Quantum Mechanics
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Lecture – 63
Clebsch-Gordan Coefficients - I

So today my theme is to get you to know how to compute Clebsch-Gordan coefficients, this you need to know, without this lot of problems in atomic molecular physics, nuclear physics you will not appreciate okay. So this is why the stress in this course is that you should have your hands on in evaluating Clebsch-Gordan coefficients or CG coefficients as it is known. So we have already seen in the last lecture.

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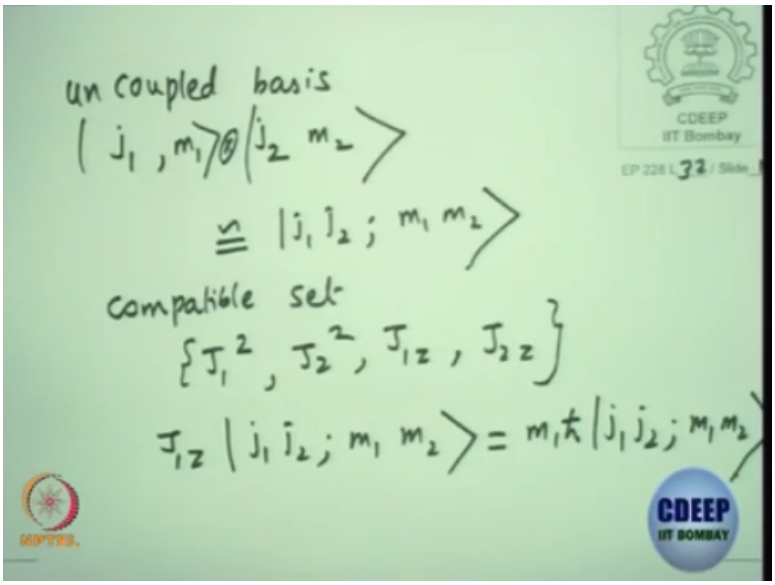

What have we seen? we have seen uncoupled basis right by this I am saying that there are 2 particles one with angular momentum j1 and another one with angular momentum j2, with the corresponding magnetic quantum number, you could write a tensor product of these states which you can equivalently write it as formally people write j1 j2 m1 m2, 2 systems with angular momentum j1 and angular momentum j2, if it belongs to a state j1 m1 and j2 m2, you can take a tensor product.

What is the compatible set of operators for this uncoupled state? The compatible set is it is J1 squared, J2 squared, J1z, J2z. So this state is a simultaneous eigen state of all the 4 operators. If you take J1z and operate it on j1 j2 m1 m2, that is going to give us m1 h cross j1, j2, m1,

m2. Similarly, for J2z is that clear and j1 squared will give you j1 * j1+1 h cross squared and similarly J2.

So it is a simultaneous eigen state of this compatible set okay. So this was uncoupled basis and we also had a coupled basis.

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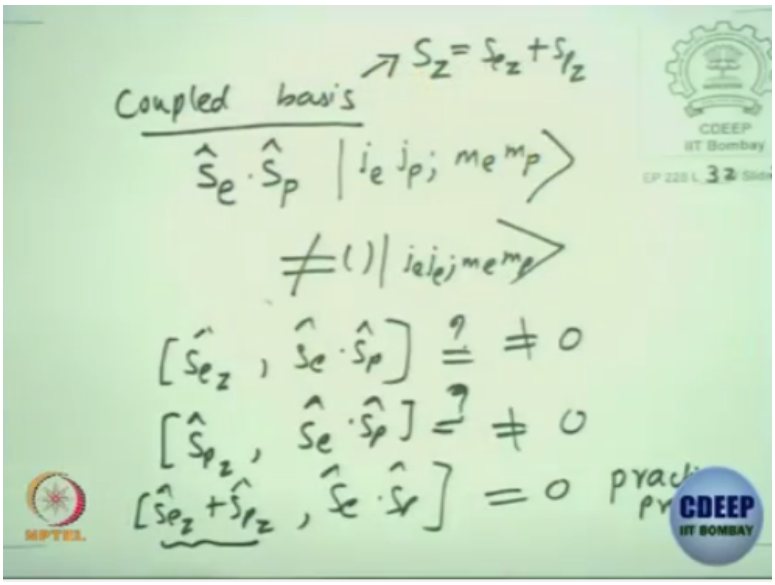

We had a coupled bases where you had seen some problems practice problems where you have spin orbit coupling or you have Sc.Sp where it is the proton electron system which are interacting, I am sure you have seen some of these problems where there is an interaction between the 2 systems and interaction between the orbital angular momentum and spin angular momentum which is a spin orbit coupling.

