Quantum Mechanics
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Lecture - 52
Oscillator Algebra

Okay so today I am going to use the oscillator algebra applications whatever we learnt in oscillators to solve problem which we have done explicitly using the position space wave function for a particle in a magnetic field and then later on we will study angular momentum algebra using this oscillator approach okay. So that is the plan. If you recall, we did this in the last lecture last week.

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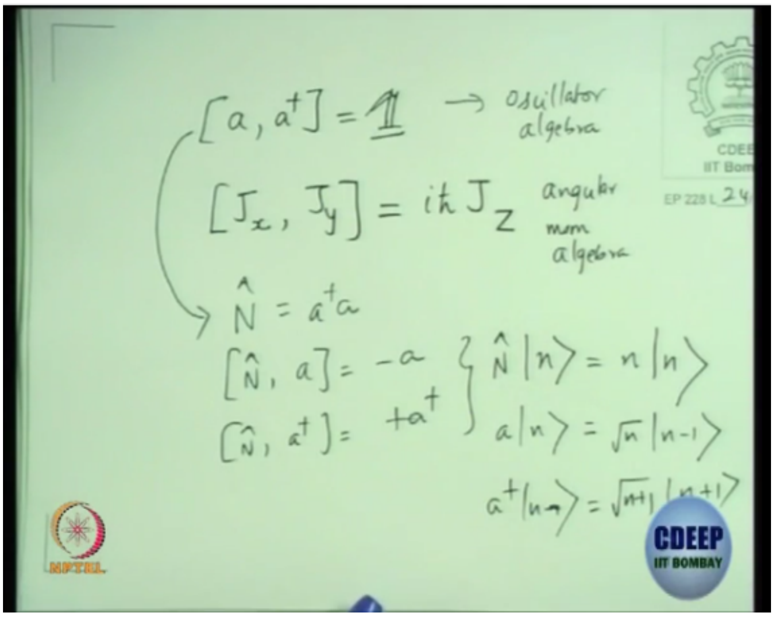

The harmonic oscillator we had a with a dagger is identity, this is what we will call it as an oscillator algebra and Jx with Jy is ih cross Jz. This is what we will call it as an angular momentum algebra. So the number operator for the oscillator algebra is an eigenstate n with the number operator with eigenvalue n and with the number operator the two ladder operators has this commutator relation.

And from this we can determine a on n is square root n times n-1, it is a lowering operator and similarly a dagger is the raising operator okay. So using this oscillator algebra, we would like to solve this problem of particle in a magnetic field and then can we use this oscillator algebra to determine the spectrum for the angle of momentum also. So this is the motivation for today's lecture okay.

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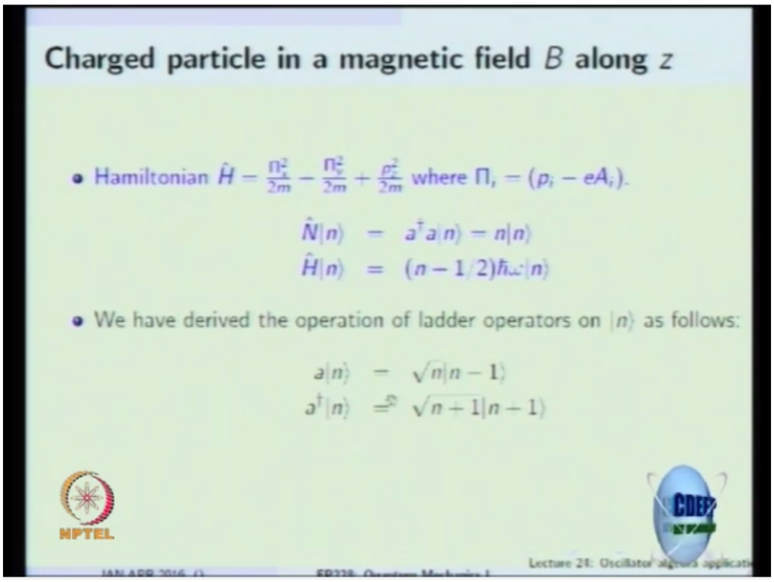

Recall the particle in a magnetic field. Let us take the magnetic field along z-direction. So this is given by this Hamiltonian which is pi x pi y are in terms of it is proportional to momentum minus the shift due to the vector potential. It pi-Ai and as I said already recall the number operator eigenstate is the simultaneous eigenstate of the Hamiltonian with eigenvalue proportional to h cross omega.

So in this problem this is for a general harmonic oscillator. What does omega will correspond here we will figure it out. So the ladder operator in any harmonic oscillator has this property as I have already discussed for you.

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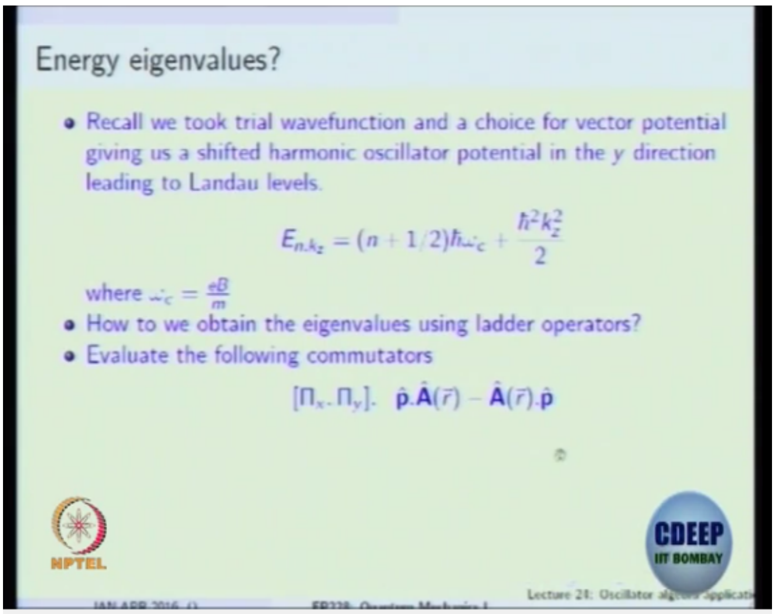

And in one of the earlier lecture, I explicitly went through a trial wave function and a choice of a vector potential and derived the energy spectrum to have the frequency with which we will call it as a cyclotron frequency and it is like a free particle along the z-direction. So this is what we called as Landau levels. How do we obtain such an energy eigenvalues using these ladder operators which we saw in the harmonic oscillator?

