# Quantum Mechanics

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## Lecture - 51
## Stern-Gerlach Experiment- II

So let us try to write the Sx and the sigma x as the next step using the experimental data. We have written Sz assuming the upstate is an eigenstate of Sz and downstate is an eigenstate of Sz with the eigenvalues as +1/2 h cross with the upstate and -1/2 h cross with the downstate okay. In the same eigenstate of Sz, we would like to write the Sx operator and Sy operator and we know that those operators should be such that it should be traceless.

If you try to find eigenvalues since it splits into two beams, the eigenvalues of those states should be again +1/2 h cross or -1/2 h cross. All these data we know but we still have to find what is Sx. How can you go about it? One is to use this property and try to figure out what is the 2 x 2 matrix which will satisfy this and you can figure it out. Other one is to use logics using the Stern-Gerlach apparatus.

And try and figure out what is Sx and Sy with these stringent conditions. They have to be traceless, eigenvalues have to be of the sigma x and sigma y have to be +or-1 okay. You can use various methods and try and figure this out okay.

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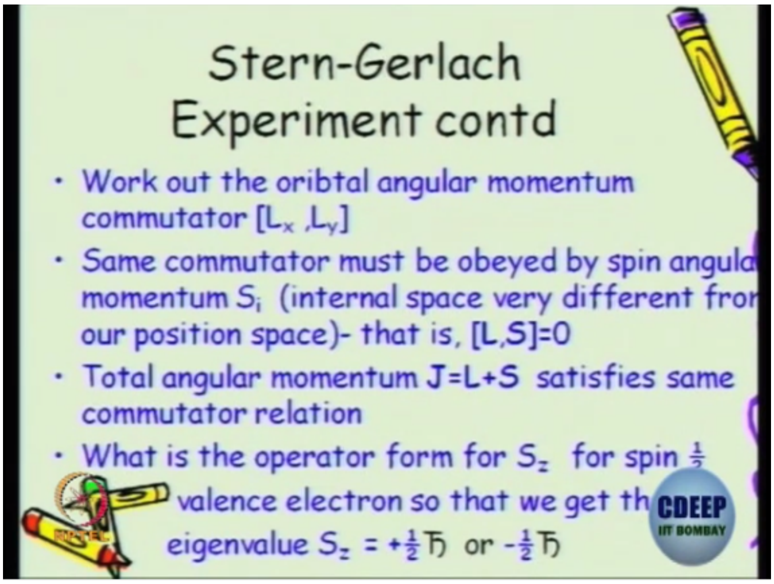

So what is the operator form for Sz for spin 1/2 is the question. Just look at only the valence electron that we get two states with eigenvalues Sz=+1/2 h cross and Sz=-1/2 h cross.

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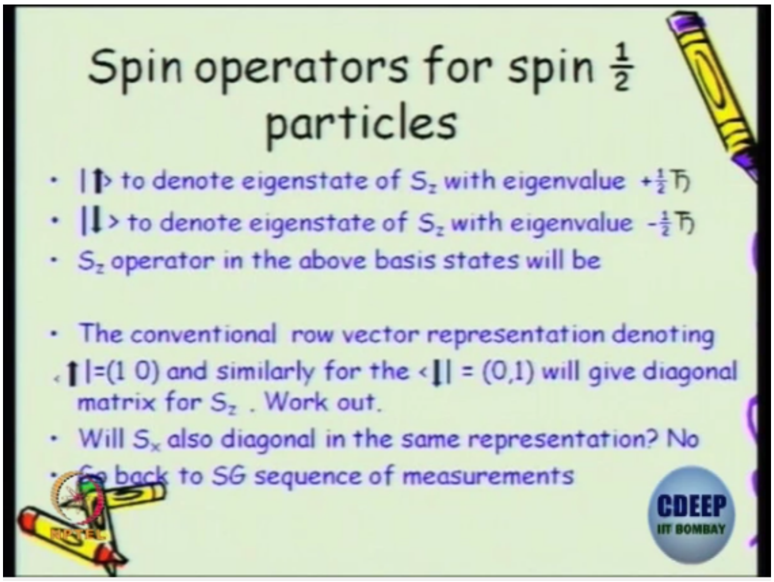


We denote symbolically as an up spin for the eigenstate of Sz with eigenvalue +1/2 h cross and similarly down to denote eigenstate with eigenvalue -1/2 h cross. Sz operator in the above basis you can write upstate and downstate the corresponding dual vector as a row matrix, row vector and you can work it out. Will Sx also be diagonal in the same representation? No, that we know because they do not, they are incompatible observables.