In such cases it is convenient to work not with the uncoupled basis, in the problem which you did you know uncoupled basis is not an eigen state of l.s right I am sure you would have checked this. If you did for the proton electron system you are taken Sp.Se and if you try to operate on the uncoupled system which is the j electron, j proton, m electron, m proton was it an eigen state.

It was not an eigen state especially for me which is half and mp which is –half right, I am sure you have checked this. So this is not in general = the same state with some coefficient. So it is not an eigen state, the uncoupled basis is not an eigen state of this. What do we do for that, what is the another way of seeing that this will not be an eigen state? If you take Sez component operator, Z component and you do Sc.Sp what is this commutator.

Scz commutes with Sp operators, you do not need to worry about it. What we have to worry about is this is only the electron spin operator, does this commute? It does not commute. Similarly, if I take the proton operators Z component Sc. Sp does this commute? So this is not = 0, this is not = 0, what about the sum. If you take Scz + Spz and you do Sc.Sp that is the sum of these 2, what do you expect.

You can show that this is 0, please check it okay. So this is a practice problem please check. The sum of these 2 Z component of the spin operators of the electron and the proton in the electron proton system that will commute with the Sc.Sp, what is this operator? this is the total Z component of the coupled system. So whenever we have coupling like this in the Hamiltonian, it is nicer to work with the compatible set which belongs to the couple basis.

Just trying to motivate you why couple basis becomes important when there are interactions. So this tells us that the couple basis will have a total Sz formally you can write it as Scz + Spz, but I have told you when you operate it on states in the uncoupled basis you should remember how to do this. It will operate on the electron state only and it will be an identity operator on the proton state and here it will be identity operator on the electron state and proton state it will operate.

This you should remember, even though it is not written this side is on the couple basis. You do not distinguish what is the magnetic quantum number of the electron, what is the magnetic quantum number of the proton when you look at this side. You only know the total Sz, is that okay. You do not really resolve what is the individual pieces making up the total Sz. So what is the compatible set for couple basis?

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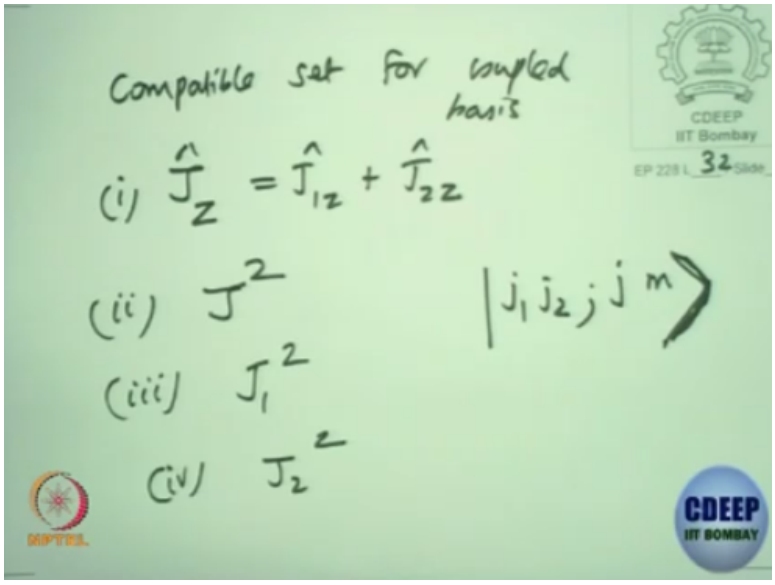

For couple basis is the total Jz, which is formerly you can write it as J1z + J2z, that you should remember appropriately on this how you should operate. This is one operator, what about compatible set, second one is J squared, total J squared and J2 squared. So you can write a compatible set here as j1, j2; J and m and I try to put just to remember that it is a couple basis I just curved it.