So couple of things I want you to work it out is commutator of the x component with y component for pi x and pi y and similarly we know that p with A and A with p that order really matters, p dot A is not same as A dot p. So just to take you on how.

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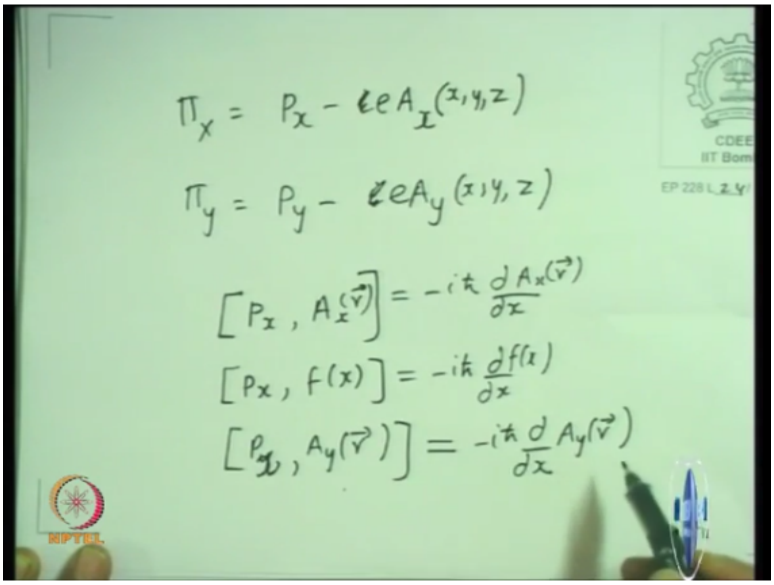

In general, the potentials are functions of positions. So pi x is Px-eAx which is a function of positions. So if you take Px with Ax that commutator is not 0, it will be –ih cross del/del x of Ax. This is because you know x, p commutator, px with x is nonzero and if you have any function of x, it should be del/del x of f of x. Similarly, if you try to find the Px commutator with Ay, you are expected to be –ih cross del/del x of Ay.

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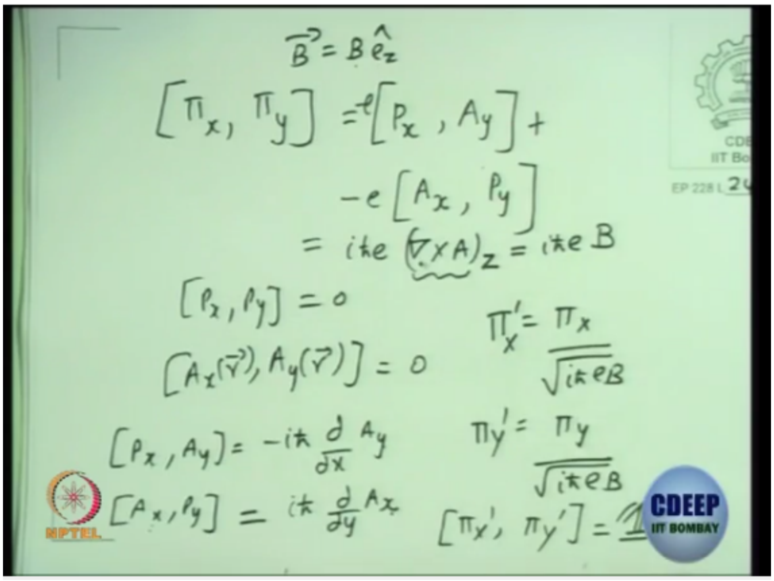

So if you want to work out using these data, use these data and let us work out for a particle in a magnetic field along z-direction magnitude is B along z-direction, try to compute the pi x with pi y, the nonzero commutators are Px with Ay and Ax with Py and that gives us a constant okay. So the pi x with pi y is not 0 like Px and Py but this is nonzero and it is the constant depends on the magnetic field.

You can normalize the pi x by this normalization, so that you can make pi x prime with pi y prime to be identity. So this is what I said in the earlier lecture, earlier slide Px with Ay is del/del x of Ay and Px with Py is del/del y of Ax and if we use that here you get curl of A which is nothing but it is a (()) (05:47).

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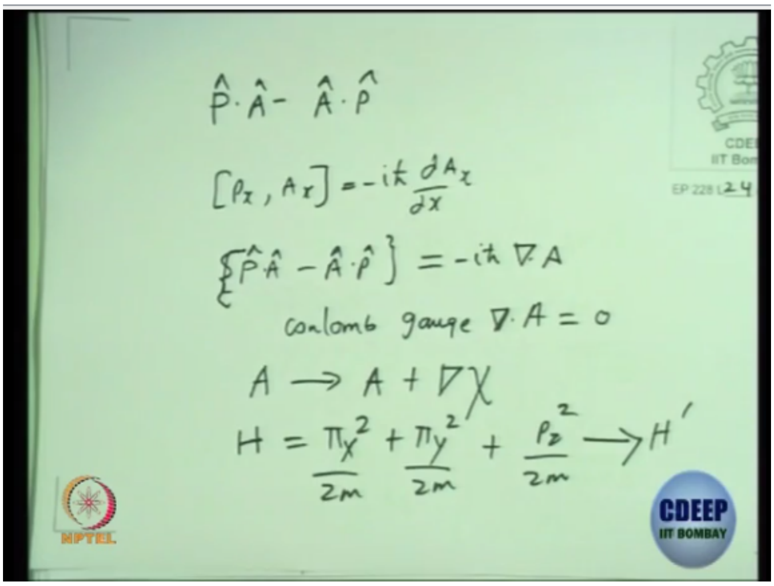

Similarly, the P dot A-A dot P Px with Ax is del/del x of Ax. P dot A-A dot P is nothing but – ih cross divergence of A or del dot A. We can choose a gauge which is known to us as Coulomb gauge where del dot A is 0 and recall that the vector potential can be changed from by a gauge transformation by a gradient of chi but this change will not affect your magnetic field, magnetic field will remain same under this gauge transformation but what happens to the Hamiltonian. Hamiltonian can in principle change to H bra.

