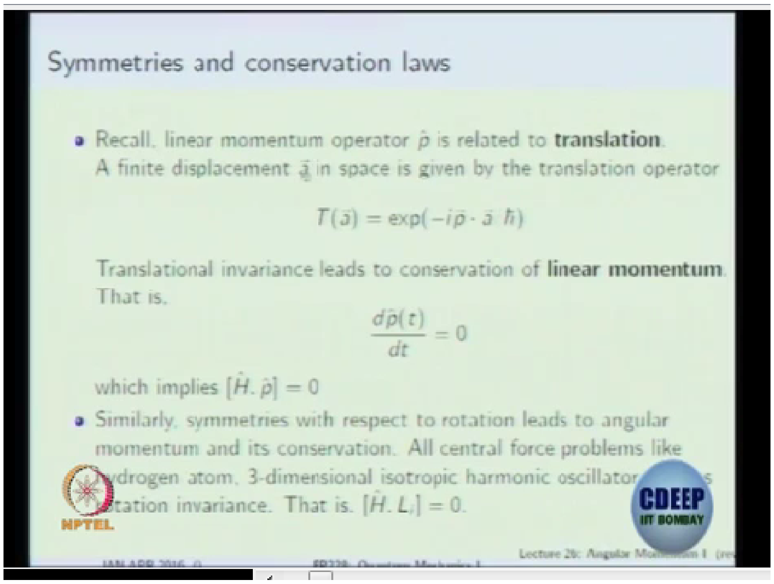
Quantum Mechanics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology - Bombay

Lecture – 56
Angular Momentum - II

Okay, so just for completeness, I taught a couple of things and keep telling you, but I thought I will put it on the PowerPoint today.

(Refer Slide Time: 00:29)



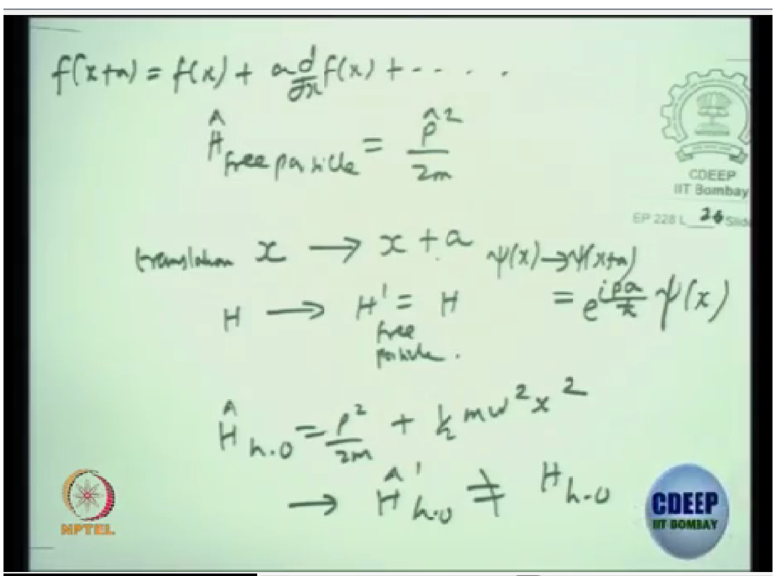
Linear momentum operator is supposed to be related to a symmetry. What is the symmetry? Sure you all know. Take a cube and if I rotate it by 90 degrees, it will remain the same about any axis, right. So before I show one cube without all the vertices are identically coloured, I rotate it by 90 degrees without you seeing it and I put it back, will you be able to see that transformation? You will not be.

So that is what is the symmetry. Symmetry is a sense where if you do a transformation and your system sees it as if it is an original state, then your Hamiltonian should commute with what were forms that operation, okay. What is the one which performs that operation? You wanted a translation to go from a position x to a position or an x factor to a position x+a. And which operator helped you to do that in a Taylor series expansion you have checked, right.

