Quantum Mechanics
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Lecture – 64
Clebsch-Gordan Coefficients - II

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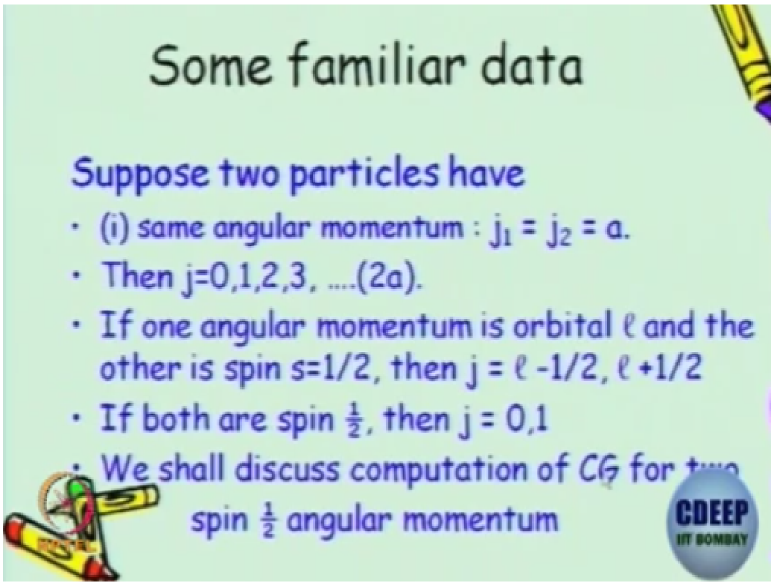

Okay, so some familiar data we just to corroborate what I said formally if suppose you take both the particles to have the same angular momentum like 2 spin 1/2 particles or 2 spin 1 particles let me call the angular momentum of particle 1 and particle 2 to be the total j, maximum will be 2a which is j1 + j2, minimum will be j1 - j2 which is 0 and it goes in steps of 1.

So this is the rang of the coupled basis angular momentum it is not a fixed value it goes from 0 to 2a. If one angular momentum is orbital and the other one is spin then what is the range, spin is just 1/2, spin 1/2, j1 + j2 is l+1/2, you have to go in steps of 1 to j1-j2 which is l-1/2. The difference between these 2 is only step of 1. So only 2 possibilities, the total angular momentum j when you couple angular momentum orbital angular momentum to the spin 1/2 there are 2 possibilities for the J.

If both are spin 1/2 by the same argument if you put a to be 1/2 then it is 0 and 1. So now what we will do is for, let us take the simplest one which is 2 spin 1/2 angular momentum addition and look at what is the CG coefficient for 2 particles which are spin 1/2 which is coupled.

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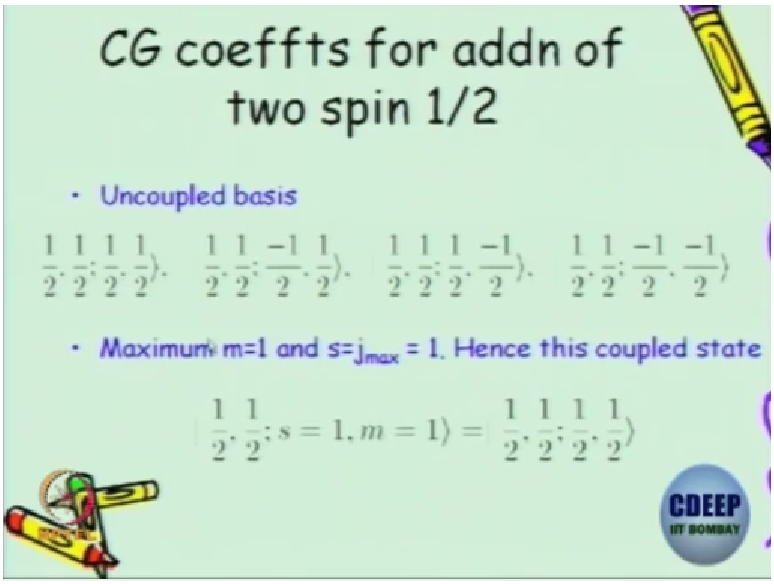

So just you recall uncoupled basis it is a 4-dimensional one spin 1/2 is 2-dimensional another spin 1/2 is 2 dimensional linear vector space, the tensor product of 2 spin 1/2 will give you a 4-dimensional vector space and we can write the 4-dimensional vector space all the basic states, what are the possibilities? m1 is 1/2, m2 is 1/2, m1 is -1/2, m2 is 1/2, m1 is +1/2, m2 is -1/2, what is the last one, both being -1/2. This is also important when you do quantum computers also.

