Quantum Mechanics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology- Bombay

Lecture – 58
Rotations Groups - II

Group of these matrices which do not commute, they are in general called non abelian groups, okay if you do a rotation about one axis, only one axis, will I commute or not, they will commute, so that is called abelian group, the group of matrices, which you write about only one axis will be an abelian group but if you have 3 axis, 3 independent; linearly independent axis, the rotation matrices in general do not commute, the reason for them not commuting is that the generators which are angular momentum, they do not commute, okay.

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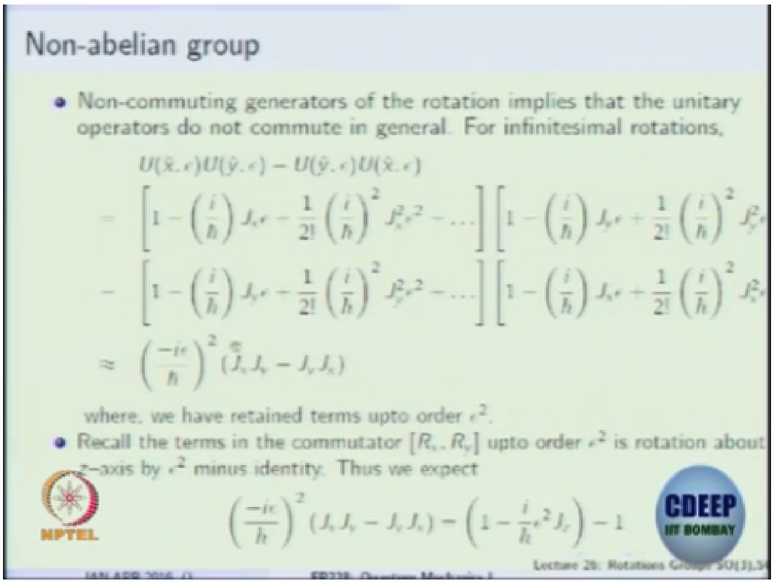

So, such is what you see naturally coming up here, even from the rotation matrices, so you redo the exercise which I did for r about x axis with epsilon angle and r with by the same angle about y axis and keep terms up to order epsilon square, okay please redo this if you keep terms up to order epsilon squared, you will see that you get commutator, this is nothing but it is the commutator of Jx, Jy.

But you know that this right hand side was rotation about z axis by an angle epsilon squared – identity, so try and write that out if you do that 1 – pi/ h cross epsilon squared angle times Jz - 1 they cancel and if you compare the epsilon squared terms, you get automatically that the commutator of Jx, Jy is nonzero and it is exactly what you are being studying giving you h cross times Jz, okay.

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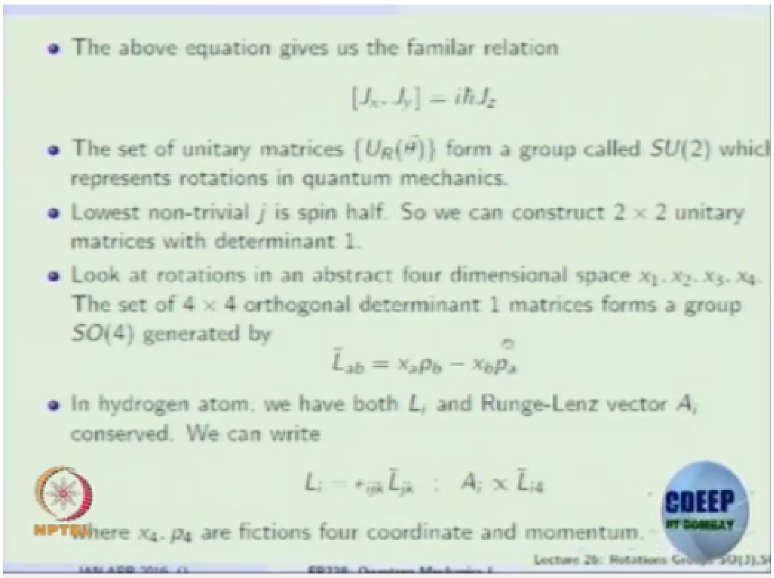

So, the above equation gives us the familiar relation, now comes this the lowest angular momentum you can have is a non-trivial one, spin 0 is a trivial one, right why is it a trivial one; it goes as if you do not see it right, if you try to take a spin 0, you can write it as a zero element one by one matrix, there is nothing non-trivial about it, okay you cannot write numbers if you write a 1 cross 1 matrix for a spin 0, you cannot make it to be non-commuting.