It is not necessary but as long as you remember it is okay that this is a couple basis which has, it cannot tell me exactly whether the electron spin is magnetic quantum number is what and proton magnetic quantum number is what, but it will tell me the total one. So that is why it is called coupled, you cannot resolve what are the, you can resolve it, but then you need to do the change of basis to go from one to the other and this change of basis from one to the other is called the Clebsch-Gordan matrix and mainly we are going to evaluate them today.

So this is a simultaneous eigen state of J1 squared, J2 squared, J squared and Jz, both are equivalent bases, why equivalent? The dimensionality of this will be exactly same as the dimensionality of the earlier one, uncoupled basis.

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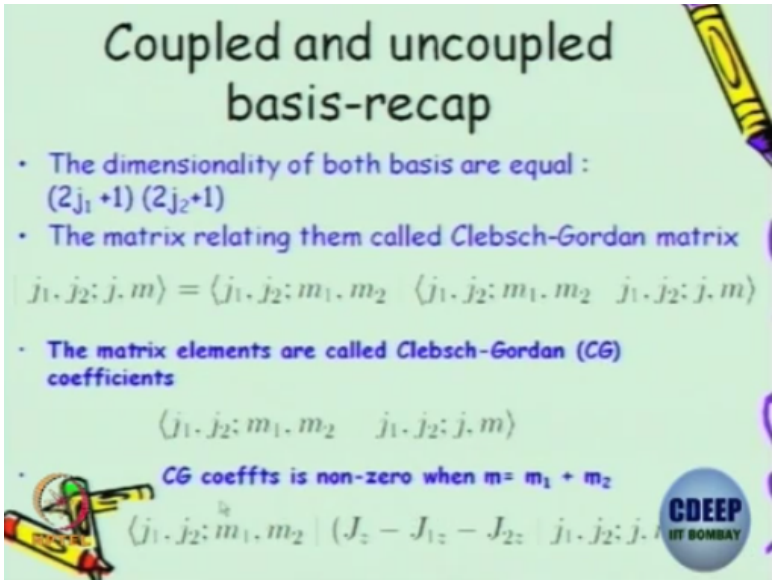

So just a recap which we did dimensionality of both the basis are equal, what is the dimensionality? It is 2j1 + 1 * 2j2 +1, the matrix relating the 2 basis is what we call it as a Clebsch-Gordan matrix and formally this is something I wanted to correct probably I have not corrected it, I should have done this okay. So let me write it for completeness since this is the first time you are learning.

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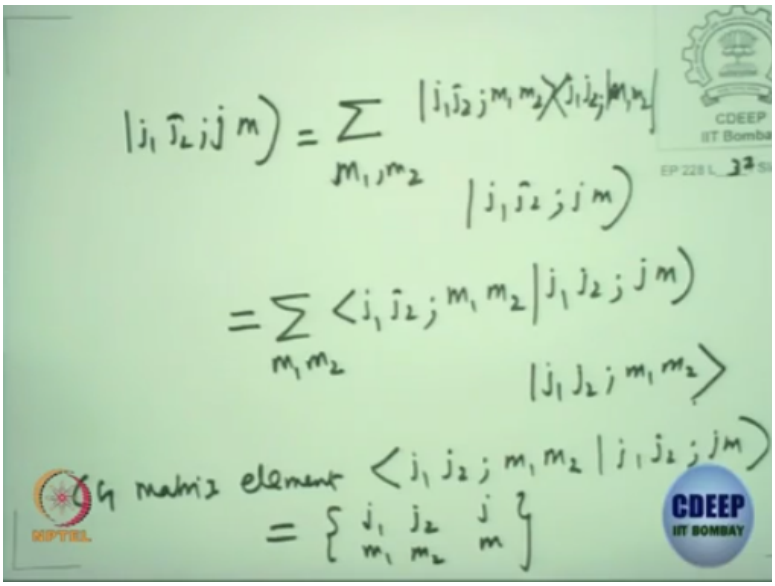

So you can take j1, j2, jm, you can write an identity state here okay. So what you can do is, you can take okay, you understand what I am writing right. This is an identity state multiplying j1, j2, jm, this is the complete set of basis, this is an identity operator. So this can be rewritten as summation over m1 m2. You can take this inner product and write I have just taken this with this and I have brought the ket this side, because this is the number.