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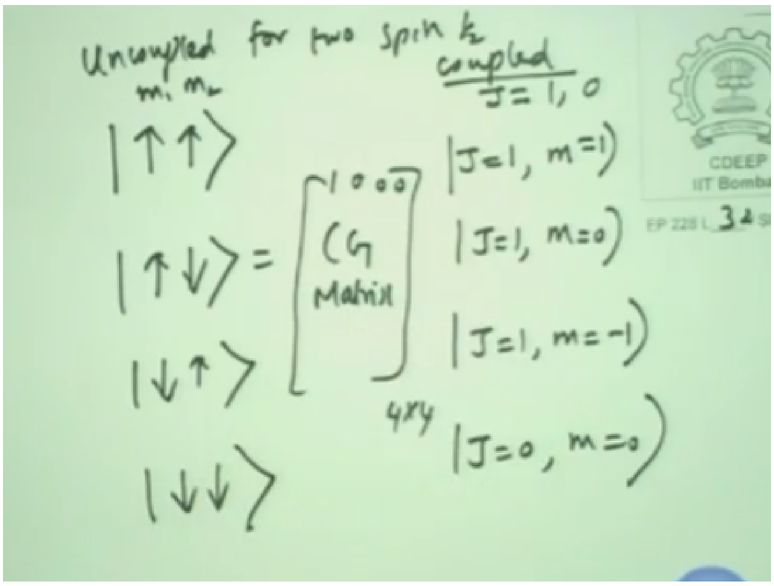

So let me just, so we can write, we can suppress the j1 and j2, write up up, up down, down up, and down down. This is the uncoupled basis. I am not going to put j1 and j2 which is 1/2

1/2, but we can just remember that for 2 spin half. I am only writing the m1 and m2, I am not writing the j1 and j2. What about coupled? Coupled, what are the possibilities of j? it is 2 spin 1/2, 1 and 0.

So couple basis will be j =1 and m = 1, j =1 m = 0, j =1, m=-1, anything else, j = 0 m = 0, interestingly this is what we expect also, the dimensionality of the uncoupled basis should be same as the dimensionality of the coupled basis, this is 4-dimensional, this is 4-dimensional. So this should be related by a matrix which is 4 cross 4 matrix. This is CG matrix. We are interested in finding the CG matrix.

The first element here is 1 0 0, you agree, this state is a stretched state, this is a stretched state, I said that the coefficient of this is 1, provided m1 + m2 is m =1 others are all 0 so these are all 0 in the 4 by 4 matrix. Now we need to work out the other 3 rows okay. So here just for explicit things I have written the 1/2 1/2 m1 and m2 the quantum numbers rather than putting the arrows, but you understand.

So these are the 4 states and the maximum state which we call it as a stretch state m =1 and s = 1 that is going to be exactly this, so that gives you the coefficient or the first row which is 1 0 0.

