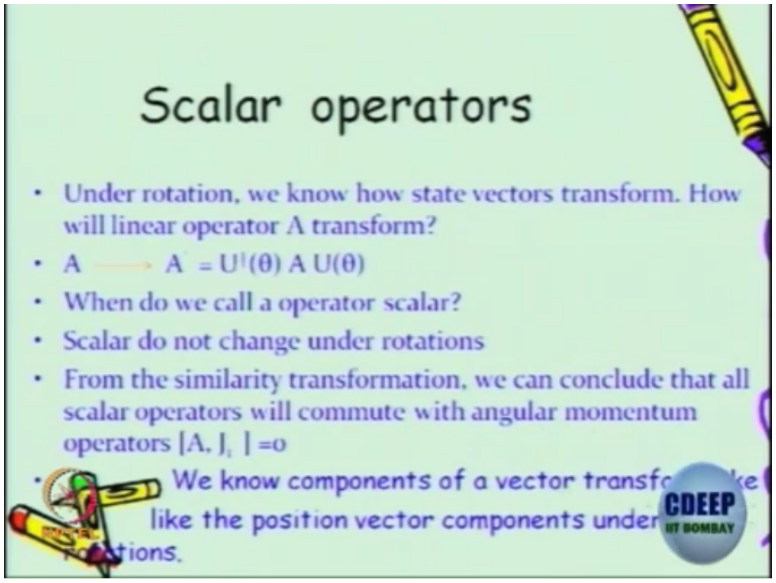

So this is something which we already discussed in the last lecture. How do we define scalars in classical mechanics gets promoted to? What are scalar operators in quantum mechanics that operators under any transformation will have the similarity transformation and you know the operator which does the rotation, which is a unitary operator which is dependent all the theta angle and in doing this transformation, if it is a scalar operator should remain the same.

A prime should be same as A, which means if you expand the u of, if you expand this u of theta as e to the power of ij dot theta, you can show that scalar operators will commute with all the components of angular moment and you have verified this for r dot r or A dot B and it is nice to know that this is the formal way, in which you can give a proof that scalar operators will commute with angular moment. Any questions on this?

This step I probably indicated in the last lecture, but you can do it. So now what happens to the vector operators. Vectors in classical mechanics transforms like the components of a position vector, any vector and the corresponding vector operator should transform like the position operator.

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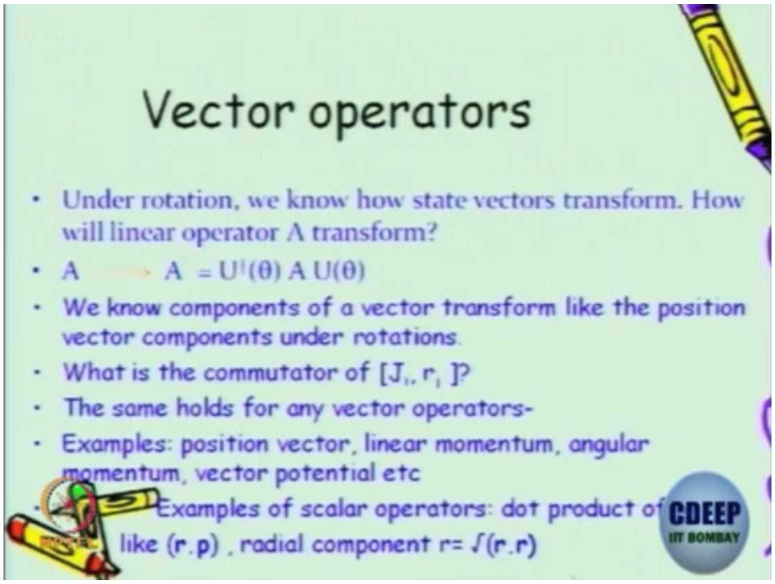

So if you try to put that in, then you have how the A prime will transform. You know how the A prime or a position vector transform, this we have elaborated. So it still from there you can deduce how the commutator or the generator of rotation which is angular momentum and the position components, how do they behave? Whatever is this behavior, it is the same behavior for any vector including angular momentum.

The angular momentum is also a vector, but angular momentum is an axial vector, that to see that you have to do a parity transformation and the proper rotation, there is no distinction between a polar vector and an axial vector. Only when you do parity things will start showing differences, right. Is that clear to all of you? In classical mechanics, I am sure you would have done this, know.