The only way you can satisfy this relation is that take Jx to be 0, Jy to be 0 and Jz to be 0, so that it satisfies trivial, so it is the trivial, spin 0 is the trivial representation, the next one is spin 1/2 that is the first non-trivial representation, okay, so for spin 1/2 you can construct unitary operator, spin half will have 2 states possible; J = 1/2 will have, m = + 1/2 and m = - 1/2, there are 2 states possible, you can denote J = 1/2, m = 1/2 as 1 0 and J = 1/2 m = -1/2 as 0 1, you can write Jx, Jy, Jz as 2 cross 2 matrices.

We have done this in Stern-Gerlachs, I am just putting it from the group theory picture here and you can have these set of 2 cross 2 unitary matrices and you can check those matrices which you did, sigma is the poly matrices were all traceless matrices right.

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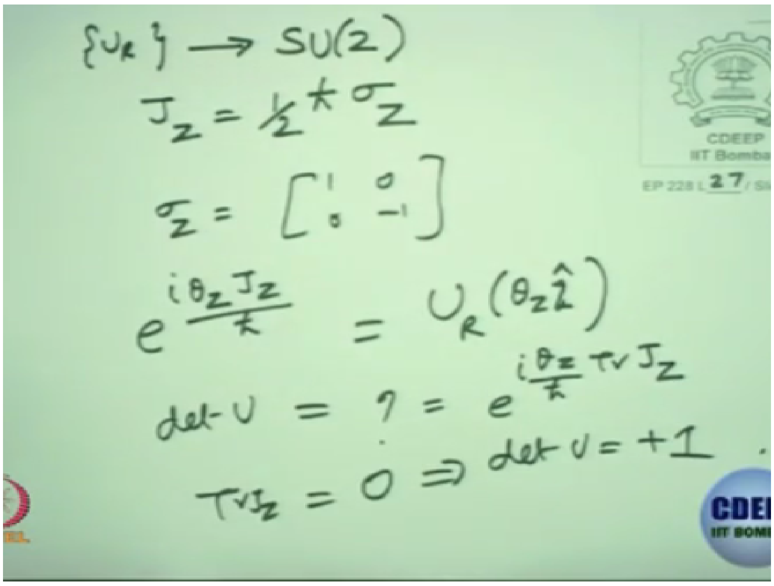

So, we can take Jz to be 1/2 of h cross times sigma z, where sigma z is the poly matrix which is 1 0 0 -1, if you take exponential of i theta z Jz/ h cross, this is the unitary operator for rotation about an angle theta z; z hat if you want to put it, will just be determinant of U, what is it? This will be e to the power of i theta z/ h cross trace of Jz and they are all traceless matrices, the poly matrices are all traceless, other factors does not really play a role.

So, e to the power of trace of Jz is 0, so which implies determinant of U is + 1, so this collection of Ur matrices will be called as S, U for unitary, the first non-trivial dimension of the vector space in which it plays a role is 2 dimension, so we write the notation as SU, this 2 is for the first non-trivial angular momentum which we study, so the lowest non-trivial J spin 1/2, we can construct 2 cross 2 unitary matrices with determinant 1.

And this is what we call it as an SU2, okay, not really write, written it here but I have said it here, the set of unitary matrices form a group SU2, which represents rotations in quantum mechanics, look at rotations in an abstract 4 dimensional space, take x1, x2, x3 which are your xyz and that add one more coordinate, you can still construct 4 cross 4 orthogonal matrices, nobody prevents you from doing by determinant +1.

If you do that then that collection of r matrices, you can call it as SO 4, should be a complete exhaustive set and similar to what we were being talking about any transformation or any symmetry transformation should be generated here SO 4 can be generated by rotation about a plane, what I was telling that earlier you are talked, if you are confined, if you are an (()) (06:53) and you are confined to be only on the plane.

You say you are doing a rotation about a point through the center of the plane, when you make it 3 dimension then you talk about rotation about an axis, when you go to 4 dimension, it is rotation about a plane and so on, so it keeps going so typically, when you do a rotation only 2 of the coordinates are getting mixed, so the coordinates xa and xb, they get mixed and the generator can be formally written with those coordinates which gets mixed rather than worrying about the remaining plane or so there is a nice neat way of writing.

In hydrogen atom, this is what I was telling for completeness, I thought let me put in here, we have both angular momentum is conserved because it is the central force problem, you also know that the Runge-Lenz vector is conserved, this Runge-Lenz vector is being conserved, you may not have it you know symmetry transformation to understand it that is why sometimes we call this to be a dynamical symmetry.