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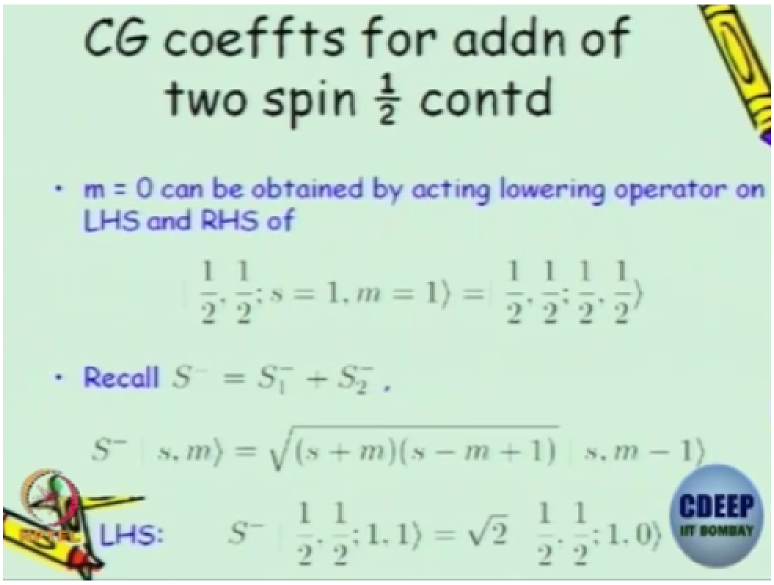

And the next thing is we need to do one step lower than m maximum, m maximum is 1 right. The next one is m = 0. So m = 0 can be obtained by acting lowering operator on both left hand side as well as right hand side. But the lowering operator here will be the lowering operator in the coupled basis and the lowering operator here will be the uncoupled lowering operator. So that is the very important thing which you should remember. So how do you do that can you try it. Left hand side is similar to any s, m, the lowering operator.

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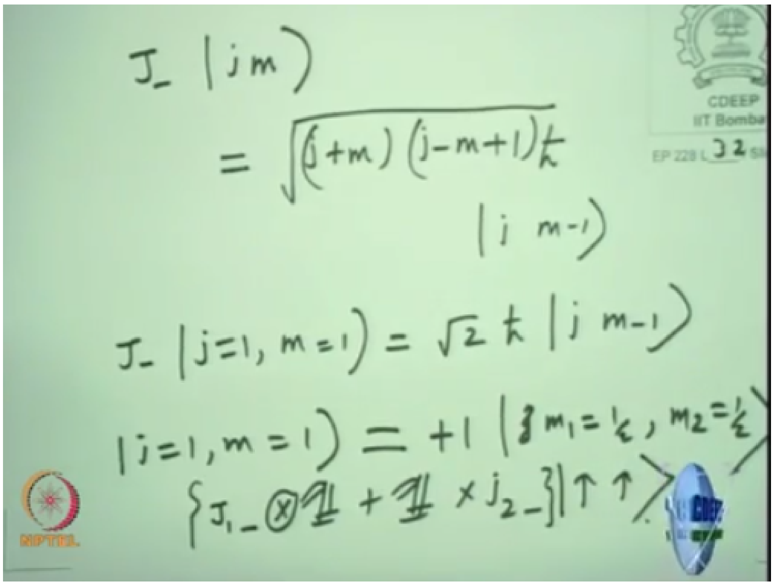

I am going to suppress the j1 and j2, I am writing only the coupled state which is jm, this will be tell me j+m * j-m +1 is that right, am I right or wrong. So J- on j=1 m=1 will be so this is the left hand side but I have already told you that j=1, m=1 is a stretched state with the +1 coefficient which is going to be m1 = 1/2, m2 = 1/2 right or both are up states. On the right hand side if I want to do the lowering operator I have to do it as if I am working on the uncoupled basis.

The lowering operator there will be j1- cross identity + identity cross j2 -. You have to do that on up up state.

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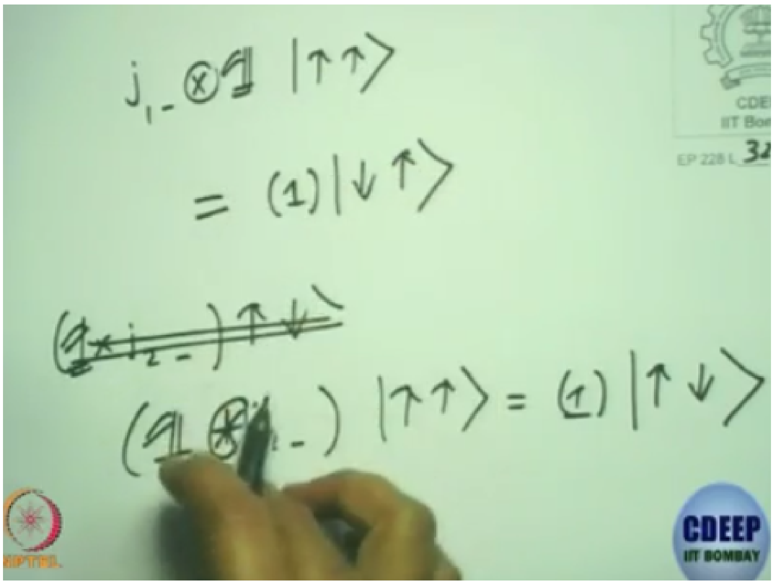