If you have not done it, please go back and check it out. You have seen that it is the momentum operator which is responsible for translating a position from x to x+a, okay. So this is what is the linear momentum operator which performs a translation. Once you perform the translation, if the system remains the same, what do you mean by in quantum mechanics or classical mechanics is that we look at the Hamiltonian and see when we take x to x+a, whether the Hamiltonian changes or not changes.

For a free particle, does it change? Because d/dx remains the same under constant translation. What happens to harmonic oscillator? Harmonic oscillator, it changes, 1/2kx square you have. x shifting by x+a will change it. Correct? So it is basically if you take the known data, so free particle is P square/2m in quantum mechanics, classical mechanics is just.

(Refer Slide Time: 03:31)

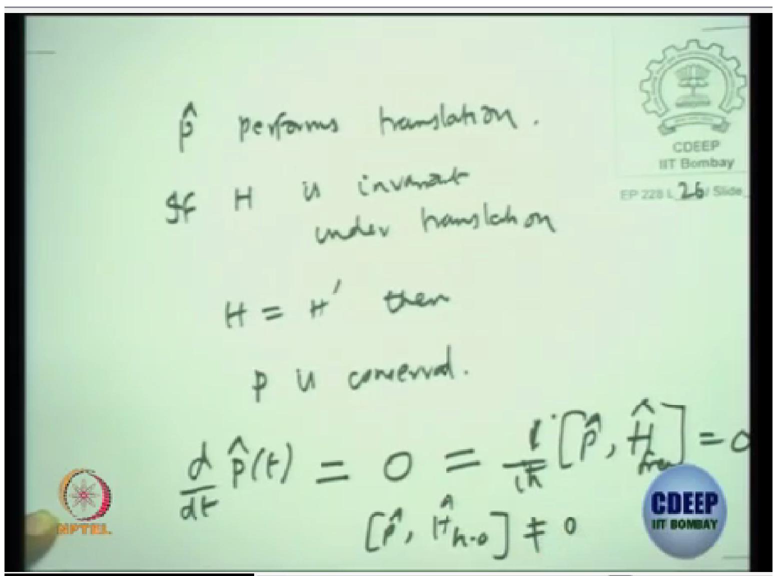


If you do this transformation in 1-dimension as x is going to x+a, H goes to H prime which is H for the free particle. You all agree? Translation. If you take harmonic oscillator which is p squared/2m+1/2m omega squared x squared, that goes to H prime, harmonic oscillator which is not equal to H. And when I do this transformation, any wavefunction, psi of x goes to psi of x+a and I could rewrite this in terms of a to the power of ipa/h cross on psi of x, right.

This is what we have done in your assignment. It is a Taylor series expansion which I have compactly did. I am sure you have all done f of x+a as f of x+a*del/del x f of x and so on. And instead of del/del x, I am going to use momentum operator. Because I use the momentum operator, I will have an i and h cross coming up. This is what you have done in your assignment, okay.

What is the meaning of this free particle has the translation symmetry, where as harmonic oscillator does not have the translation symmetry? If there is a translation symmetry, the one which the operator which performs the translation is a linear momentum. Linear momentum has to be conserved. In quantum mechanics, so let me write that.

(Refer Slide Time: 05:36)

