Quantum Mechanics
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Lecture – 65
Tutorial 11 - Part I

We are now to the end phase of our tutorials and this tutorial, remaining tutorials will be more focused on Clebsch-Gordan coefficient and Wigner-Eckart theorem. So there will be again the set of 5 problems and this tutorial I will just give you an outline and you can just fill in the gaps and solve each and every steps because Clebsch-Gordan coefficient has been discussed in the class.

And Wigner-Eckart theorem will be in the following lectures you will see how Wigner-Eckart theorem can be used and these 2 tutorials will actually give you a flavor of how powerful is the raising and lowering operator and how these can be used to obtain the Clebsch-Gordan coefficient and let us now start with the first tutorial, first problem of this tutorial.

And as you have already seen in your regular class that Clebsch-Gordan coefficient for a system of 2 particles of spin of angular momentum 1, 1/2 and 1/2, how it can be obtained. So for 1/2 and 1/2 you had 4 cross 4 matrix that you have to obtain and here what we are going to do is.

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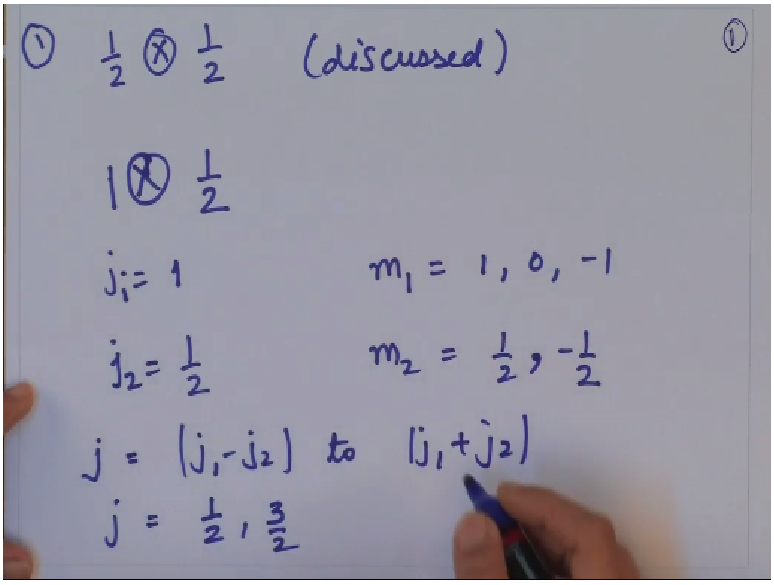

So this 1/2 is discussed in the class right and on the similar line actually you can calculate the first problem that is you have to calculate 1 cross 1/2, when j1 = 1 and j2 = 1/2. So when j1 = 1, m can take values, m1 can take values 1, 0 and -1 and m2 can take values 1/2 and -1/2 okay. And you will see how easy it is to calculate these Clebsch-Gordan coefficient, once you get the idea how to proceed then some of the terms you can just evaluate by orthogonality condition.

So we start with j1 is 1, j2 is 1/2 then capital J or j takes values from j2-j2 to j1 +j2. So in this case our j can take value 1/2 and 3/2 agreed? and what will be the value of capital M? capital M will also take value from m1 – m2 to m1 + m2 okay. So here you can see that you are writing this coefficient in terms of a coupled and uncoupled basis. So when you are calculating let us write the uncoupled basis first.

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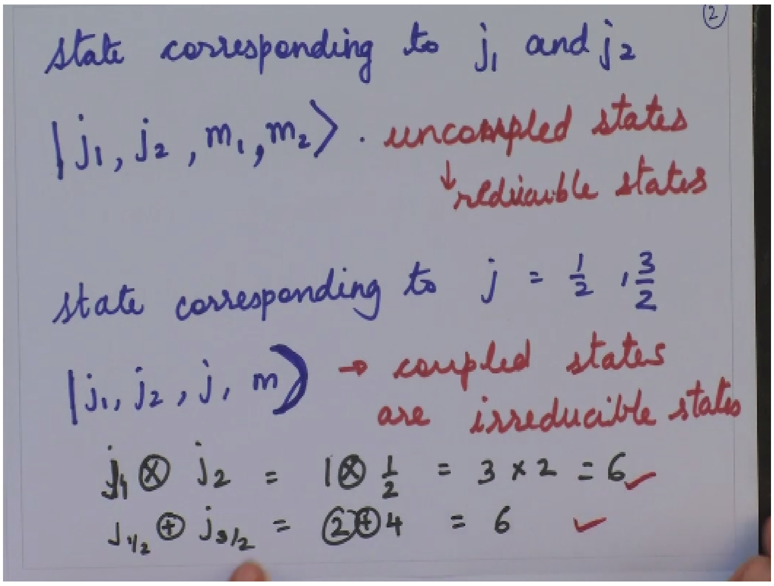