State will be, what is this coefficient? you should remember this 1 and 2 I am just putting to know it should operate on the first state and it should operate on the second state. As an operator it is just a lowering operator okay. What will this be? Some coefficient times, it is identity on the first state and the lowering operator on the second state, that is why I am putting this notation.

What will that be? coefficients are both are 1 because your j1 and m1 are 1/2 and 1/2, so 1/2 +1/2 will be 1, okay so this is 1 and this is 1, but you have to sum up both.

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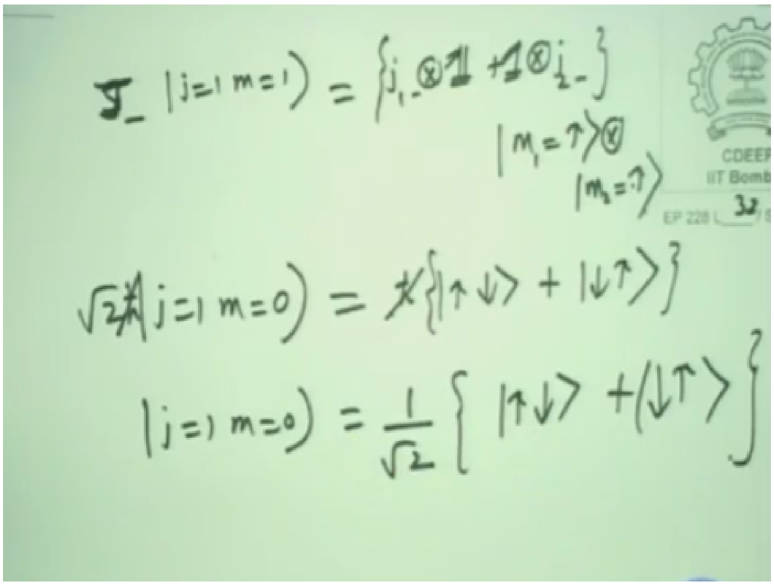

We had j = 1, m=1, we had an s- operator or j- operator. This has to be j1- cross identity + identity cross j2-, this has to operate on m1 = up spin and cross product with m2 = up spin, you all agree? I am writing in all possible notation so that you get familiar with everything.

Uncoupled basis is a tensor product of 2 spin. This is another way of writing. This we have already done.

This side is square root 2 j=1, m=0, this side h cross, thank you yeah, and this side is going to be up-down + down-up with a h cross. We did this separately, the first one we operated on the state and we got down up, the second one we operated on up down, up up which gave you up down, but it is a sum of these 2 operators, you have to take the sum of those 2 states. So what do we do?

We need to find out J =1 m =0 that state is 1/square root 2, take the square root 2 outside from here below, remove the h cross on both sides times up-down + down-up. So this side is a coupled state, coupled state can be written as a linear combination of the uncoupled basis and that coefficient should give me the CG coefficients of the CG matrix. So let us write the CG matrix again. You have to help me.