We may not have this physical geometrical symmetry like translation or rotations to understand them but it still conserve it, so it is some kind of an abstract symmetry which is called dynamical symmetry and you can compactly write your Runge-Lenz vectors as if you are looking at a fictitious coordinate which is the fourth coordinate and fourth momentum and rewrite your angular momentum Li.

Here this i; this a and b denotes 1, 2, 3, 4 and the ijk will only denote 1, 2, 3, so I have put in here where this one is 4 and this one is i so again, there are 3 components here and 3 components here, so there will be 6 generators for the SO 4 group, right this is anti-symmetry, how many independent possibilities are there here is 6, so 6 generators using with you can start studying even the hydrogen atom problem as if it has the fictitious fourth coordinate, it has no meaning.

But as mathematical problem, you can solve them and see whether you can get this spectrum as proportional to; energy spectrum proportional to -1 over L square, so I am not doing this if anybody is interested please go and look at Schiff symmetries in quantum mechanics where he talks about this dynamical symmetry and work out the spectrum very nicely okay, quantum mechanics does not distinguish whether you do only orbital angular momentum or you do spin angular momentum.

SU 2 can also be for those orbital angular momentum but that is an higher dimensional representation of the same just because you write 3 cross 3 matrices, you can still write unitary operators okay, so now that you have got it, let me explain a bit.

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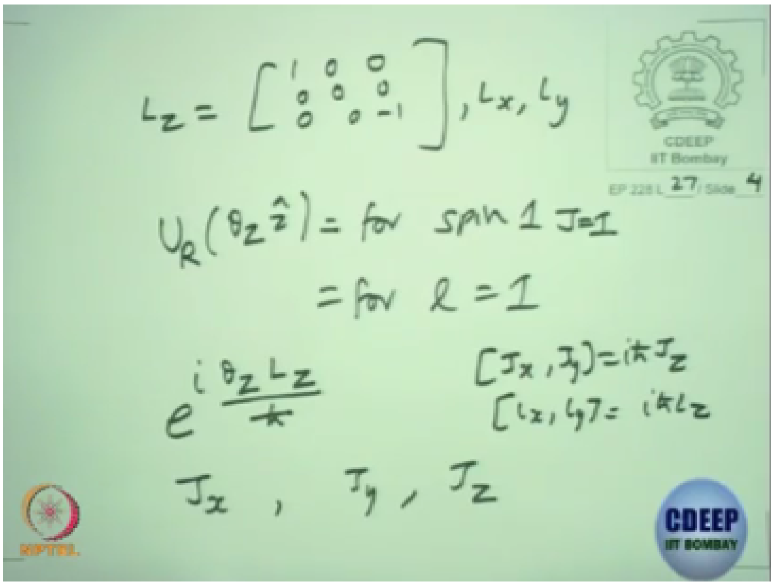

So, what you can do is; you can write your Lz as 1 0 0 0 0 0 0 0 -1, right, you can write a unitary operator for rotation theta z upon z hat, for spin 1 or for l = 1, both are same, spin 1, I mean J = 1 and here I mean l =1, when I write an exponential of i theta z Lz over h cross kr, so I am not really separating what is S and what is l; j for sake can be 1, how the S and l goes into the J, I am not worried.

What I am worried is whether I can give a matrix, you have to write Lx and Ly, this I think is there in one of your tutorials sheet or assignment, right you can also write a similar thing for Jx, Jy and Jz, you do not need to say what are the components of S and l, we will come to it at some point, we will do the Clebsch Gordan. Right now, when I say I give you in physical rotations, if I give you the orbital angular momentum, you can still construct unitary operator for it.

But if I do not say it is orbital angular momentum but it is the spin J, by that spin J, it can have a piece which is orbital angular momentum, we can have a piece which is spin angular momentum, together it could be a 1, J = 1, so it does not mean that you are trying to say spin J = 1 is not SU 2, it is here when you write 3 cross 3 matrices for this, for spin 1, why do we call still a SU 2, I was very careful when I said look at the lowest non trivial dimension right.

But you can do it for spin J =1, you can write a 3 cross 3 matrix, need not be poly matrix, right, J = 1 will have 3 states that also you agree, right but there is something more, you are only looking at rotations in that space, there are only 3 possible parameters, okay there are only three possible parameters and you have only 3 possible generators just because you exponentiate with 3 parameters and the J which can be a 3 cross 3 or a 2 cross 2 or a n cross n that is not mean you go to SU n.