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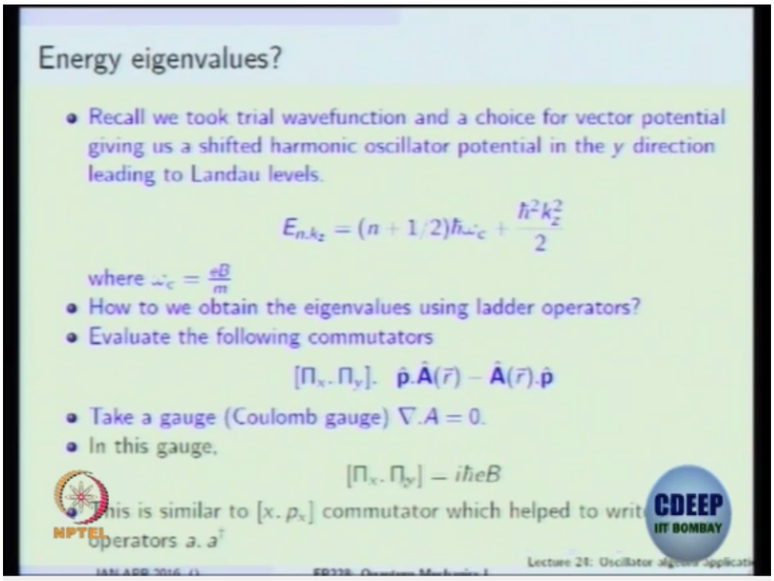

So we worked out pi x and pi y and P dot A and A dot P. Take the gauge which is the Coulomb gauge and if this gauge pi x pi y is ih cross eB and this is similar to our familiar x, px commutator which helped us to write a and a dagger okay. So that is what is the motivation, just like we had in the harmonic oscillator x squared+px squared putting n, omega everything to be 1 here.

We have a pi x squared and pi y squared and similar to x, px commutator which is constant and nonzero we have pi x with pi y to be nonzero and a constant. So using this we can try to construct a new ladder operators involving pi x and pi y.

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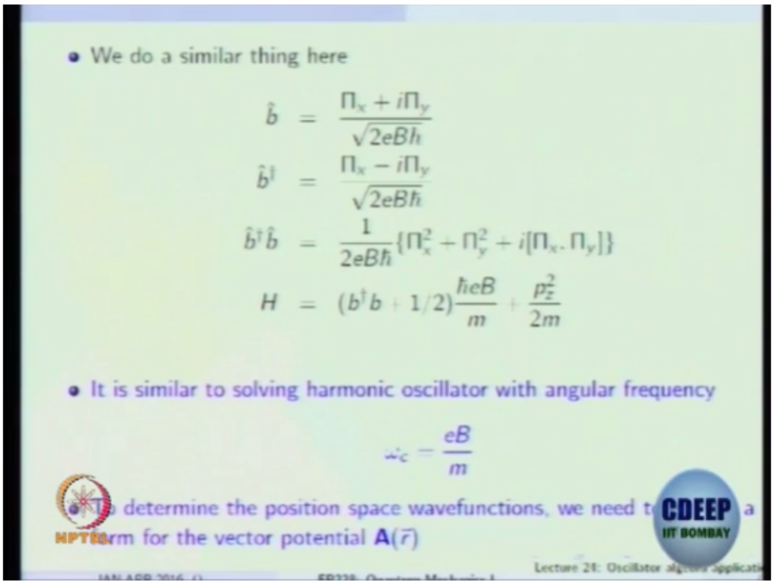

So this is what we do here. We try to take the ladder operators as pi x+i pi y with this normalization and similarly the conjugate Hermitian conjugate of that and if you try to find the commutator, please check the commutator and you will be able to show that this is identity and also construct the number operator which is b dagger b and turns out to be please verify this but we need in the Hamiltonian pi x squared+pi y squared.

So you can take this term to the left hand side and try to rewrite your Hamiltonian and rewritten Hamiltonian is exactly similar to what you would have seen in the harmonic oscillator with the b dagger b is like the number operator, the frequency omega is eB/m which is your familiar cyclotron frequency. So once you have this Hamiltonian rewritten looking like a harmonic oscillator you can read off the spectrum as n+1/2 h cross omega c+free particle along z direction which is h cross (()) (08:56).

The thing which we did if you recall, we worked out the position space wave function and we took (()) (09:05) trial function for it but if you want to work it out, you have to make a great choice okay.

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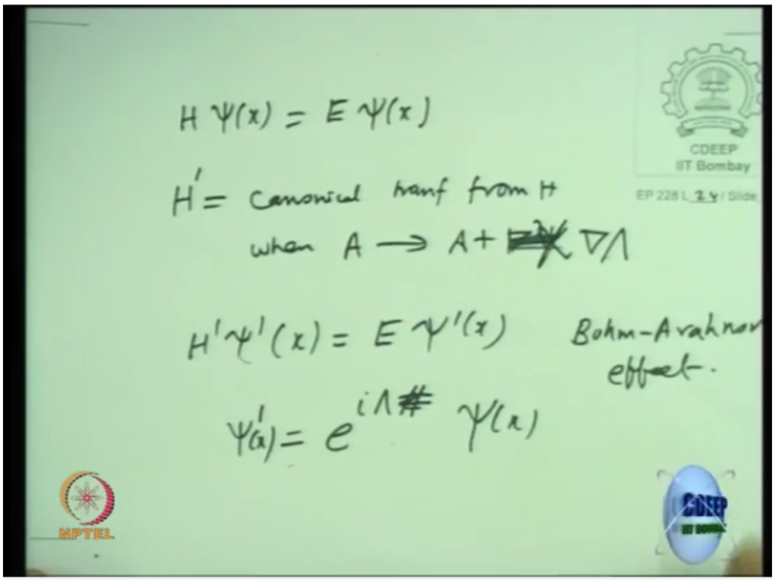

So just for completeness, the Hamiltonian the particle in a magnetic field Hamiltonian, suppose we take the wave function energy eigenvalue is E. By doing a gauge transformation A to A+ gradient of lambda, you could get a new Hamiltonian but this Hamiltonian is what we could call it as a canonical transformation of the original Hamiltonian because the energy spectrum remains the same, it is just that wave function will pick up an overall phase factor and no physics changes if you pick up the phase factors.