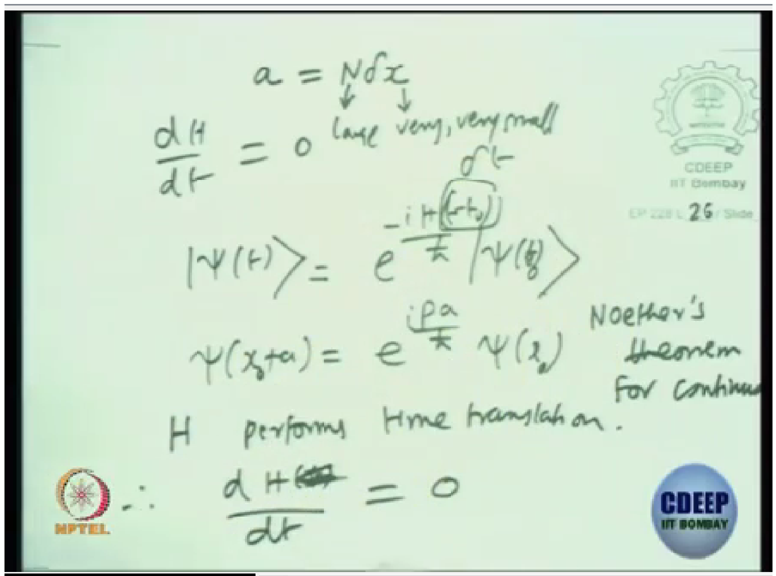


So p performs translation. If Hamiltonian is invariant under translation, what do we expect? H is same as H prime. Then what is conserved? Momentum is conserved, right. So this is what we call it as a translation symmetry for the system. Associated with a symmetry, the one which performs that symmetry operation, will be conserved, classical mechanics. Quantum mechanics, what do we say? d/dt, Heisenberg's picture, p operator should be 0.

By Heisenberg's picture, this is nothing but 1/ih cross p operator with H operator is the 0 for a free particle. For free particle, it is 0. If this is 0, we know momentum is conserved or matrix elements of the momentum operator does not change with time. It is a constant of motion. And this operator is the one which performs for you the transformation which is translation. Let us do the same thing. Harmonic oscillator you can check.

What is p operator with harmonic oscillator? Will this be 0 or non-0? Non-0. What does this mean? It does not possess translation symmetry. So for time-independent systems which we are doing, you can argue that dH/dt is still be a 0, right.

(Refer Slide Time: 07:47)



Equivalently, the function psi of t when you write, we write this as e to the -iht/h cross/psi of 0. This is similar to what I was trying to say x+a is e to the ipa/h cross on psi of x. If you see x is like the parameter t, then the one which does time translation, the operator which performs time translation is the Hamiltonian and Hamiltonian better be conserved. Is a very beautiful way of seeing.

So we can say h performs time translation, okay. Therefore, dH of t/dt in Heisenberg which anyway where in any picture Hamiltonian is same, it is independent of time you know, has to be 0. Which means time translation invariant is loss if your Hamiltonian has time dependence. So all the systems which you are studying, dH/dt is 0 because H's are all time independent. Because it is time independent, doing time translation does not affect your Hamiltonian at all.

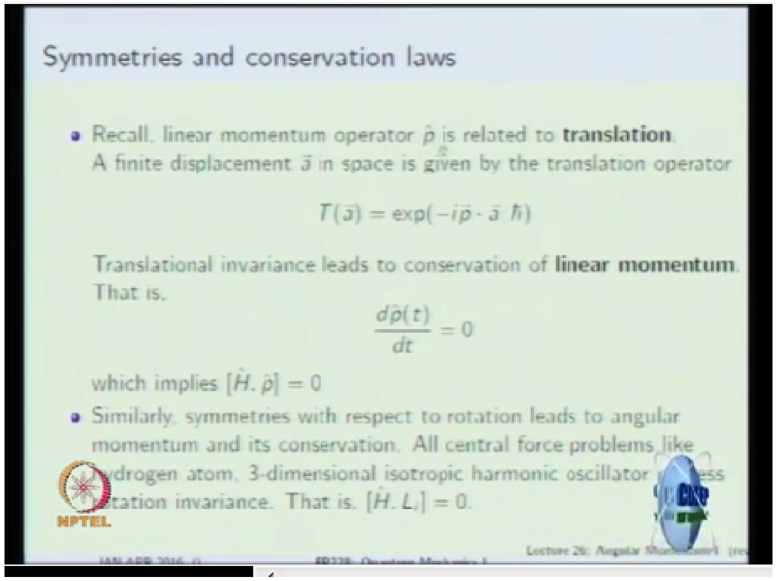
So I am just saying see if you want you could put it as t0 here and you can see this as t-t0 when there is a shift. So here a is the shift from what you can call it as x0 and x0 and a is like the t-t0. This is delta t if you want. So every symmetry, the question is will it be associated with a generator S for continuous translation, not for parity. Parity is x goes to -x. You cannot write some generator for it, some discrete transformation you do not have.