We have seen from SO3 to SO4, you have seen, what have you seen? SO3 had only 3 generators and 3 angles, SO 4 has what; the minimal non-trivial matrices which you can write is 4 cross 4 with 6 generators, so just because you write Jx, Jy, Jz as 2 cross 2 or a 3 cross 3 or a 4 cross 4 matrices does not mean you go to SU that is not right because there are only 3 parameters and 3 generators, you are still doing rotation in an abstract space.

And the matrix representation of the abstract space which satisfies the SU2 angular momentum algebra is strictly this, you are not going out of this, SO4 algebra and SO is also similar to your SO3 algebra, right, SO3 algebra has Lx Ly = ih cross Lz but SO 4 algebra is different, I can always write a set of rotations in 3 dimension as a subset of set of rotations in 4 dimensions, those will involve only these 3.

But they will look like a 4 cross 4 matrix, looking like a 4 cross 4 matrix does not mean it is SO 4, it should be a complete set exhausting all possible generators, okay writing a 3 cross 3 matrix for a spin 1, you cannot judge that it is SU 3, you have to see how many parameters are involved, how many generator are involved, if I write that as e to the i theta dot j, with J could be 3 by 3 for spin 1, then I should know it is SU 2.

You have to only look at the first non-trivial lowest dimension and fix it, this is also an exercise for you, I do not even tell you about rotations, I just ask you to use this condition that r; r transpose = identity, determinants r = 1, try to find the number of independent basis possible in the case of a SO 3, in the case of SU 2 and in the case of SO 4, you can actually check and show that the number of independent basis or the generators will be 3 for SU2, 3 for SO3 and 6 for SO4.

Sure you would have done this, number of orthogonal matrix entries when you write symmetric matrix entries you write, you know how many nonzero entries are there, right anti-symmetric matrix in 3 dimension has only the upper diagonal and the lower diagonal is negative of the upper diagonal, so there are only 3 entries similarly, for unitary matrices where it is exponential of Hermitian.

If you put hermiticity condition on 2 cross 2 matrix, how many independent entries are there, if you put determinant to be = 1, how many will get reduced, please check it independently okay that will also clarify for you just by writing higher dimensional representation for SU 2, does not mean you can call it as SU3 or SU4, depends on how many parameters and how many generators okay.

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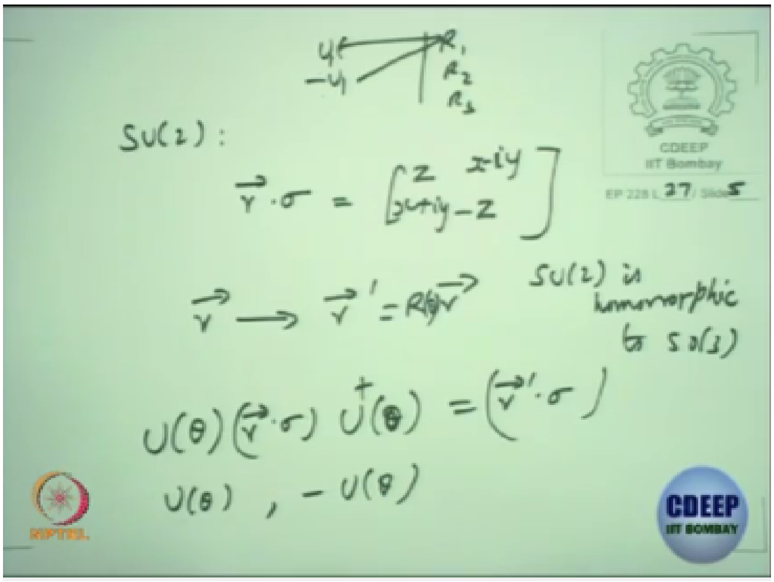

So, for SU2, you can denote a vector as r dot sigma okay, this is one way of writing it, what is r dot sigma; it is nothing but z – z, this is x + iy or x - iy and x + iy something like this, what is rotation do for you? A vector r goes to r prime which is r times r, what is this 2 for the SU 2, is that if you take u of theta, this is r of theta r dot sigma, u of theta u dagger that will give you r prime dot sigma.

So, the analogue of this equation in SO3 will be this, even in physical space, something now you can note that if you take the 2 signs here okay, you can take u with an identity; u of theta and – of u of theta will both give you the same equation, right because there are 2 u; u u dagger is there, so the identity matrix here can be either represented by identity or – identity, it is the homo (()) (18:28), it is a 2 2 1 matrices okay.