So the state of the system, state corresponding to j1 and j2 that is j1 as 1 and j2 as 1/2, what do we obtain? How do we write the state of the system? j1, j2, m1, m2 and please note that there are various ways of representing this states. Sometimes you see people write it as j1 m1, j2 m2, the order is not so important, but you need to remember whatever you are writing whatever form or whatever order you are writing you have to stick to that order.

So j1 j2 m1 m2 is the uncoupled state, so this is an uncouple state or the uncoupled basis and the uncoupled basis is a reducible state or it can be called as a reducible state and the state corresponding to j okay can be written as that is j is 1/2 and 3/2 can be written as j, j1, j2, j,

m. So in your lecture you have seen that it is denoted by a curved bracket okay. So this is the, or the angular bracket, so this is a couple basis or the couple state.

And these couple states or coupled basis are irreducible basis or irreducible states okay and before we go ahead let us check the dimensionality of these 2 bases, are they dimensionally correct or equal okay. So here I can write for j1 =, so j1 cross, let me write 1 or j1 cross j2 is nothing but 1 cross 1/2, this is what we are evaluating. So for 1 cross 1/2 what will be the dimension for j1 = 1, the dimension will be 3 and for j2 = 1/2, the dimension will be 2.

This is nothing but 6, similarly I will find out the dimensionality for j when j is 1/2 and when j is 3/2. So when j is 1/2 dimensionality of j will be 2 and when j is 3/2 dimensionality is 4. So this is nothing but, sorry so 2+4 that is 6. So dimensionally you can see that the 2 dimensions are matching.

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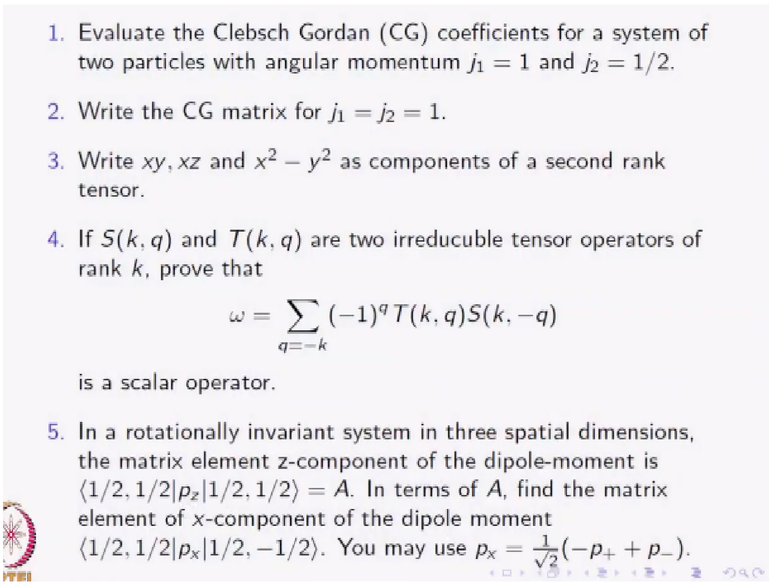

So in general if I want to write the dimensionality of the coupled and the uncoupled basis.