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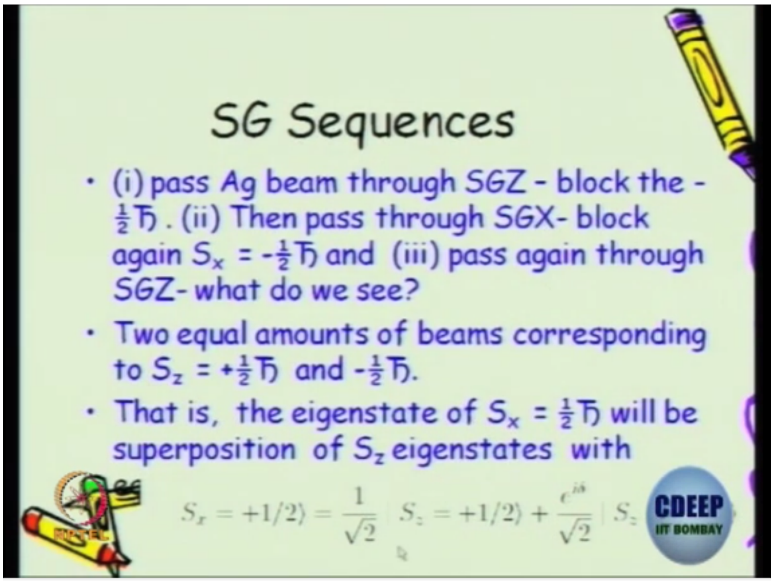


Go back to the Stern-Gerlach sequence of measurements which we did and what do we say that the Sx eigenvalue is +1/2, I am suppressing the h cross now, let us not worry about it, you can put it back later. Sx=+1/2 h cross after you make it go through the SGX, the state with +1/2 will be a superposition. In general, will be a superposition of the two basis states okay. Experimentally, we saw that it was equal intensity.

So we are going to put the magnitude as 1/root 2 and 1/root 2. We want to normalize the states. There could be a theoretical phase factor here which will not be experimentally detected but you can in principle have a relative phase factor. Overall phase factor is useless but you could have a relative phase factor. This is the most general state, one of the eigenstates of the Sx operator can be written.

The other eigenstate should be orthogonal to this state. The Sx=-1/2 will again have equal intensity but the coefficient should be such that those two states are orthogonal and also that state per say should also be normalized. In fact, I could have put an up spin here and a down spin here just to make notation simpler but you know both the notations okay.

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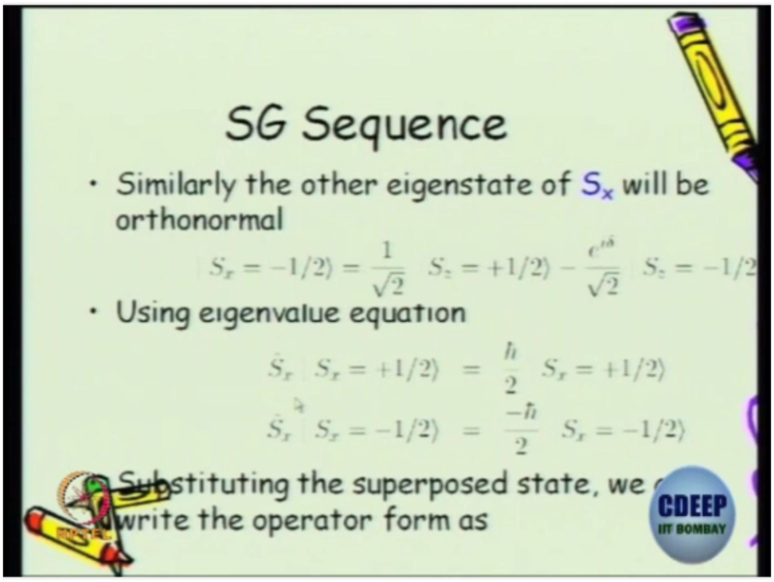