This is for convenience I have used a curve, but actually it is angular only, it is an inner product of 2 base state that will give you some kind of a coefficient and this coefficient is what we call it as a Clebsch-Gordan matrix element. So we write CG matrix element either this way or some books writes it as j1, m1, j2, m2, j, m. So our aim is to evaluate these elements.

Our aim is to evaluate elements of the CG matrix and that we will do. So the matrix elements of the CG coefficients which is formally given and this also we checked in the last lecture coefficients is nonzero only when the sum of the magnetic quantum number with the uncoupled basis adds up to be the magnetic quantum number of the coupled basis and this also I said you can work it out by taking the total angular momentum - the individual Z component angular momentum.

This is actually 0 and if you substitute the Jz will prefer to act on this basis, J1z and J2z will act on this basis, you will get m-m1-m2 times the inner product of those 2 states. That inner product of those 2 states of the CG coefficients and that is nonzero only when m = m1+m2 okay.

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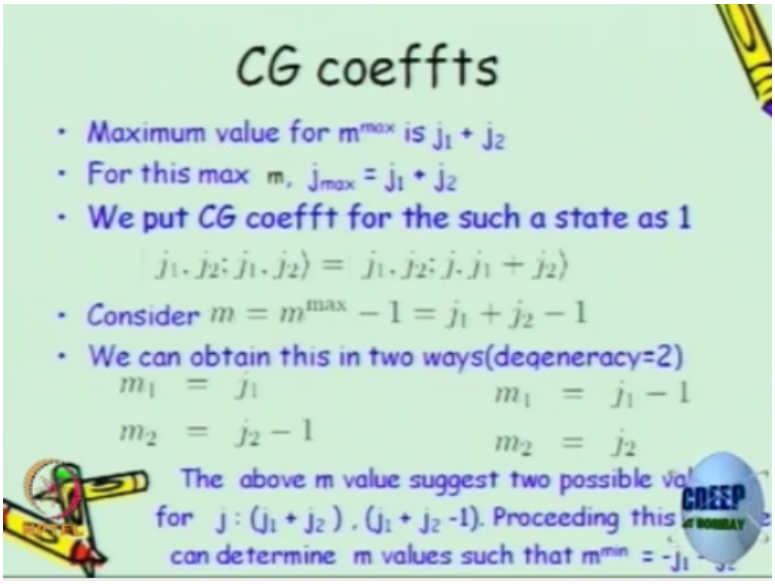

So this is where we stopped in the last lecture. What is the maximum value of m, m max in the total angular momentum cannot exceed j1 + j2 is that correct?

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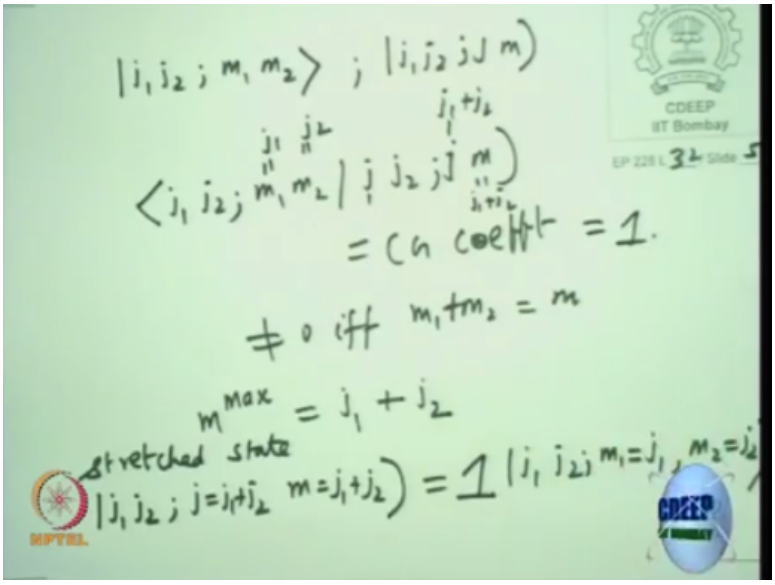

So we have one basis j1 j2, m1 m2 and then another basis, equivalent basis, which is jm which I am putting a curve just to remember it is couple basis and CG coefficient is nothing but in a product of j1 j2 m1 m2 with j1 j2 jm, that is the CG coefficient and this is nonzero if and only if m1 + m2 + m. Now with this condition what is m maximum, these states m1 maximum is j1, m2 maximum is j2.