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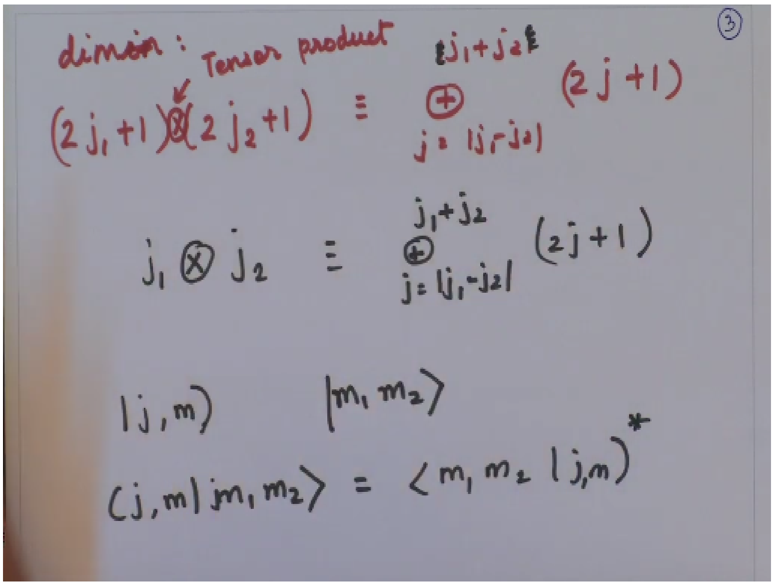

One can equivalently write the dimensionality or dimension is 2ji + 1 * 2j2 +1, so this will be a product okay, but this is a tensorial product remember, it is not a direct product, it is a tensor product. So the tensor product is equivalent to this is the sum, the direct sum from j = j1-j2 to j1+j2 which is 2j +1, this is what we have done in this example, j1 cross j2 = j. So I can rewrite actually this in the form.

Here what I mean is j1 j2 is equivalent to direct sum of j1, j1 going of j, j1-j2 to j1 + j2 sorry this will not be there because you do not need a mod when you are taking the sum only when the difference. So 2j + 1, this you need to remember, this is the first step in which you will understand actually how you can compare the reducible and irreducible states. So you should have the idea how these dimensionally match okay.

So the uncoupled basis are reducible basis or the reducible states and the coupled basis are irreducible states. So with this now you need to go a step ahead okay before we go to the table or what you call as Clebsch-Gordan coefficient or CG coefficient table you prepare a table and write the coefficient that is more convenient to understand and look and interpret also.

So before we go to the table let me tell you some places j1 and j2 in the state is suppressed and we write it as j, m for the coupled state and m1, m2 for the uncoupled state, this is how these states are represented. So when I write jm m1 m2 and if I take the reverse of this that is I do this the other way, this will be nothing but the complex conjugate of this, but if your coefficients are real there are be no change.

So they are equal, the complex conjugate as well as this number what you obtain is the same. So you will see how these table is prepared and how one can actually write in a beautifully in the block diagonal form. So let us go to the table part. So in this table part to obtain.

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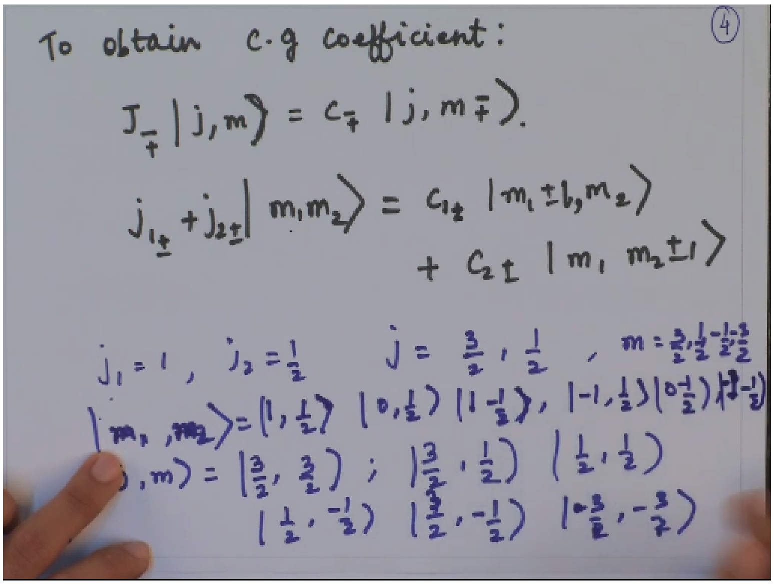

This is the point you should remember, to obtain the CG coefficients what we will do is we will require or we will use ladder operator. So you just recollect J – or J + when operated on a state, say j,m it will give you C – or + that is this is the CG coefficient that you will obtain and now you know actually how to obtain these CG coefficient j, m – or + okay. Similarly, if the ladder operator is operated on m1 m2 so when I do this j1-+ j2 – m1 m2.