So, this tells you that it is not 1 to 1, so if you write a set of R's; R1, R2, R3, you can have a u1, -u1 both of them correspond to this okay, so that is the subtlety, algebra is the same but if you are looking at the physical rotation which involves integral angle of momentum, those matrices can be made isomorphic to SO3, if you are looking at half odd integral, then you have this way of looking at it.

And you can see that there are; there is a 2 to 1 matrix, so that why we say that SU2 is homomorphic to; even though the algebra is (()) (19:43), there is still 3 parameters and 3 generators just because it is 3 parameters and 3 generations you should not come to the conclusion that you are studying, Su 2 is an necessity if you want to look at a space of both integral and half odd integral.

But you can live with SO3, if you are only looking at a space of integral which is what happens in your hydrogen atom problem with only orbital angular momentum, there it is no half odd integral coming up, the Stern-Gerlach experiment is the one which forced us to look at spin half, why only spin half; the algebra which we did using Schwinger oscillator method told you that you can have a half odd integral.

And we try to conclude that J could be integral or half odd integral, you want to have everything as a universal J which is integral and half odd integral, you have to work with SU2.

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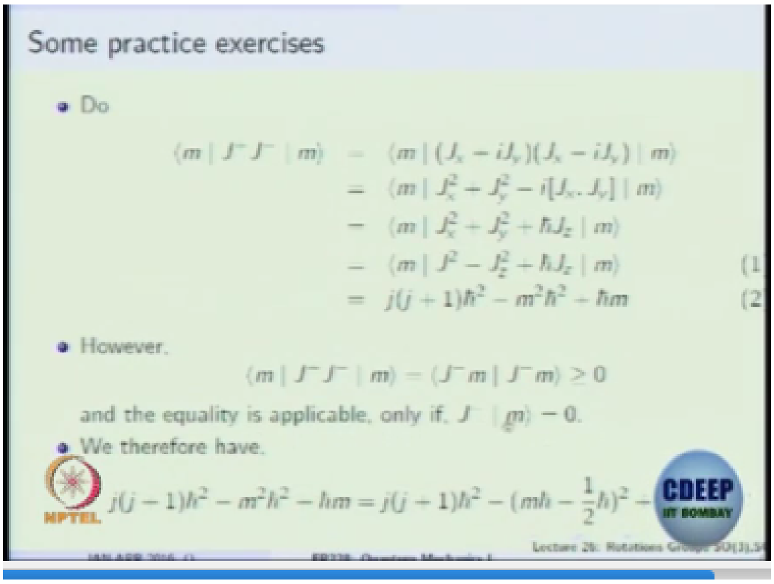

So, some of these are practice exercises for you which I want you to do okay, what will this matrix element be; J -; J + is Hermitian conjugate of J -, so this matrix elements should always be; I have suppressed the J here for just to make it look compact but you can put this small j here, what will this matrix element be, what was the, why was this argument that a dagger a expectation value in harmonic oscillator was positive definite?

The same argument will go through here, J – on m, let it give you a new state, then the dual vector state will correspond to J + on the dual raw state m right, so this has to be always > or = 0, so rewrite this explicitly in terms of your Jx + ijy Jx - ijy and you can write this explicitly, can we do; Jx squared + Jy squared can also be written as; you can add a Jz square to it and subtract a Jz square to it that way you know it will help us to write this state to be Eigen states of this magnetic quantum number.

And I have suppressed the J, you can put the J, okay you can do this, you can rewrite this Jx squared + Jy squared as J squared - Jz squared and that will be was the j * j + 1 h cross square okay, so please redo it again, redo it with the J + and J - interchange also that will also be positive definite right, please do this similar exercise for J – J +, this is what I was arguing for you that J – on m, you can put it as a ket.

And then that the dual ket will be like this and this is positive definite and equality = 0 will happen when you have a state which has a lower bound, so J - on that lower bound state is 0, this also comes automatically from here, just like the harmonic oscillator where we took the number operator, matrix element is positive definite, you can do the same exercise without resorting to the Schwinger mechanism; Schwinger oscillator method, you can do this here.