And this condition m1 + m2 should be m will tell me m maximum cannot exceed, it has to be j1 + j2, is that clear, m maximum is the sum of the angular momentum of the 2 particle, if j1 is particle 1 angular momentum, j2 particle 2 angular momentum then the maximum value magnetic quantum number in the coupled basis has to be this. So if you have this to be the maximum? This jm state is meaningful if j = m maximum right.

So j max also will be j1 + j2. So we will take that maximum state, let us take that maximum state, j1, j2, j = j1 + j2, m = j1 + j2 this state, this can be achieved only with j1 j2, m1 = j1, m2 = j2. This is the only possibility. This coefficient we will put it to be 1, this is what we call it as a stretch state or the maximum state. The stretch state in the uncoupled basis is this and the coupled basis is this and we will equate those 2 states, that is the starting point.

So the coefficient is CG coefficient which relates the stretch state in the couple basis to the stretch state in the uncoupled basis is 1 always. So what is this meaning in this language if you put this to be j1, if you put this to be j2 this will be j1 + j2 and this will be = j1 + j2 that is the meaning. This is going to be 1. So that is what I have written here, that we will put the CG, it is = so this is uncoupled state and this is coupled state and these 2 are equated.

The m value will decrease from maximum value in steps of 1. So you reduce from the stretch state by unit 1. How do you do that? You do that by the lowering operator, which is J - the total J - is the couple basis and in the uncoupled basis it is j1 - + j2-, if you do that you can go reduce your m maximum to m by 1 unit. This is what we are going to do okay. So let us consider state with a magnetic quantum number which is 1 less than the maximum value.

So that is j1 + j2 -1, once this m becomes j1 + j2 -1 and we have fixed the total j to be j1+j2 that will still allow the state, you agree, that will still allow that state okay. So we can obtain this in 2 ways, what is the first way? In uncoupled basis if you take m 1 to be j1 and m2 to be j2-1 that is one way to get a specific m, m has to be m1 +m2, m has to be j1 + j2-1, there are 2 possibilities m1 =j1, m2 = j2-1.

Another possibility is m1 is j1-1 and m2 is j2, so this is a systematic way in which we can keep lowering the steps and seeing what are the degeneracies, which 2 states can contribute. So above m value suggest 2 possibilities for J, this is also I said, the total J is 1, it could be j1+j2 because that is the stretch state condition, I cannot go above that, other possibilities I could also have j to be j1+j2 -1 because the state jm permits j to go from –m to +m or m should be </= j, is that correct, any other possibility?

So proceeding this way you can keep determining all possible degeneracy and states in the uncoupled basis which can contribute to the total magnetic quantum number which is m and you cannot go below the total M, it has to be –j1, -j2, you cannot exceed that, agree. If j is j1 + j2 then you can go from –j1 + j2 to +j1 +j2 and steps of 1, that is the angular momentum algebra restriction.

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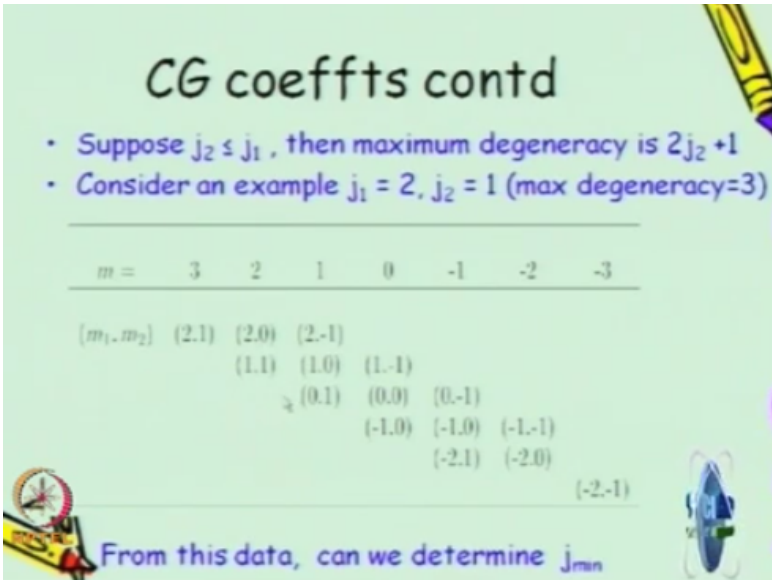

Suppose we take this for convenience can take both of them to be equal angle of momentum or you can take one of them to be less than the other particle angular momentum and this kind of degeneracy evaluation which we did here you can try to do it and you will find that the maximum degeneracy will not exceed the lower angular momentum okay, these are checks which you can do.