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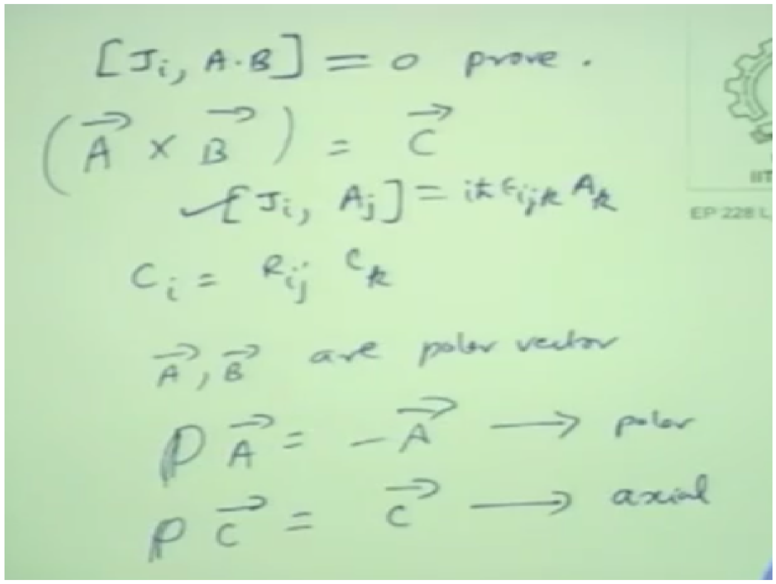

So if you have A x B under the rotation, call this to be a new vector, under rotation Ci will be Rij Ck, where Rij is the rotation matrix in classical mechanics right. So this is also a vector under proper rotations. The only difference between A, B and C is that A and B are polar vector, under parity, you call parity on A will give you –A, but parity on C will be C itself. It does not change sign. So this is why, this is called polar.

This is called axial and all C as well as A or B, how it transforms under rotation is exactly the way the position components transform, that is no difference okay. So that will tell you how to find the commutator of ji with Aj. So this will be ih x epsilon ijk and it will be the Ak, okay. This I wanted you to check. I did not do this for you, but I gave you an indication that you know how to do it for R and instead of R, you replace it by A and you can do this.

So what is this commutator and the same holds for any vector operators. So for example position vector, linear momentum, angular momentum, vector potentials are all examples of vectors. They are also vector operators in quantum mechanics. Scalar operator you can construct out of vectors by taking a dot product, right. If you take a dot product of 2 vectors, you get a scalar which does not change under rotation. So this is also something which I want you to check.

So you have to show that ji with A dot B, prove this. With Ai, you know what is the transformation, with the components of the A vector. With the components of the B vector, you know what is the transformation and you know A dot B behaves like a scalar. So you should be able to show this that ji with A dot B is, what is it, zero or non zero? Zero right. It has to be zero. So please prove this also using this.

This way you will get a handle on how to play with commutative bracket and you know the answer, you are verifying the answer, R dot P is a scalar. So R dot P has to commute with j, can also do R dot B with R x P also, but instead I am just asking you to do how any vector operator transforms with respect to the angular momentum you know that algebra. So why do not you use that and see why A dot B commutes with ji.

It is not obvious right, but there will be product of 2 epsilons, and you can use the, you have done the algebra of epsilon, right. Levi-Civita products use those and you will be able to show this. Do a little bit of algebra and you will be able to show it to be 0. So scalar products examples radial component, which is square root of R dot R, R dot T, A dot P, A dot B, all kinds of products. They are all scalar okay.

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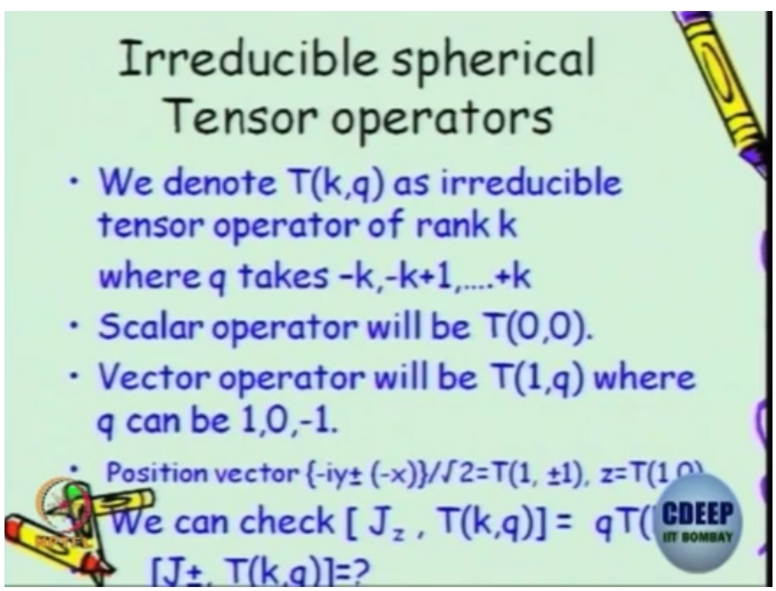