So once it will operate on m1 and the other time on m2. So I will have C1 +/-, so it will be same for + and – component, this we will be using and I will show you one example and then the remaining you can fill up the gaps. So c1 will give me m1+/-, m1, m2 + c2 +/- m1, m2 +/-. So the state will be either lowered or it will be raised by a factor of 1 depending upon which operator you are operating.

J plus will raise the state by 1 and J - will lower the state that is m/-1. So with this now, I will make a table, so we will do some working on this page, but when I prepare this table what I can do here is I will have horizontal okay.

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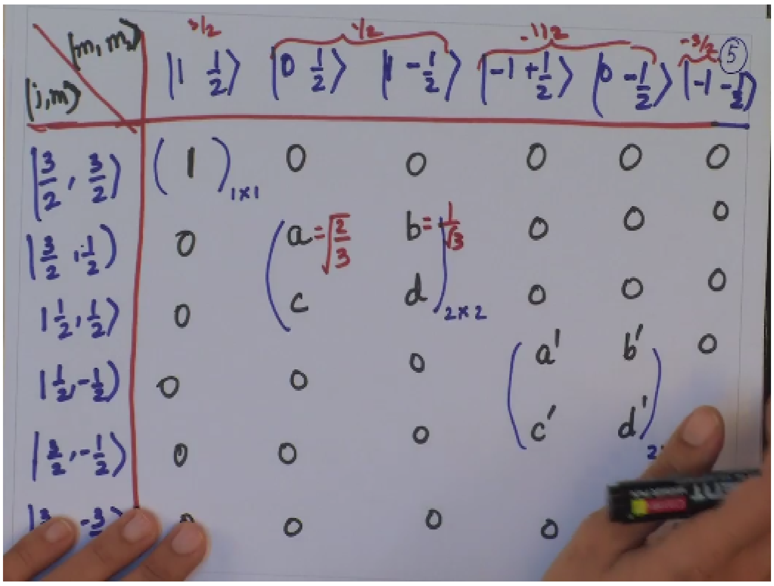

So here what I am doing is I will write the CG coefficients so this will be m1 m2, so these are the reducible states and j, m are the irreducible states okay and corresponding to these states we will have to write the corresponding coefficients. So as we have seen in the previous like j1 is 1 and j2 is 1/2. So j can take value 1/2 and 3/2 or 3/2 and 1/2 whatever. Similarly, m1, m2, if I want to write them in terms of the states.

And M can take values here I will have, m can take values, 3/2 1/2-1/2 and -3/2 so 3/2 will have 2 states 3/2 and -3/2 1/2 will give me 2 states 1 and -1/2. Similarly, for m1, m2 you can have different states. So this let me write this for the uncoupled state, I will have, this I can write it as 3/2, 3/2. So what we will do is we will write these states in the decreasing order of m. So 3/2, 3/2 will be the first term in the column and the next one will be 3/2 1/2 then comes 1/2 1/2 then comes 1/2-1/2 then 1/2 3/2 and 1/2-1/2 and 3/2 and -3/2.

So decreasing order of m, so 3/2 1/2 1/2 -1/2 -1/2 -3/2, so you can have 6 such combination. Similarly, for J1 and J2 I mean m1 and m2 we will have combinations. so m1, m2, let me rewrite it here only. So m1 and m2 I will have m1 can take value, 1, 0 and -1, m2 can take value 1 and -1/2. So this will be 1, 1/2, this is the highest state which can be achieved okay. The second will be 0 and 1/2 then third will be 1 and -1/2.

Then fourth you can have -1 and 1/2 or you can have 0 and -1/2 or you can have -3/2, sorry -1 and -1/2, so that will give me -3/2. So whatever I have written here it looks very clumsy. So when I write it in the form of table it is very clear, very neat and very much understandable, that is the reason why I showed her that when you write the state it is very difficult to evaluate this way, but now when you look this and rewrite in the form of table.

So here I write first state highest state then I have 1 0 and 1/2 then I have, so I will write in the order of decreasing m, 1 and -1/2. So there are various ways in which you can group these states. So here my intention is to group the states in terms of decreasing order of m okay and you will in a minute you will understand why is this choice being taken. So this is 3/2, 3/2, this is the highest state.