For example, let us take j1 to the 2 and j2 to be 1, this tells me that because j2 is 1, maximum degeneracy has to be 3, this would have said that degeneracy is 5, but when we try to write we will not exceed the degeneracy as 3 as plotted here in this table, m max is 2+1 which is 3. So m minimum is -3, you can go in steps of 1, decreasing steps of 1 till you hit m minimum which is 2, 1, 0, -1, -2, -3.

What are the m1, m2 contribution for a given m? m1+m2 should be equal to m right, you are all following, m1 + m2 should be = m and that shows that for a specific 3 it is a stretch state, it is one-dimensional. If you take m = 2 there are 2 possibilities, you can put m1 to be 2 and m2 to be 0 or the other way round. If you go to m =1 you can have, you should remember that m2 will take values from -1, 0 and +1 and m1 will take values from -2 -1 0 +1 and 1 2. You understand know.

If you take that then this is the only possibility, this is convincing that if j2 is < j1 the maximum degeneracy of states which can contribute to a specific m in this case is 3. So you will not get a degeneracy of 5 anywhere, just to put your hands and check out for something

else also take j1 to be 3/2 and j2 to be 1/2 or something and check it out what is the degeneracy or j1 to be 5/2 and j2 to be 1, check out what is the degeneracy.

Just for you to see this table, how the degeneracy, from this data can we determine j minimum, a degeneracy 3 will kind of indicate the j minimum has some you know which is the one which is 3-dimensional. So roughly we know J minimum from this elaborate example.

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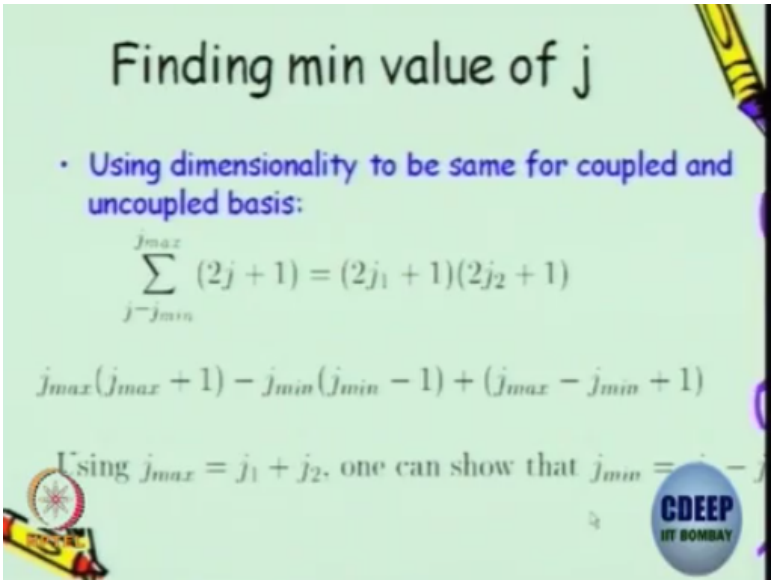

But I have also said as an exercise for you last time, what did I tell you, use dimensionality to be same for coupled and uncoupled basis. Take the j max to be j1 + j2 and determined j minimum that is what I said. So if you do this just a simple summation what will you get? So you get this to be the left hand side and that has to be equated with J max to be j1 + j2, if you equate it to this then you can show that j minimum is mod of j1-j2.

If j2 is < j1 you could write j1-j2, this will also ring the bell that for j1=2 and j2 =1, j1-j2 is 1 and this is 3-dimensional that is why the degeneracies did not exceed. This is some kind of an indication, but independently the J value, maximum J value is the sum of those 2 angular momentum of both the particles, minimum J value will be the positive of the difference okay. If j1 is > j2 it is j1-j2 that is the minimum j value, if j1 is < j2 it is j2-j1, clear?