So now I am slowly getting on to you on a notation and then I will give you some examples. That way you will understand what is going on. We are going to call a tensor operator to be a tensor operator of rank k, okay. So take this as a definition. So we write T in bracket we have 2 integers or half odd integers. Integers, because operators are always going to be integers. So k is always a positive integer that gives you the rank.

And q just like your angular momentum runs from –j to +j, q is restricted to take values from –k to +k. All operators was integer k which I will call them as a tensor operator of rank k and the Q will take possibilities from k to -k in steps of 1. For example, the familiar scalar operator can be called as the tensor of rank 0. Once it is ranked 0, the q will also be trivially 0. So it is 1 component. It is just 1.

If I take a vector operator, vector operator will be called as a rank 1 operator and q will take values -1, +1 and 0, is that right? According to that definition, which I have given in the first bullet. Now we will ask, suppose you take the position vector, you know that the components are x, y and z or to make contact with this formal notation, it is better to write as complex coordinates. We will see why we are doing this.

Z component is still T(1, 0) z component is T(1,0). The T1 with q+/-1, we will take I to be this and we will see why, why this is true okay. We will also see why we are doing this. I want you to do this for the position vector, at least check for the position vector where T(1, +/-1) is this T(1,0) is z. Please check what we get. So let us do it right away.

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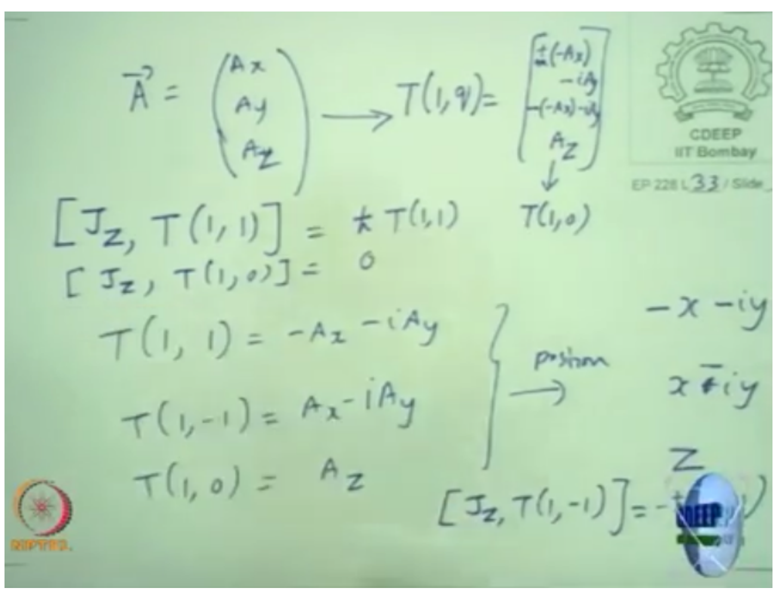

You can write A vector as Ax, Ay, Az, okay. You can also rewrite this as, you can write this as a tensor of rank 1,q with Az, which is just, which belong to T(1,0). And let us write this as +/- -minus Ax, -iAy, so this is + and other 1 ne is minus, so that becomes okay. So let me write T(1,1) for this vector A is -Ax –iAy, t(1,-1) is Ax-iAy and t(1,0) is Az for any vector including position vectors.

If you want to write it for position vectors, for position vectors, this will be x-iy, x+ iy and z. This is just a convention. We will see why this convention is C, then I would like to look at jz with t of. Something wrong, did I make a mistake? Yeah this is –i, thank you. So now do Tjz with 1, 1, what I get? So take it to be –x, -iy and jz and do the commutator. So T(1,1) with h cross, what about jz with T(1,0) that is 0 x 0, is that right? The jz with this is 0.

What about the last 1, -h cross 1 -1 okay? Is that right? So can we write this compactly? We have 3 data here. So can I write this as, tell me how do I write this compactly.