For continuous symmetries, translation is a continuous symmetry and finite translation by a vector a can be obtained by dividing into infinitesimal delta x done n times. The n times delta x where n is very large, you can get a finite translation of n*delta x where n could be taken large and this is very small, okay. So any continuous symmetry which can be broken up into infinitesimal such transformation, you can find a generator or an operator that is what is called as a generator of a transformation.

It is called Noether's theorem, good point. So many of you know things, Noether's theorem in classical mechanics you have done. In quantum mechanics also, it is applicable for continuous symmetries. By continuous I mean any finite transformation can be split up into infinitesimal transformation and you can add that. That is what I mean. A rotation of a cube which I said by 90 degrees is a discrete transformation.

There I cannot apply this generator which performs that operation. On parity also, you cannot apply this but in general for any continuous transformation like time translation Hamiltonian is conserved, that is an operator which generates time translation. Similarly, space translation, momentum is conserved. Momentum is a generator of space transformation and so on, okay.

(Refer Slide Time: 12:02)



So linear momentum operator is related, is an operator which performs the translation and a finite displacement when we write, it is this operator always coming with what is the amount of translation you have done, okay. So this is sometimes called as a parameter. This is the operator and if you are in 3 dimension, you have 3 operators px, py, pz. Similarly, you will have 3 parameters ax, ay, az because you have to say how much amount it is translated along x, y and z and you get the translation operator by finite unit written as an exponential.

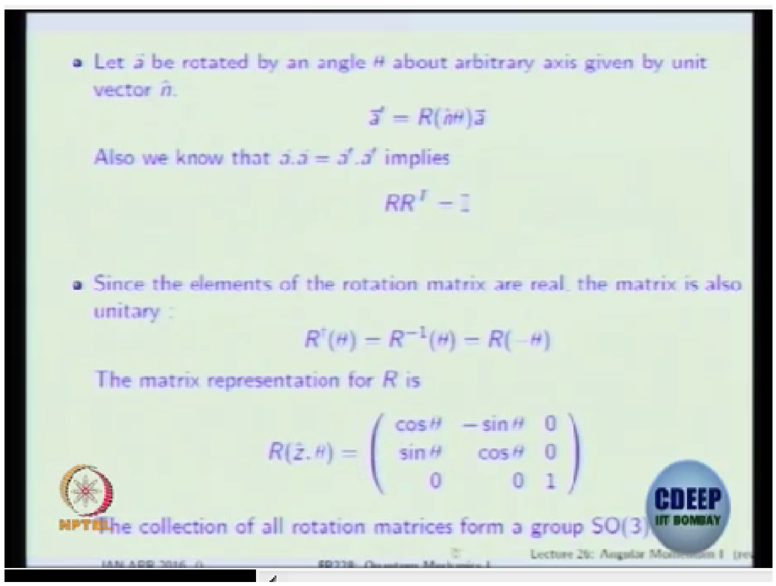
And translation invariance of a system actually means that the momentum, linear momentum is conserved which we have seen in both the ways in the Heisenberg picture. It is equivalent to saying that if a system has translation invariance when the operator which performs translation, better commute with the Hamiltonian. So symmetry is with respect to rotation just same thing. Hydrogen atom is invariant under rotation, right.

Is it invariant under translation? Hydrogen atom problem is written in relative coordinates. There is a proton at position xp vector and there is a electron with coordinates x electron vector and when I write the V as the difference between the proton coordinate and the electron coordinate, if I shift proton coordinate by a units, if I shift electron coordinates by the same a units which is what is called the same translation done on all the objects, relative coordinate will not change.

So you can show that x will commute. Check it. I am just trying to say this from the physics point of view, okay. So 3 dimensional isotropic harmonic oscillator, isotropic in a sense its frequency of oscillation along x, y and z are the same. You can write x square+y square+z square+r square, that is also like a central force problem. And all these cases you can show that it has rotational invariance.