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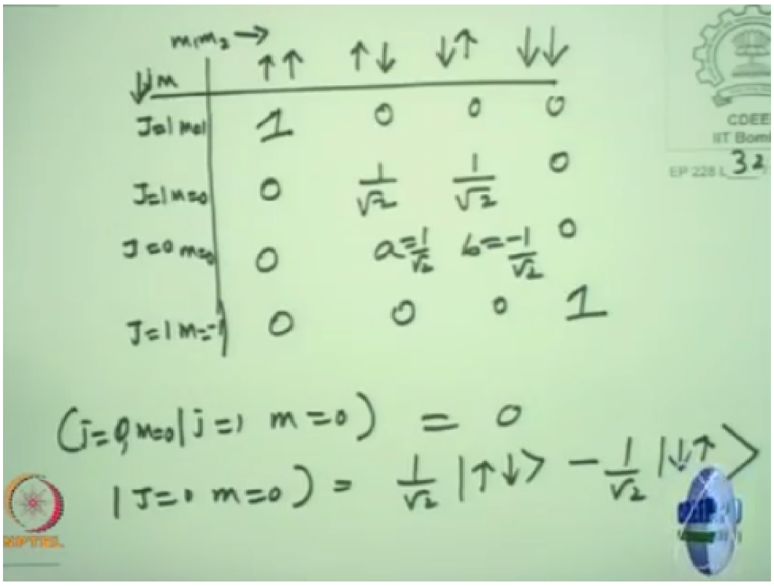

So I am going to put here m1 m2 this side and this side let me put j and m. So m1 m2 is up up, up down, down up, and down down, clear. So let us write the state as j=1, m=1, j=1, m=0, j=0, m=0 and tell you why I am doing this, j=1 m=-1, both the m=0 can happen for both j=1 and j=0, I am just putting it together. The j=1, m=1 contribution will only come from m1 +m2 + m, only this one at a stretched state.

So this element is 1, others are all 0, j=1 m=0 we have written down, if i/square root 2 times up down + down up, so which means this is 0, this is 0, this is 1/square root 2 and 1/square root 2. What about j=1 and m=-1? Which one will contribute? Will this contribute, will this contribute, will this contribute? Why it would not contribute? m1+m2 should add up to -1, which is the only possibility, this one.

So this is the lowest stretch state that also I can put it to be 1. What about j=0, m=0, I want m=0, I cannot get m=0 from both up right, this is 0, you all agree? This is also 0, but this I can fix it as a and b. How do I fix this a and b, if I call this to be a complete set of basis, I would like my J =1, m=0 to be orthogonal to j=0, m=0, so that is what will happen right. So that will fix us, there can be some phase factors and stuff like that or the sign factors.

You could fix one convention and write it this way, up to a sign these 2 are correct. So this way I have actually showed that j=0 m=0 is 1/square root up down – 1/square root 2 down up. The left hand side is a coupled state; right hand side is in the uncoupled basis. Only these 2 states will contribute to this state and this has to be orthogonal to the j=1, m=0 state, okay. So as an exercise we also check that j=1, m=-1 is down down right.

But you can also do independently just to make sure that in whichever way you do it, it should not give us the wrong result right.

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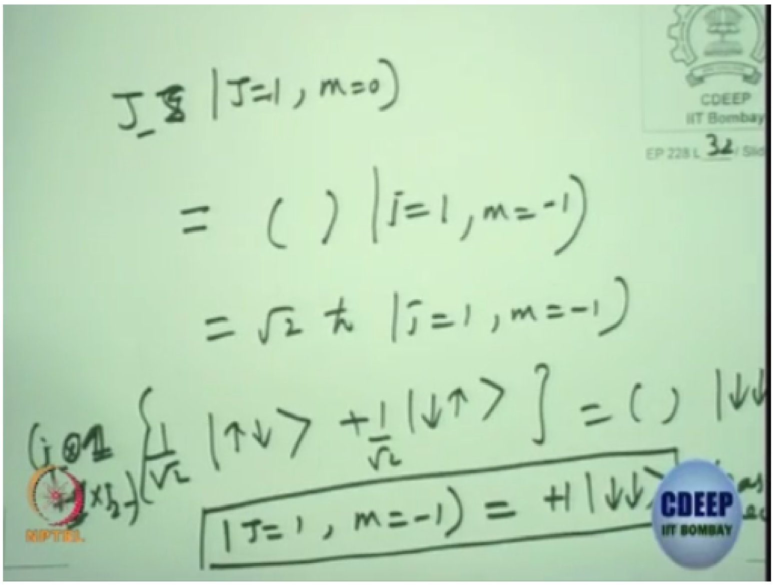

So you take j=1, m=0 and operate your s- or j- operator, what will that give you? Something coefficient j will not change, what will change? m will become -1 okay, what does that coefficient yeah, what is this number? root 2 h cross. So do the same thing j=1, m=0 is actually 1/square root up-down + down-up on this you operate your j1 – cross identity + identity cross j2 -.