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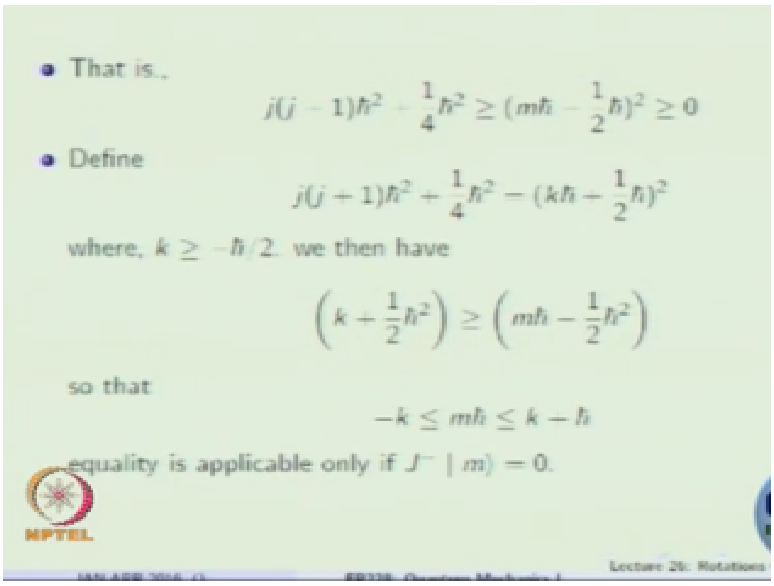

So, what are the requirements, then this being > or = 0, will put kind of a restriction here, okay simplify it, this has to be > or = 0, the question in a neater fashion that it is > or = 0, to find a new integer K is just 2, see that your m takes values from - J to +J that we did there by a different argument but you can go through this argument, okay, so you find that the magnetic quantum number m has some kind of a restriction.

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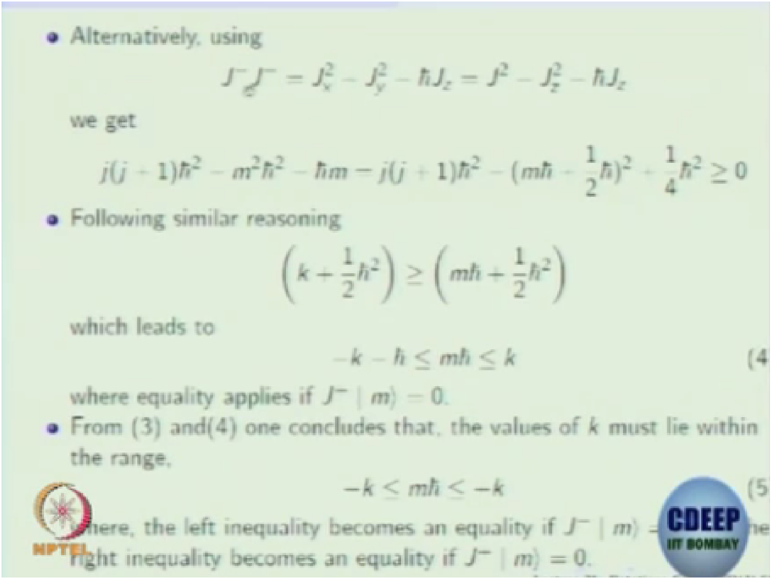

So, in fact which is going from - K to - K + h cross and equality is applicable when J - m is = 0, so you can do the same thing with the interchange J + and J – okay, leave it to you to recheck, tell me if I made any typos in it okay and looking at equation 3 and equation 4, you can try to find what is the common range for the magnetic quantum number and that will turn out that this common range will be from – k to +k.

I think this h cross is not correct, it is not, yeah or you put h cross everywhere, okay, so this is a systematic exercise where we know that J - on - K is 0 and J + on + K is 0 and we get the range of magnetic quantum number going from - K to + K.

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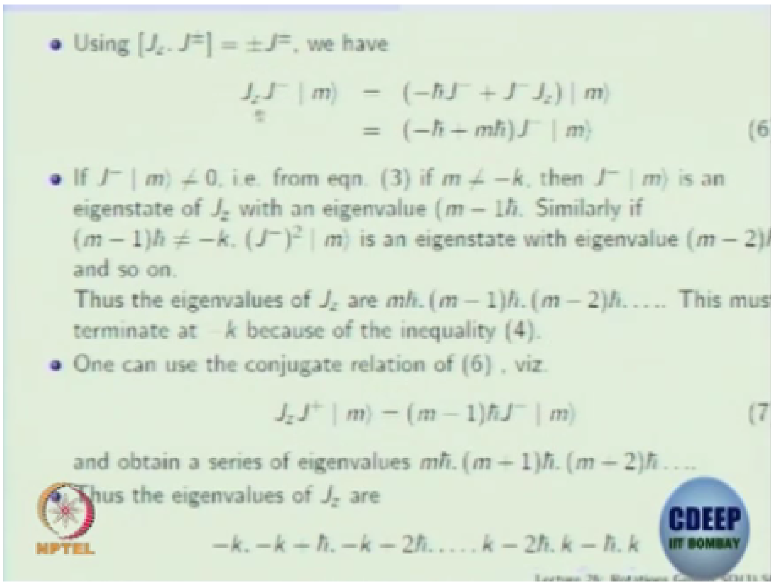