Rotational invariance he had been harping on this for many times in all your assignments, is equivalent to saying angular momentum is conserved or equivalently Hamiltonian should commute with the generators which perform this notation which are the components of your angular momentum and we are trying to put it in rotation in physical space. So it is orbital angular momentum. If you can understand this, this follows straightforward. And we will go over this a little throw slowly.

(Refer Slide Time: 15:05)



So take a vector a, rotate it by an angle theta about any axis given by a unit vector at n, can take this at axis to be z axis for convenience but you can take it to be any arbitrary axis. Will the vector a remains a? It will not. It will change under rotation but you also know that a dot product of 2 vectors will not change in the rotation, right. Radial coordinate does not change in the rotation or any vector a dot b will not change.

The reason is the rotation matrix, what is this condition called? Called orthogonal matrices, right. R and R transpose, its identity is the orthogonality condition or orthogonal matrices, R is supposed to be orthogonal matrices. Why? It has real entries only. No complex entries. If it had complex entries, then it would have become a unitary matrix. So orthogonal matrix is also a subset of unitary matrices.

Once you put them to be real entries. Otherwise, you have to do complex conjugate and then the transpose. Since they are real entries, you do not need to do a complex one. Just do a transpose. So they are also unitary matrices and since the elements of the rotation matrices are real, the matrices are also unitary and you can write the inverse of a rotation by an angle theta, can be written as rotation by a negative value, right.

A clockwise rotation by delta theta will also be like looking at it as the other rotation by a, in the other sense as a minus, you all agree with that, right, okay. So the explicit matrix representation so far it is just formally written but you can give a matrix representation for it involving rotation angle theta and I am doing it about z axis for you to see this. So you can collect all these R matrices, okay.

If you take a product of this R matrix with another R matrix for some other rotation about x axis, you will get a new 3*3 matrix but each of those new 3*3 matrix you can show that it is always satisfying this property R transpose=identity, okay. So whenever you have a collection, so this will be a set of infinite matrices you can have. You agree? You can change theta. You can change the axis but the 3 axes are the 3 independent axes which you have, product of any 2 R matrices will be a new R matrix which again satisfies this property.

So the set of matrices and we also have one more condition, what is the determinant of this R matrix? 1. The determinant of this matrix is 1. So this collection of matrices for every matrix you also have an inverse matrix. You just have to change the sense of rotation, theta to -theta. So this set which is the collection of all these R matrices and which includes the inverse and the product of 2 R matrices again an R matrix which satisfies this property, that is symbolically denoted, that set is symbolically denoted as a group.

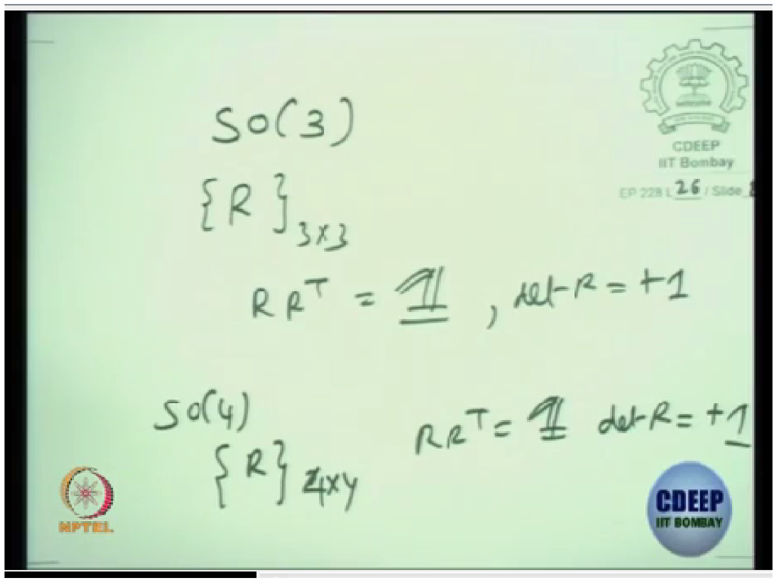
It is an orthogonal group, that is why they put an O. Orthogonal group of rotations in 3 dimensions. So there is a 3 here. The S is because we are doing proper rotation which is determinant of proper rotation matrices are all 1, okay. Formally they call this collection of all the rotation matrices, that set satisfies all the properties which are required that is why the set is called a group.