But we will see in some situations like Aharonov-Bohm effect where such phase factors will become important. If anybody is interested, can take a look at Bohm-Aharonov effect okay. So this is what we did in the class a couple of lectures ago.

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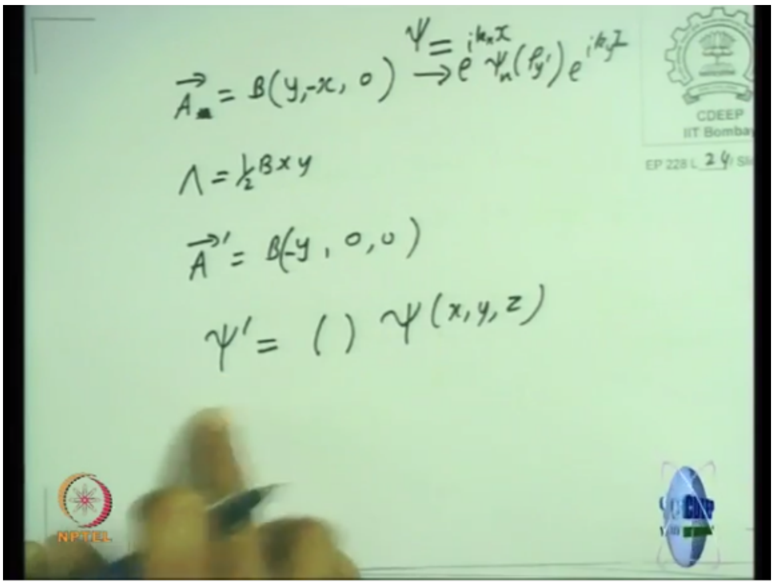

Maybe yeah where we took a choice for A and a trial wave function but you can also see by doing a gauge transformation using the lambda gauge parameter. You could actually get a new vector potential and try to see how the wave function changes by a phase factor okay.

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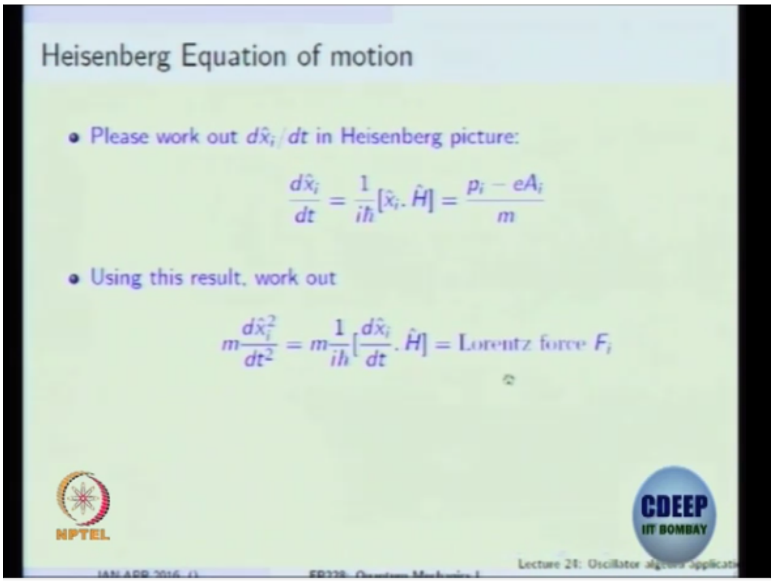

So now that we have got the energy spectrum if you recall for this Hamiltonian, I said you should get the Lorentz force, I do not know how many of you have already worked it out but if you have not recall Lorentz force theorem and that should be obtainable in the Heisenberg picture writing the Heisenberg equations of motion. So in the Heisenberg picture workout dxi/dt which is given by your familiar commutator of xi with the Hamiltonian.

So if we work it out, you will be able to see this. We would like to work out the m d squared xi/dt squared okay. So this is something which you need to work it out. So if d/dt of dxi/dt so the operator is dxi/dt which will show up on the next right hand side with the commutator with the Hamiltonian. So this I want you to work it out and what is your expectation. This is the force law for a particle in a magnetic field.

So this should be proportional to your Lorentz force Fi yeah. So here this is your classical force equation where F is the Lorentz force.

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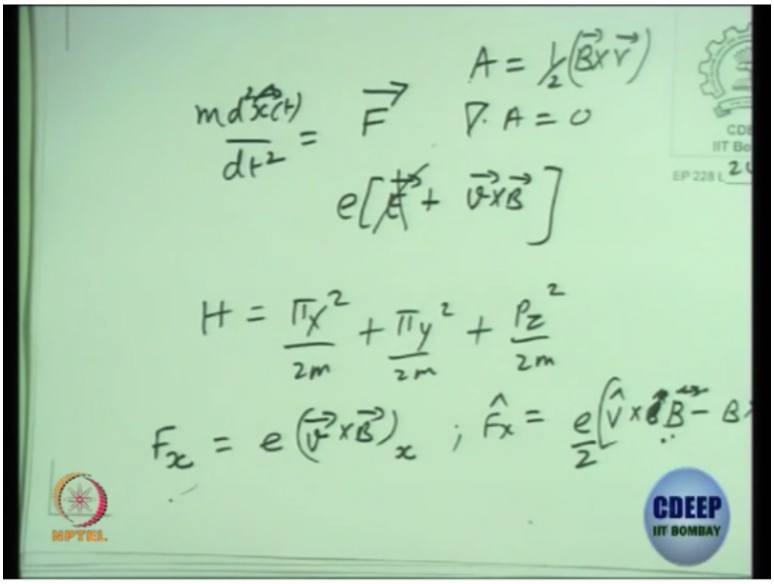

And let us work with the case where the electric field is zero, this is what we are doing now. If you are taking a particle in a magnetic field and this is the Hamiltonian x component of the force, I would have written it as P cross P in classical physics but if you go to quantum mechanics you have to write a Hermitian operator. In fact, when you write the Heisenberg's equation of motion for d squared xi/dt squared you will exactly get this combination V cross B-B cross V will show up there okay please check it.