Do this, what do you get? If you do a minus on the state it will make it down, so it will become down down state, but if you do a j2 – on this state it is 0, cannot go below, but if you do the j2- on this state it will make it down, j1- will not act on it, so it is going to be some coefficient times down down and then you compare and then what do we get? J =1, m = -1 is down down with +1 coefficient, please check this.

I argued by the lowest stretch state to be coefficient 1, you can also do it by the ladder operation and you can show this. I have not done the in between step but please do it. So now CG coefficient looks interesting, easy. So m =0 can be obtained by acting lowering operator on LHS and RHS. LHS is the stretched state, RHS is also the stretched state in uncoupled basis.

You do S- as S1-, I am just summarizing for completeness. You can do the S- on the coupled basis, lowers it to m-1 with this coefficient. Sorry I forgot the h cross, please plug in the h cross. So there will be a square root 2 times h cross here, we have done this on the;

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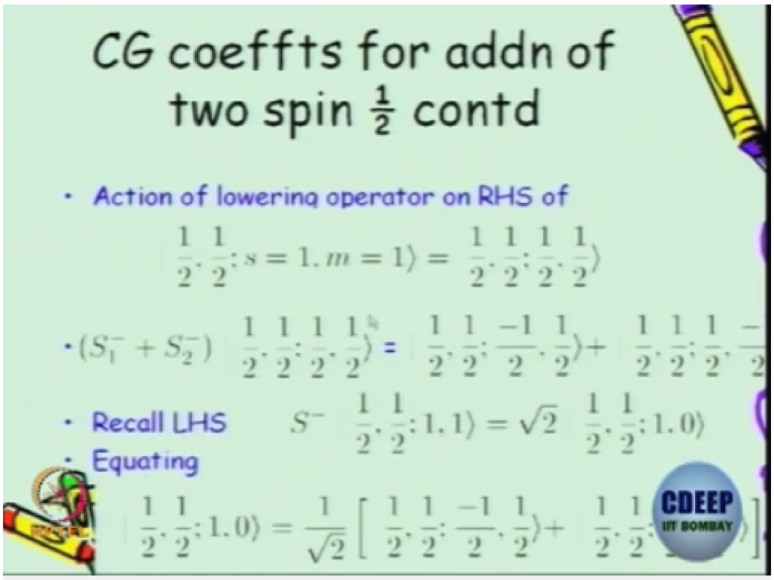

So and do it on the uncoupled basis, I am not putting the cross identity, identity cross, but you remember this. So this one will lower 1/2 to - 1/2 and this one will lower the second 1/2 to -1/2. So it is some of those 2 states and using this we have written the uncouple state with the magnetic quantum number as 0, can be contributed by m1 and m2 which are 2 possibilities and it is the sum of those 2 with the coefficient 1/square root, is it normalized?

It is normalized, so it becomes automatically normalized because we have taken care of that even in the construction okay, because the left-hand side couple states whenever I wrote those coefficient they are also normal.

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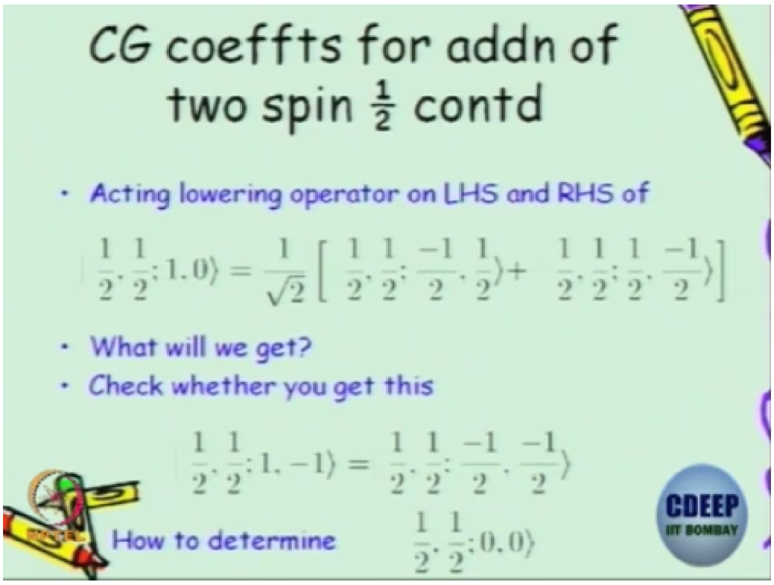