Otherwise, you cannot call it as a group. You need a set of axioms. And symbolically this is a short hand notation orthogonal matrices of rotations in 3 spatial dimension and we are looking at proper rotation collection and that is denoted by S. S denotes determinant of these matrices are +1. So SO3 is a fancy language. If you go and see any book, you will see SO3 is a group of rotations.

So this is nothing but a collection of all the rotation matrices about any arbitrary axis, okay. Do not get scarred by this notation. So I will ask you what is SO4? The same way set of proper rotations in 4 spatial dimension, 4 spatial you are not able to see but suppose you think that we are in 4 spatial dimension, you will have a 4*4 matrix, right. And can I talk about in 4 dimension as a rotation about an axis? See in 3 dimension, whenever you do a rotation, what exactly happens?

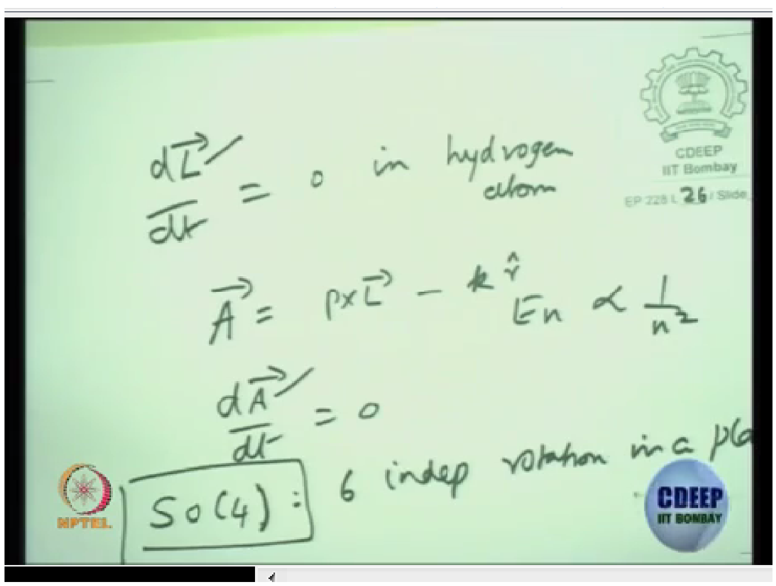
When you do a rotation about z axis, what exactly happens? The xy coordinate of the a vector gets modified. So the one which is orthogonal to the xy coordinate, we call it as z axis but if you are in 4 dimension, xy getting modified. There are 2 more coordinates. So you cannot talk about rotation about an axis once you go beyond 3 dimension, okay. So we will always say it is a rotation in the xy plane. Similarly, rotation in the xz plane and so on, which you do not need to do when we are doing 3 dimension.

(Refer Slide Time: 21:35)



So SO3 you have all understood now is a collection of rotation matrices 3*3 which satisfies RR transpose to be identity, right. I am just trying to tell you it is SO4 is again a collection of some R matrices which is 4*4 with RR transpose to be identity and determinant of R is 1, okay. That is the definition law. But why I am trying to push you on to this so for is that if you know your hydrogen atom problem, okay, now hydrogen atom problem you all know angular momentum is conserved.

(Refer Slide Time: 22:24)



So you have angular momentum in hydrogen atom. Classical mechanics I am sure you have all done, Runge Lenz vector, right. What is Runge Lenz vector? They denote it by p*L-kr hat, something like this. Is this also conserved? Classical mechanics, I am sure you would have done this in the Kepler law. So da/dt is also conserved. If you want to put all the 6, there are 3 which are here which is conserved.