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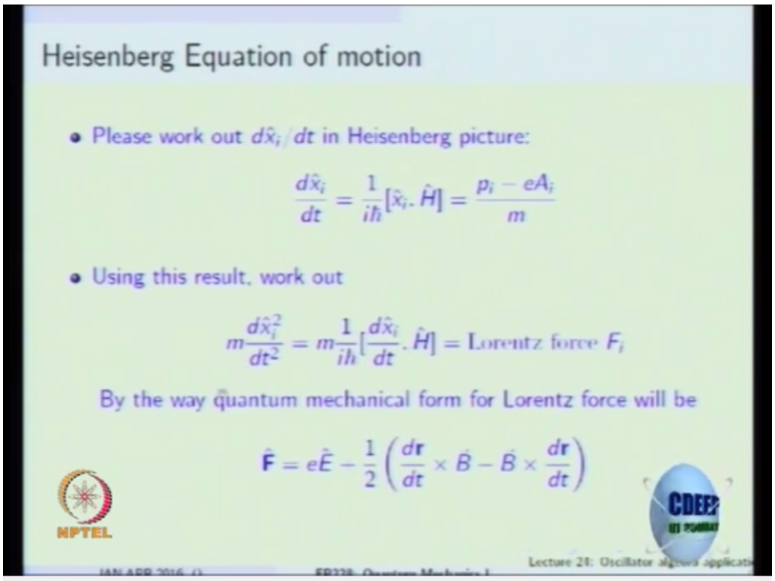

So please check that this force the quantum mechanical force form for the Lorentz force should have the Hermitian operator which is the combination of dr/dt cross B-B cross dr/dt for the Lorentz force and we can set E to be 0 because we are working with (()) (13:03) is 0 okay. So far I have tried to show how using the harmonic oscillator algebra, trying to write the Hamiltonian for a particle in a magnetic field in terms of the ladder operators we could get the spectrum.

And I have also tried to use the Heisenberg's equations of motion and derived for you the familiar Lorentz force.

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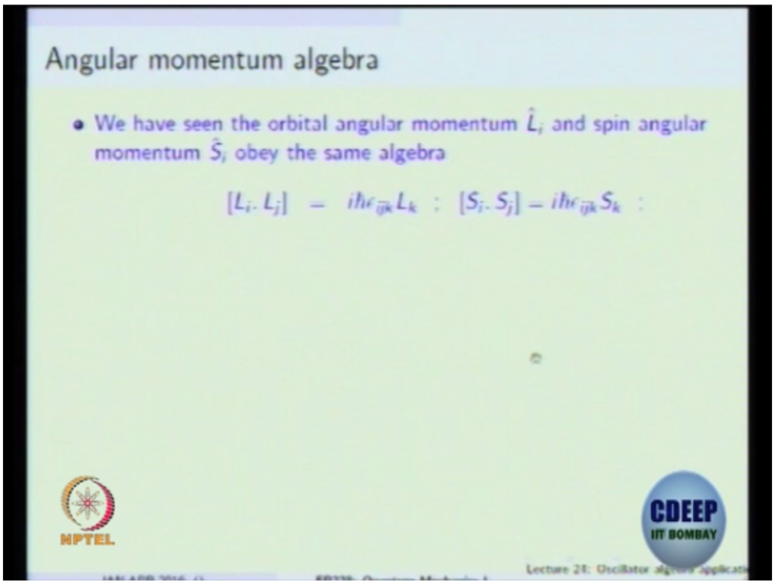

This will move on to the angular momentum algebra taking some of these data of these ladder operators and whether we can find the states for the angular momentum operators. So recall the orbital angular momentum and the spin angular momentum which we discussed last week for the Stern-Gerlach experiment.

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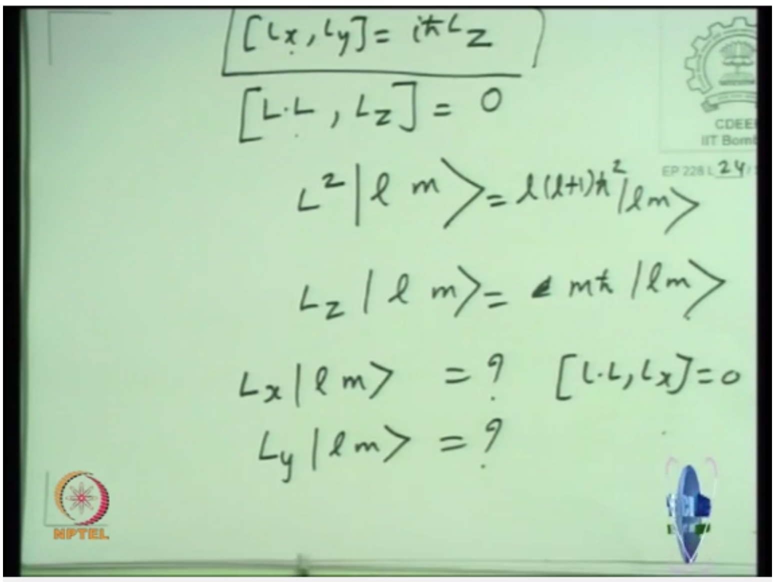

So Lx Ly is ih cross Lz and L dot L commutes with all the components, for conventions we are going to take it with respect to Lz and we write it to be a simultaneous eigenstate of Lz and L dot L and these are the eigenvalue equations with eigenvalues which we have seen when we discussed this hydrogen atom and the questions we should ask is how Lx operates on lm and similarly how Ly operates on lm okay. What do we get are the questions which you would like to know?