Similarly, the other eigenstate as I said I put a negative sign here, so that they will be orthogonal and use the eigenvalue equation for the Sx operator. What is the eigenvalue equation for the Sx operator? Same similar to what you had for the Sz states but now it should be eigenstates of Sx operator but these eigenstates are not eigenstates of Sz operator. That is why we wrote them to be a superposition of Sz states.

Use this and using this data, I want you to fix what is the matrix form or the operator form for Sx. I will leave it to you as an exercise. Do it and check it out the 4 elements in the 2 x 2 matrix. Please try to work it out the way I did it for the sigma z using these equations and the previous data. These two data you use and using these two equations try to figure it out what should be the matrix representation or the operator representation for the Sx operator.

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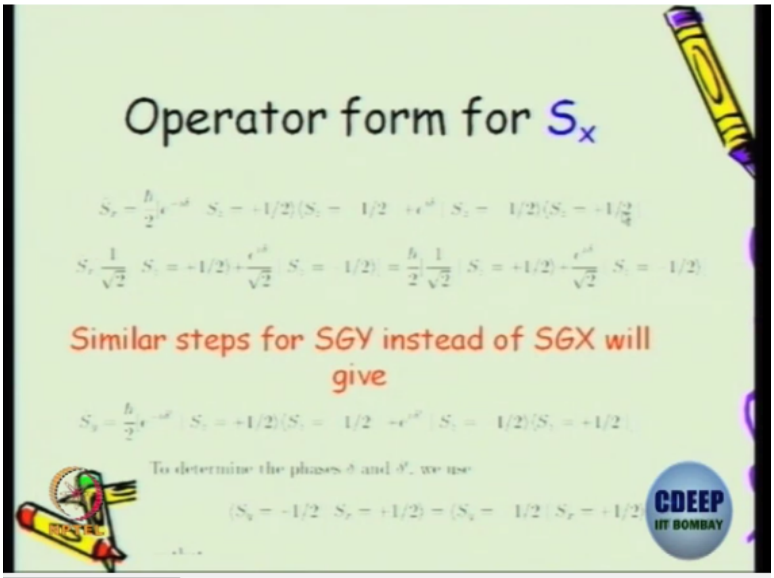


It is off diagonal matrix. Please verify, after you have done that work you can verify whether I get this with these appropriate phase factors okay. So this is an outer product of the up down and similarly down up. How will you write the matrix for this?

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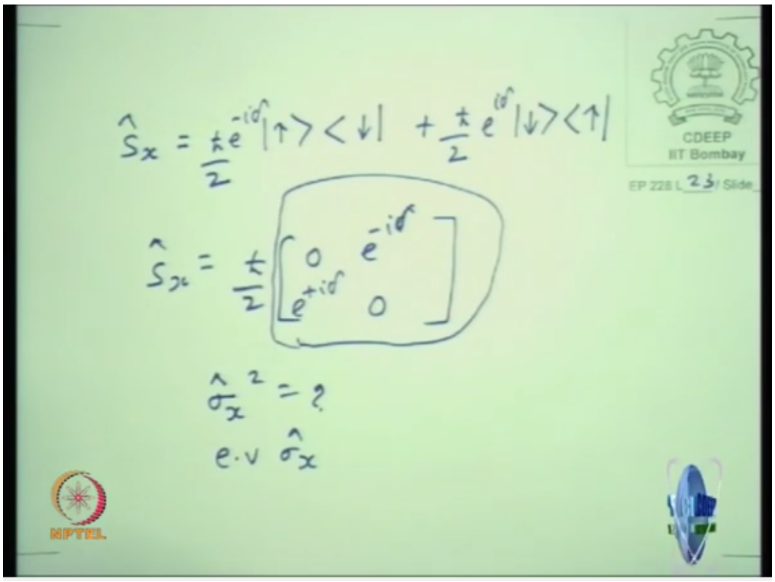