And then we have to do the ladder operation, this is what I was telling you in the last lecture, you can go and do it with an angular momentum algebra itself that is what I am reproducing it for you here, so use this algebra, so this is the algebra which you would like to solve and then if you try and solve this, you will be able to show that the J - m is an Eigen state of Jz with m - 1 Eigen value and the series of Jz should go from - K to + K will also tell you that you can keep doing this lowering operator.

How many times; should not go below – K, so that will dictate for you how many times you can do this, then you can do that and get the eigenvalues of Jz, you can fix this to be mh cross m - 1 h cross m – 2 h cross and it should terminated – K, you can also do the conjugate relation, this is done with Jz and J -, you can do it with Jz and J + and you can go in steps of mh cross m + 1 h cross and so on.

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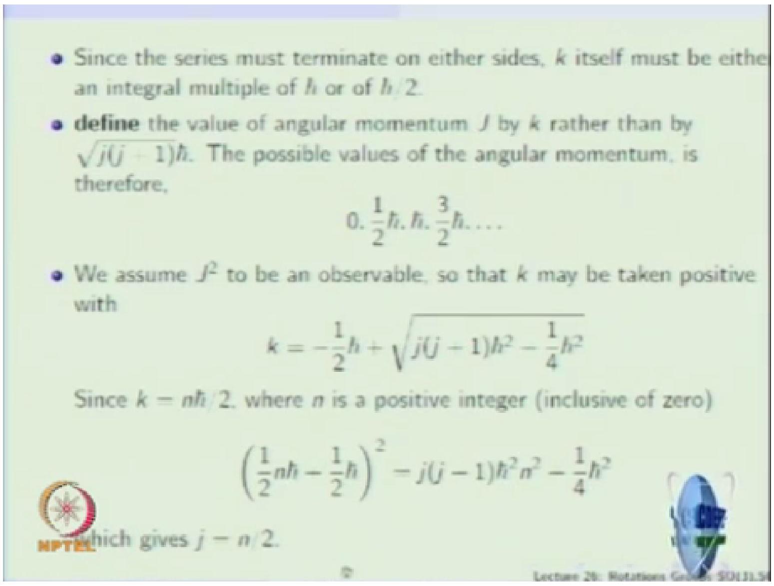

So that will give you the span of Jz Eigen values which goes from –K to + K and we have already see that it should terminate on both sides, the J – on - K is 0 and J + on + K is 0, you can try and say that it is always the K value could be either integral or 1/2 order integral, these are the only 2 possibilities and you can define your K, the angular momentum J by K rather than j * j + 1 h cross.

So, the possible values of angular momentum will be the same, so the K; the h cross is included in the K, okay, so K is nh cross by 2, where n is a positive integer and you can see that if it is odd then it will be 1/2 order integers, if n is even then it will be integers and you can fix what is J from here; J will be just in terms of this, you can fix it to be n/2.

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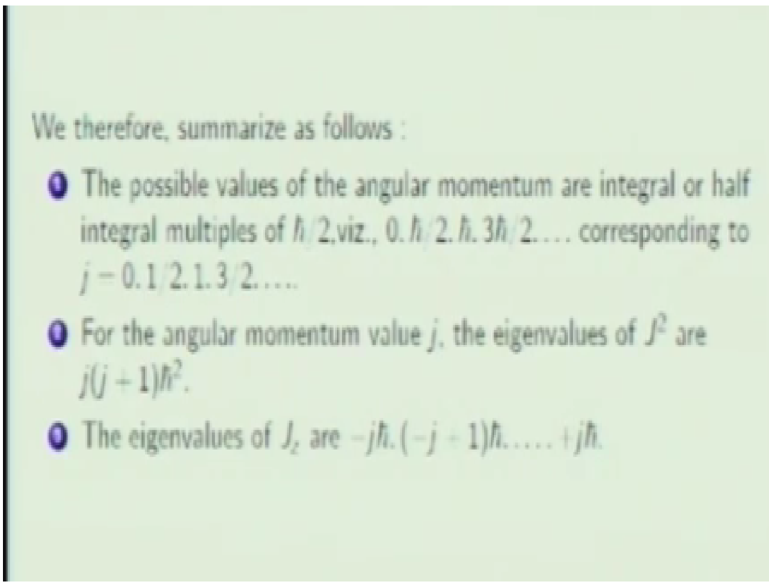