Then I have 3/2 1/2, then I will have 1/2 1/2, then I will have 1/2-1/2 and then I will have 3/2-1/2, then I will have 3/2-3/2. So this is the lowest possible state, this is the highest possible state. Lowest possible state I can have maximum m value as 3/2, and here you can see, the maximum m value that is m1 + m2 is 3/2. Here the maximum m value that is 1/2 and this will also give me 1/2.

So here m is 3/2, here m is 1/2, here m is – so this will give me, I have a + here because otherwise I will not get a -1/2, these are repeated. So here this will give me -1/2 because -1 +1/2 will give me -1/2, this will be also -1/2 and the last one is the lower most state which is -3/2 okay. So now next step to proceed would be that you will have first term okay. So there is only one way or the answer to this we know is 1.

So CG coefficient of this would be 1 because here capital M or m is m1 + m2 and this is the only possible state and same holds for this last term, the lowest possible state. So these 2 corners you are done and the corresponding terms here it is not possible to have this 3/2 combination anywhere. So you will have 0 0 0 0 same way I cannot have 3/2 in these. So I will have a 0 0 0 0 and 0.

Now corresponding to this 1/2 state there are 2 possibilities. So let me call this as a and this as B and it is not possible to have these. Similarly, I can have 1/2 with these 2 states. So let me call this as c, this is as d. So the remaining would be 0. Again here there is a -1/2, so these are all 0 here also these will be 0, these will all be 0s, but here -1/2, there are 2 possibilities. So I can have a a prime b prime, here I can have a c prime d prime.

So this is the table we have prepared using the ladder operator and orthonormality condition you can evaluate these boxes that is this matrices or the elements of these matrices, that is a, b, c, d, a prime, b prime, c prime and d prime. We have seen it for 1/2 cross 1/2. Same way we are going to do it in this table. We are going to fill it, but before we go to anything let us see here that this is a 1 cross 1.

Very neatly you can see the table can be actually compactified as well as spread out as this is 2 cross 2, so you have a block diagonal form 1 cross 1, 2 cross 2 here you have again 2 cross 2 and here you have 1 cross 1. So this is a matrix 1 + 2 + 2 +1 so it is a 6 by 6 matrix and now you understand that you can first write the table and then evaluate these terms. So coming back to the evaluation part.

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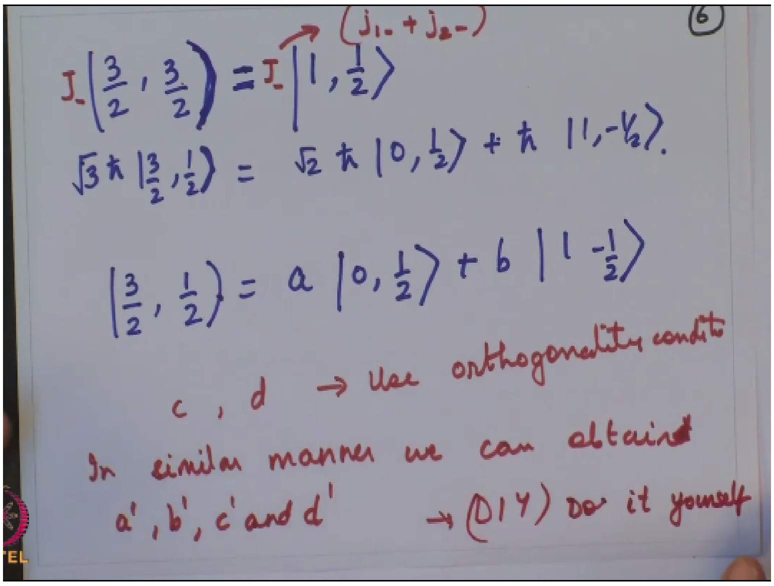

First term we have seen here I have written down that the first term that is 3/2, 3/2 is this is the uncoupled basis, the coupled basis and you can write this in terms of the uncoupled basis correct. So this is what we have, this is 1, from the table I can write this, agreed, correct. Now in order to write this state 3 1/2 I can write it as 3 and 1/2, 3/2 and 1/2 as a times this + b times this term, agreed?