Acting lowering operator, you can still do the acting the lowering operator on LHS again and RHS which we have derived now and what will we get? We will get that the 1-1, is exactly with coefficient 1. How do we determine j=0, m=0 is the question, that also I said that j=0, m=0 has to be orthogonal to, must be orthogonal to all these states.

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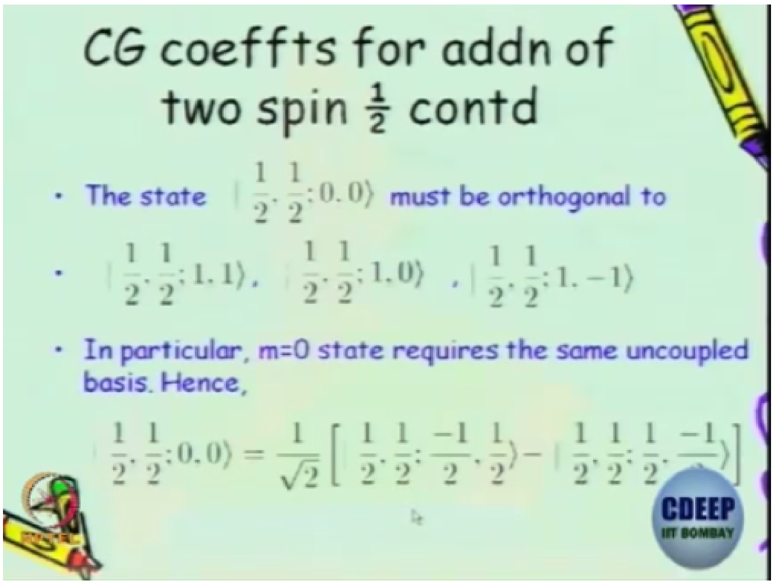

The only state which it has overlap is only with the 1 0 state we can check the orthogonality there, these things are trivially orthogonal because this one contribution with some 2 up spins, this contribution is from 2 down spins whereas to get a 0 you need 1 to be up and 1 to be down, so there is no way, they will all be orthogonal, if you need to only check that this is orthogonal to the different j, but the same magnetic quantum, which is what we did for this case okay.

Okay so I said that you can, you have to make sure that they are orthogonal and that fixes the coefficients have to be 1/square root and -1/square root. There is something which I want to ask you in these 2 states. If you look at the j=0 m=0 state, if I interchange m1 and m2, is the state symmetric or antisymmetric. It becomes, it picks up a negative sign right. So j=0 state which is composed of 2 spin 1/2 composite, the singlet state or J = 0 is sometimes called as a singlet state because it has only one component.

That state is an anti-symmetric combination of the uncoupled magnetic quantum number, it is m1 m2, if it is m1 is up and m2 is down, if you interchange you pick up this negative sign. What about j=1 and m=0, it is symmetric okay. Another way of seeing that j=1, m=2 has to be symmetric is that the ladder operation on a stretched state, a stretched state is always symmetric right, both are, so on a stretched state which is symmetric if you do the ladder operation you will always get symmetric states.

You will not get antisymmetric state, there is no way you can get antisymmetric coefficients okay. These are some things which you will see in particle physics that they will kind of control how to fix this angular momentum conservations and so on. So you just keep it in mind that, it has the orthogonal state is an antisymmetric state j=0 m=0 and j=1 and m=0 is obtained by ladder operation on a symmetric state, it will remain symmetric okay.