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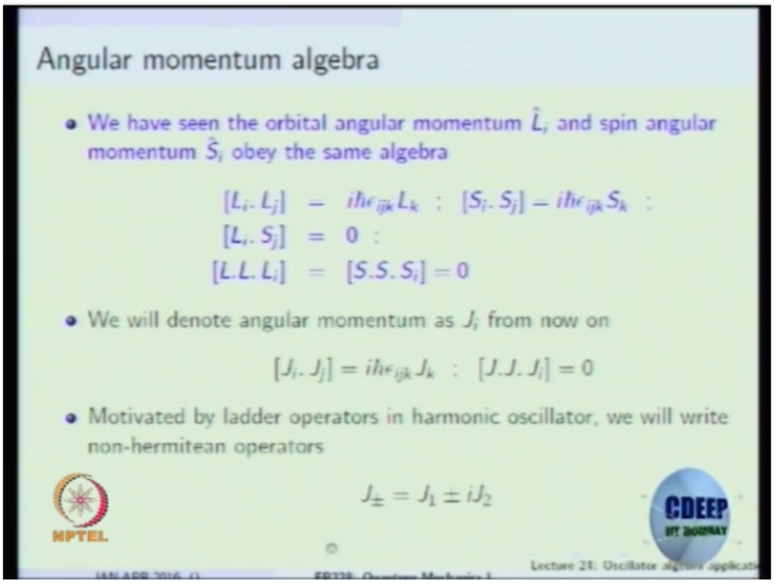

So we have seen orbital angular momentum and spin angular momentum. They obey the same algebra. This is what I tried to motivate you from Stern-Gerlach experiment. It exactly behaves the algebra of the spin operator should be exactly similar to what we have plus they are in two different spaces. They have to be 0 and we have this condition which is the simultaneous eigenstate of L squared and Lz.

We are checking the i to be the z component for convenience and universally following whatever books are doing and similarly you can have it to be a simultaneous eigenstates of S squared and this is it. (()) (15:18) trying to talk whether we are talking about orbital angular momentum or spin angular momentum. We will compactly write angular momentum by a symbol J and the J will satisfy the same property.

And we will write eigenstates of J squared and Jz to be the simultaneous eigenstates. So we have done this all ladder operators. The ladder operators are not Hermitian operators. You cannot experimentally observe them. A is not equal to A dagger that is why you have two ladder operators; they are not Hermitian. Similarly, we would like to do a ladder operator here using these angular momentum Jx, Jy and Jz okay.

Motivated by these harmonic oscillator ladder operators let us write J+ as J1+iJ2. I am using 1 2 and 3 as x y and z okay and J-s J1-iJ2. It is similar to the angular momentum operators which I am writing J+ and J- are looking similar to the way we wrote non-Hermitian operators in the oscillator algebra. What is the next thing? I had this Jx Jy to be ih cross Jz. I would like to write the commutator of J+ with J-, J+ with Jz and J- with Jz. So that I can try to play around and find how to go about this problem.

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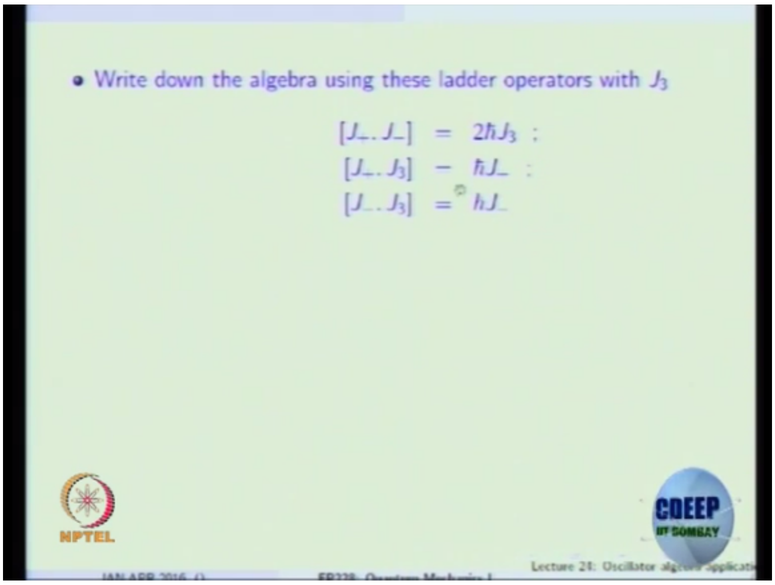

So write down the algebra using these ladder operators which is J+ or – with J3. What will this be? Can you work it out?

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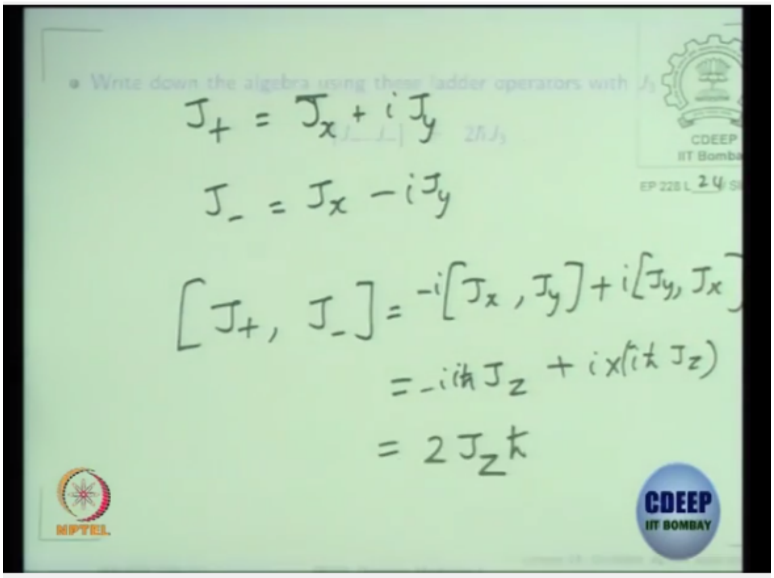

What is this? The negative side right. Is that right? What is this answer turning out to be? So it is twice h bar J3 and do the J+ with J3 excellent. So it is +h cross J+ or –h cross. So the same algebra of Jx Jy Jz which we wrote, I am trying to rewrite yeah J+ with J- what it is which is like your a and a dagger, something wrong – okay. Both of them are – yeah, sorry. The third one is + and second one is – okay I will correct it please try to.