So let me write it. So 3/2, 1/2 is written as a times 0 1/2 + b times 1 -1/2 agreed. Now in order to come from this state to the state what we will do here is we apply a ladder operator, do a ladder operation. So when you apply a ladder operator on J - that is J - on this you will obtain the CG coefficient c - and this state will be lowered by that is m will be m-1. When you are applying your J – here you J- here is you will have to apply j1-/+ j2-.

This is what you are going to do in this step. So when you do this, this term is very easy to check okay. This is very easy to check because you can easily calculate the coefficient of this, this will come out to be 3 h cross 3/2 and 1/2 okay and similarly you need to calculate this part of the equation and this comes out to be 0 1/2 + h cross 1-1/2, alright. I am going really very slow so that you are not stuck anywhere.

And then later you can actually do things much quickly, in a much smarter way actually. So from this actually you can read the value of a and b. So when you take this 1/2 here and cancel the h cross what you obtain is the value of a. So let me write the value of a for you, so it comes out to be 2/3 in the square root of 3 2/3 and this comes out to be 1/root 3. Now using orthogonality condition, you can obtain the values of c and d.

So use orthogonality condition, it is very simple to check and why we are using orthogonality condition? when you operate this on any of the other quantities 1 or -1/2 or 0 and 1/2, you will obtain the value of c and d, because again you will or you can put this 3/2 1/2 and 1/2 and 1/2, when you do the, use the orthogonality condition and use these 2 equations you will be able to evaluate c and d.

So I will leave it to you all to do this part to evaluate c and d. In similar manner we will have we can obtain a prime b prime c prime and d prime. So these are all DIYs, do-it-yourself problem okay. So this I think you will be able to manage this part of the problem. Now again you may be bored, but it is very important to go to the next problem because in the next problem.

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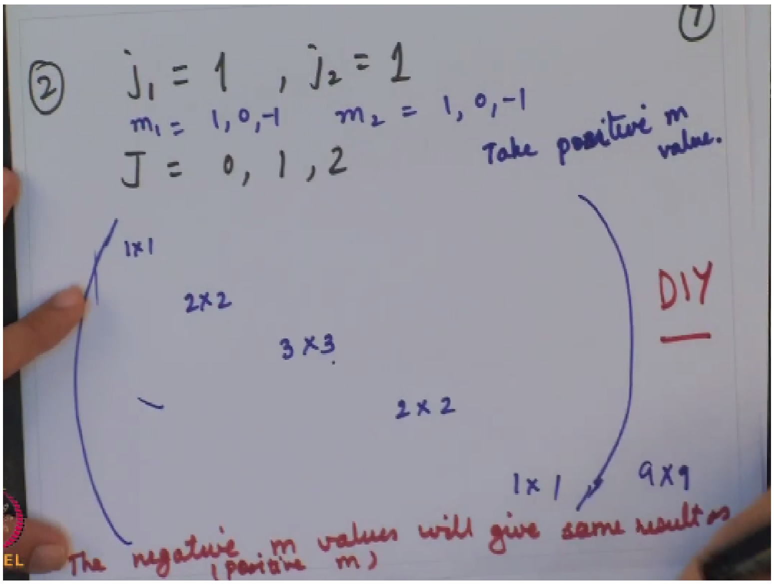

j1 is 1 and j2 is 1, so here the dimensionality has increased that is you are going to a bigger and bigger matrix once you are comfortable with 4 cross 4 you can do this 6 cross 6 then you are going a step further where in you have a 9 cross 9 matrix. So when j1 =1, j2 =1 your capital J can take value 0 1 and 2, okay and here again you will see how your m1, m2. So your m1, will take value 1 0 -1, m2 will take value 1 0 -1 and so on.

So this will be again you can solve this very neatly and you can actually obtain a 9 cross 9 matrix. Again you can guess from the previous question that it will be of the form 1 cross 1, the block diagonal form will be 1 cross 1, 2 cross 2, 3 cross 3 and 1 cross 1. So 1 + 2 + 3 + 2 +1. So this will be a 9 cross 9 matrix and one more thing is important here to know that you can actually calculate this part of the matrix where you will take m positive m's only.

Take positive m values and the negative m values or the, will give same result as positive m values, remember this okay. So negative m values will give the same result as positive m values. So you can actually DIY this okay.