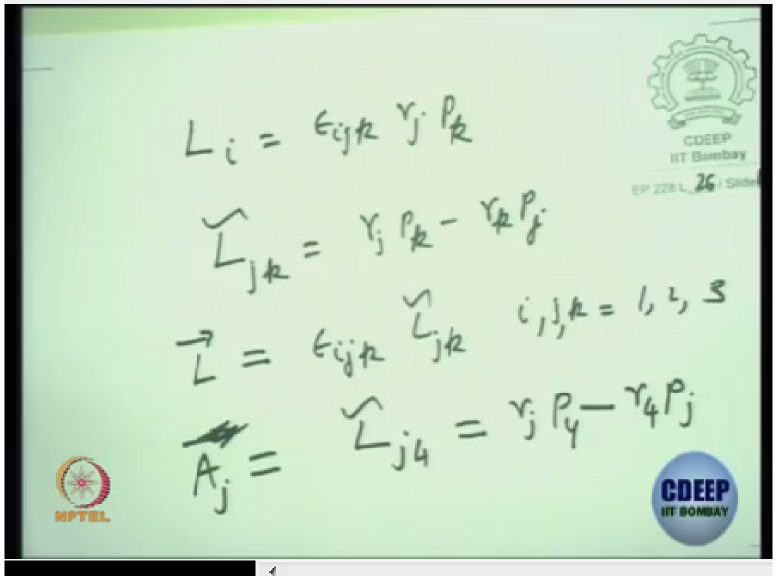
There are 3 more which are conserved. The want to put all of them together, I need to go to SO4. SO4 as I said it is a rotation in a plane. How many possible rotation in a plane basis you can have? Rotation about axis, there were 3 angular momentums. x axis, y axis and z axis. In 4 dimension if I want to look at rotation about a plane or rotation in a plane, how many independent planes are there? 6, right.

There are 6 independent rotation in a plane. So these are 6 independent parameters. I need 6 independent operators which will do. So there are 3 of them and 3 of them. They will continue to behave like I am looking at an SO4 select. So we will do this if I have time towards the end but whoever is interested, can go and look at shift quantum mechanics book symmetries in quantum mechanics, various trials, specifically address this and you automatically if you invoke this symmetry, you can get En to be proportional to 1/n squared without doing any calculation.

So this is something which is beautiful that you know how much we struggled to get 1/n squared energy by doing the Schrodinger equation, separation of variables and so on but SO3 group can be extended to SO4 but then you will ask, what is the fourth coordinate in hydrogen atom, right. That is one question you can all ask.

No, just take it to be an abstract p4 and R4 like xyz can call it as R1 R2 R3 R4 and p1 p2 p3 p4 and rewrite your Runge-Lenz vectors in terms of the p4 and R4 and beautifully, okay like this Schwinger formalism I get to showing of how to get the angular momentum J+ and J- eigen values to be determined or the coefficients to be determinant, you can do this, okay. So this has to motivate you.

(Refer Slide Time: 25:28)



So what I am saying is you wrote Li to be epsilon ijk rj pk, right. Now I am trying to say that this is not the right way to write. We will write Ljk as rjpk-rkpj. I am saying let us do that is the rotation in the jk plane, okay. And now I am trying to say that we will call the actual angular momentum vectors which you are all familiar, maybe you can use this to be ~, actual angular momentum is nothing but it is you epsilon ijk L~jk where I, j and k are 1, 2 and 3.

You can write your A vector as Lj4~ by putting a fourth fictitious coordinate. What is this? Lj4 is rjp4-r4pj. This is where the Runge Lenz vector components, so this is Aj, can be rewritten. Think about it just to kindle some of you who like to go and look at some things. So the 3 generators of the angular momentum you still can work with as if you are in 3 spatial dimension.

But if you want to understand the Runge Lenz vector which is conserved, you can introduce 1 more fictitious coordinate which is the fourth coordinate for the position as well as the momentum and write this to be the 4 components of the Runge Lenz vector, okay. So let me stop here.