So, let us just summarize, going only with the angular momentum algebra without resorting to representation in terms of harmonic oscillator ladder operators, you can also fix what are the possible values of your angular momentum and you can also find the Eigen values of J squared, this we knew is j * j + 1 and the eigenvalues of Jz goes from – jh cross to +jh cross.

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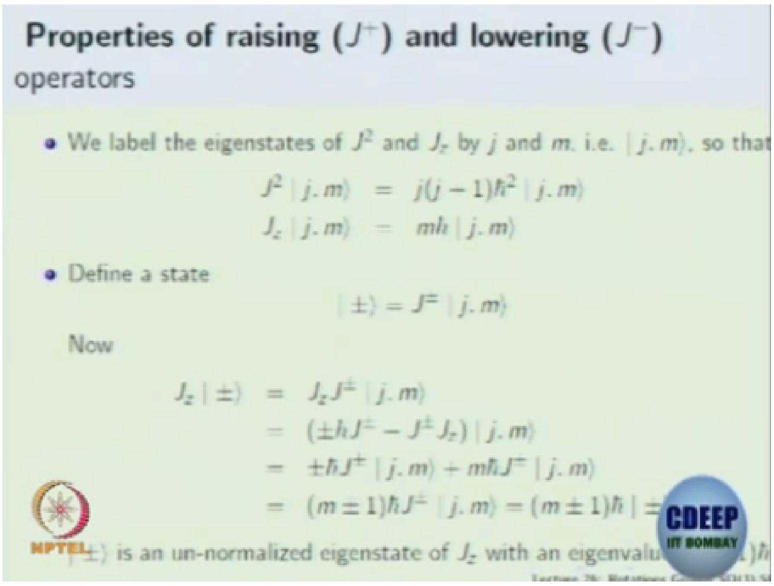

Then comes the lowering and raising operators, this also you can do, call this plus or minus as J + or - Jm and use this algebra and try and write the way we wrote C, Cn and Dn, do the same here, okay.

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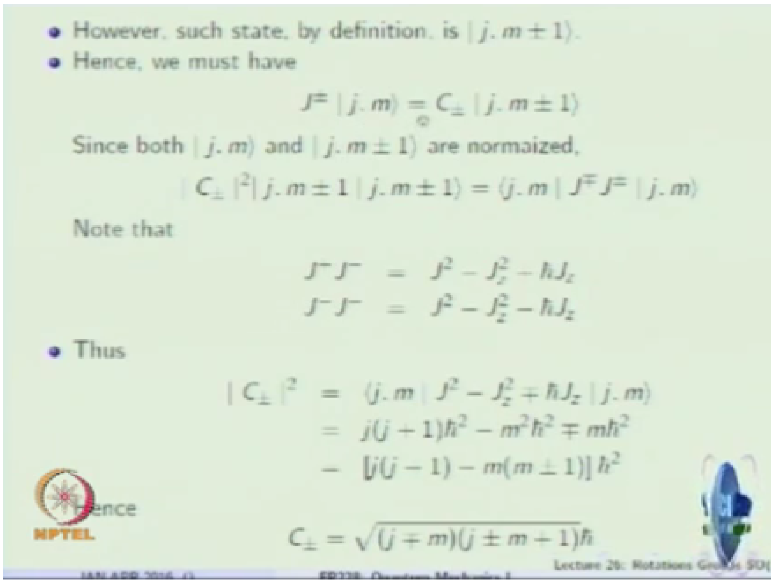

So, let us denote J + or - on Jm as C + or - on Jm + or – 1, I did it by the harmonic oscillator approach, I said you please check it out and we will discuss it today and this time I am also not discussing, I am saying this is a practice, you please work it out yourself, I am just putting it for completeness, so that you can go and look it up in case you make a mistake. So, actually I am going very fast since I said it is practice for you.

So do this and find out what is C+ or – mod square and C + or - will turn out from per say from angular momentum algebra also in this fashion, so I will leave it you here, I stop here.