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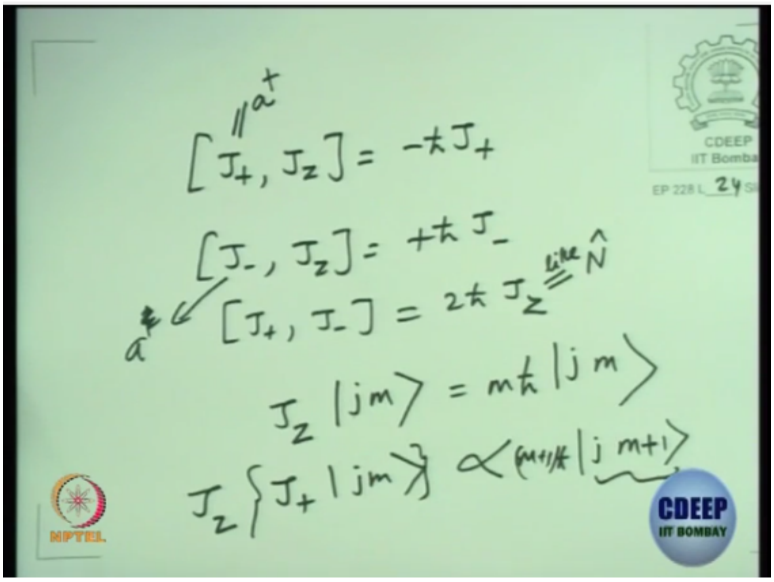

So J+ with J3 or Jz is –h cross J+ and J- with Jz is +h cross J- and J+ with J- is twice h cross Jz. Now let us compare with the harmonic oscillator, kind of said that the states are eigenstates of Jz from the hydrogen atom experience right. So this is similar to your number operator (()) (19:28) not exactly okay. It is like Jz operator is like number operator. If I write my state as jm with Jz operator, it gives you m h cross jm.

We have one more because we know J dot J it is an eigenstate of J dot J that is why I put this additional index. Otherwise, I could have just work with m okay. The maximal compatible set for an angular momentum algebra is simultaneous eigenstate of J dot J and Jz that is why I have two integers. So Jz is like your number operator in harmonic oscillator. It is an eigenstate with eigenvalue mh cross.

Then, what does it tell us? J+ and J- are like your, I do not know one of them will be a and the other one will be probably a dagger or the other way around okay. So this algebra will actually tell you what happens to J+ on jm. If you remember when we did the oscillator, we tried to say that a on n, how did we argue a on n? a on n went to I do not know whether this is m-1 or m+1 so I think it is like a dagger and this is like a.

Rewrite it that way, then this will become a +sign, that will become a –sign. Then, that will be like the a dagger and a excellent. This has to make you think also and do it. So use this algebra and try to figure out that it is like a ladder operator. It is not an eigenstate but this one will be an eigenstate right. This will be an eigenstate of Jz with eigenvalue m+1 h cross that is why we wrote the state as use this algebra.

Similarly, the next one to figure out that it will be a lowering operator. You see the power of algebra now. How we are exploiting the power of algebra. What is left? You have to determine these coefficients that is right. So that is from the hydrogen atom data but we will do it independently now and show that how the restrictions are that is all. So those things will come to, those things are not visible here.

I am just saying it is like. If it is exactly harmonic oscillators, then nothing more to do. As he is pointing it out, there is no information that m goes from –j to +j and you cannot have raising operator going to infinity like the way you did it in harmonic oscillator okay. So we need to fix those things, will come to fixing that. I am just saying that you can if jm is an eigenstate given to you which is a simultaneous eigenstate of J squared and Jz, I can actually get the trend how the J+ and J – operates on it from this algebra.

That is all I said but this coefficient of proportionality has to be fixed and also these issues that m should go from -J to +J also J could be integers or half-odd-integers. All these things are not fixed right now. I am just using hydrogen atom data and just doing this but we will now do explicitly to figure out all these informations.

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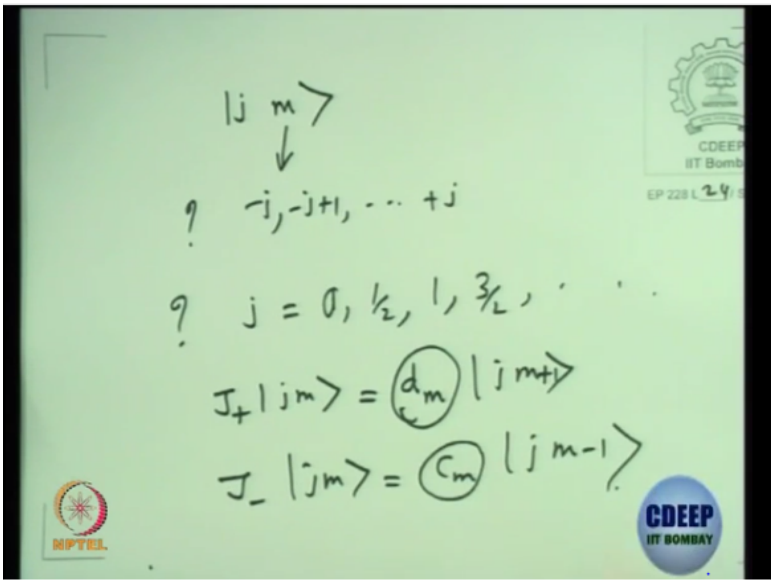

So I need to figure out that j and m, m goes from –j, -j+1, till +J. This has to be figured out. You have to also figure out whether J could be 0, 1/2, 1, 3/2 you know these are the possibilities has to be. In the orbital angular momentum in hydrogen atom, it was only integers. Now we have to naturally see whether we can get half-odd-integers. This is also to be figured out.