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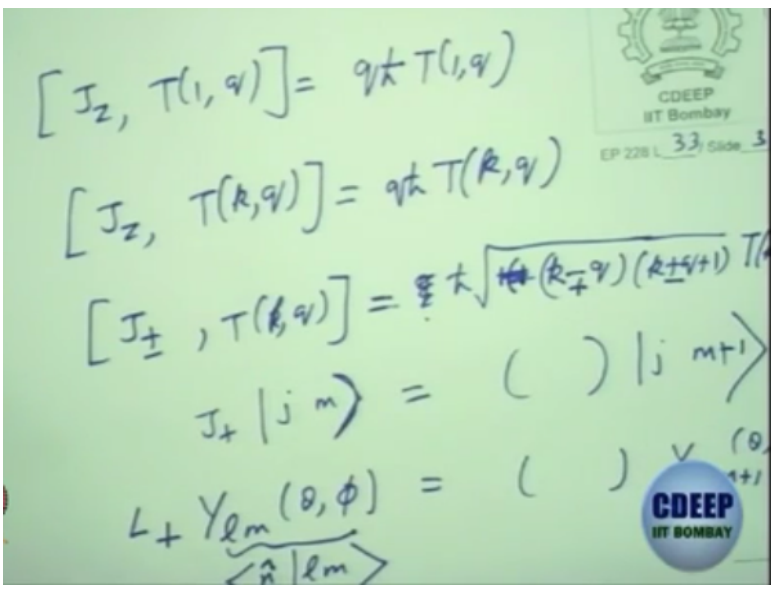

Jz T(1,q) will be qh cross, T(1,q). Anything wrong? You can do this. Now my claim is that this will hold even if I do for other tensors of rank A qh cross T(k,q). I have not proven, but it can be proved. Is that all? You need to do other things also. What are the things, j +/- T(1, q), what will this be is the next question. Why am I doing this, you know? The reason is that we will see your J plus on your Jm, jm state. This was giving you some coefficient on jm-1, sorry m+1.

I want to see whether the tensor operators also has a similar property, since you know the positive components, can you please verify what you get here and can you check whether we can write this as informally if it is k, I would have put k, k is 1. So let me write this as 1*1+, no this is J plus k+q or k-q, k-q, minus plus anyway I am very bad at all these signs, but you can help me out, which is plus and minus. This is correct, on H cross T of.

If this is K, then my claim is it will be +/-1, q+/-. So this is my guess. The guess is done from thinking that there is a one-to-one correspondence between states and operators. If that is the case, then my operators commutator algebra with the components of your angular momentum, they have to have these properties. Incidentally, you should go and check what is these things. Formally you know the differential operator for L+, right.

You can write a differential operator for L+ and you know what is Ylm of theta phi, take L=1, do the L+ and check what you get. What is the expected? You can write the Ylm theta phi as some unit vector whose direction is given and Lm and you can operate L+. This unit vector gives you the direction which will give you the theta and phi. In the angular momentum algebra L+1, Lm you all know, but I am asking you to write the differential operator position basis.

And operate it on Ylm of theta phi, which you have done in your first course or as the wave function formalism. So both the way should match right. So you can check that this one will give you Ylm, m+1 theta phi, but what the coefficient I am not writing, but you can check it out. It will be exactly same as this. There is no change, but you can verify it for at least for L-1 and L=2 you can play your hands on, seeing that the spherical harmonics are also called spherical tensors.

I am sure you all have heard that they are spherical tensors, right. So you can take these spherical tensors here and look at the commentators. So these are based of trying to show that there are lot of resemblance to the angular momentum states to your spherical harmonics and spherical harmonics for coming out naturally in writing your orthogonal basis functions, right. Need not look at writing all possible polynomials but some linear combination, which gave you the nice.

And this is what comes in your multipole expansions also. Is that correct? Dipole moment will involve spherical harmonics of L=1. Then the quadrupole moment will involves spherical harmonics of L=2. You can do the multipole expansion. I am just trying to connect the various ways in which you have seen this. There is an expansion which has this neat way of breaking into L=0, which is a scalar, L=1, which is a vector, L=2 which is the spherical tensor of rank 2 and so on, okay.

Ok this I am leaving it to you as a practicing to check how the tensors formally, do it for k=1 and then generalize it. I am not asking you to prove in general for any case, but it can also be done, but right now you do it for k=1. This we have already verified, just I want you to do. I have given you the answer. Please verify that that happens for k=1.