What else I need to also figure out jm as dm on jm, what is this dm, J- on jm is some Cm on jm-1. We need to figure this out also where some of these things have not been fixed as of now. So we need to fix this. Is this clear? The problem clear, how do we do this? Yeah, so there is another nice trick which we are going to do. This is what is the Schwinger method. Let us do that just as okay.

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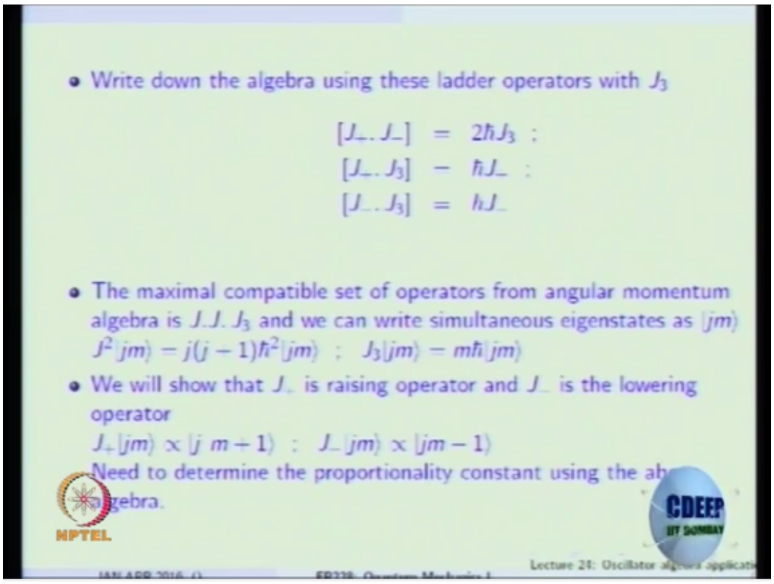

So this is things I have said except for this negative sign which is here. So all the other things are fine and we can write simultaneous eigenstates jm which is an eigenstate of J squared and J3 and we will show that J+ is a raising operator and J- is a lowering operator. This also you can see from this algebra is what I was trying to tell you. Only proportionality constant has to be figured out using this above algebra.

Not only that as he has already pointed out that we need to figure out m takes values from -J to +J. You have to also figure out that J could be integers, could be half-odd-integers. All these possibilities because if you lose Stern-Gerlach J is actually half. If you do orbital angular momentum, J is actually integers. So all possibilities are allowed has to be figured out. We have not still done that okay. So for that we use this Schwinger oscillator method.

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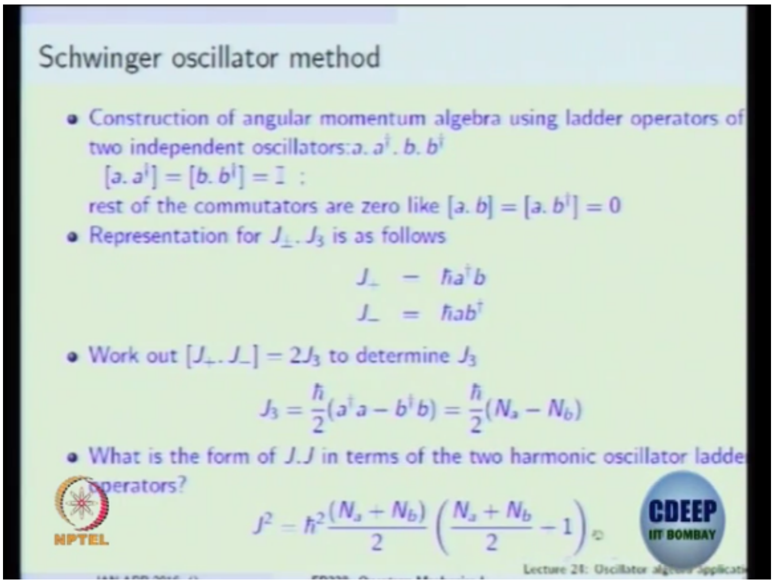

What is the theme? Theme is to generate the angular momentum algebra using two independent oscillator algebras. There are two independent oscillators for the system, one with a and a dagger, another one with b and b dagger. Do not confuse this B with what I wrote with pi x+ so this is just a two independent oscillators. You can call a1 a1 dagger a2 a2 dagger if you want but you can as two independent oscillators.

So we are going to write address. This is what is called representation right. You can represent the angular momentum ladder operators J+ J- and Jz or J3 as follows. Let us take two ladder operators with one of them as raising and another one does lowering. By independent, I mean that between A and B, the commutator is always 0, a and a dagger will is harmonic oscillator algebra, b with b dagger is harmonic oscillator algebra.

Between a and b or a and b dagger or a dagger and b you know a dagger with b dagger they are all 0. This is what I mean by rest of the commutators are 0. They do not talk to each other. They are independent. Using two independent harmonic oscillator, we could write a product of two ladder operators. If you note here, one ladder operator is like raising, the other one is like lowering.

It is just combination which we take; the J- is Hermitian conjugate of this. You can take the dagger of this and you should get the J-. Order is really does not matter because whether I put first b and next a or a and b does not matter because they commute. I can work with this. So now to find what is J3 substitute these two here and determine what you get for J3. I will leave it to you to do this. Please try this out. J3 should be a function of again involving the 4 ladder operators okay.

Please try this out and interestingly if you work this out, you will find that you get it to be a difference between the number operators. Please work this out okay. Similarly, work out what is J dot J. Please work out what is J dot J. Na is the number operator for the harmonic oscillator with ladder operators a and a dagger so Na is nothing but it is a dagger a. Nb is the harmonic oscillator with ladder operators b and b dagger.

I am not bothered about the frequencies of the two oscillators. For me I am more interested in constructing the angular momentum algebra in terms of the two harmonic oscillator ladder operators. Is it okay? So please work this out J dot J and tell me what you get okay. Please very whether you get this. Let me stop here today.