Sx is up down with e to the power of –i delta h cross/2 and then you have a +h cross/2 e to the power of +i delta down up. So what will this be in matrix form? h cross/2 I will take it out, diagonal is 0 right, off diagonal is the other way round or this way. So again traceless. What about the square of these matrices? 1 0, eigenvalue is +or-1 for this matrix. Check it, call that matrix as sigma x.

Find what this is, find eigenvalue of sigma x okay. So as of now I have not fixed what is delta as you was asking, you could have chosen delta=-0 but we will do it systematically. I want you to verify this. This I have not worked it out but please do it yourself and check it okay. So similarly instead of putting it through SGX, you put it through SGY, instead of SGX if you had put it through SGY then an arbitrary eigenstate the same argument.

There will be two eigenstates for Sy operators and you can write the states of Sy operator again as a superposition of Sz eigenstates with a different phase factor. Let me call it as delta prime okay. So I can do that.

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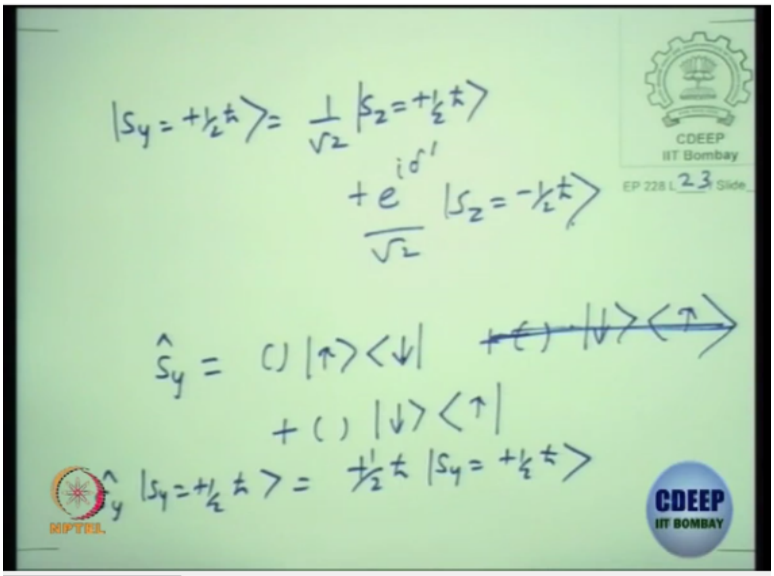


Write it as 1/root 2 times Sz=+1/2 h cross+e to the power of i delta prime/root 2 Sz=-1/2 h cross, I can do this. The way I wrote it for Sx states in terms of the eigenstates of Sz, I can also write Sy states in terms of the eigenstates of Sz and do the same things, the way we did and fixed what is the Sy operator. Again, it will be up up, up up or up down, with some coefficient with another coefficient down up okay.

What are those coefficients, fix it by using the same arguments? If I had an Sy operator, on Sy=+1/2 h cross it has to be +1/2 h cross on Sy=+1/2 h cross and +1/2 h cross state is this. Rewrite it and try to find the matrix element in the basis states of Sz. Do you all understand right? Will you do it? Again, you will find a matrix for this which is very similar to what you did for your Sx but only delta will replaced by delta prime.

So now to determine the phases delta and delta prime, we can also do the superposition, the Stern-Gerlach first along x and then along y. Again, you will have equal intensity. So the superposition of an Sx state, there will be equal components of Sx=+1/2 and -1/2 in a specific Sy state or vice versa. So that means the probability should be equal, use this fact and try and fix what is delta and delta prime choices.

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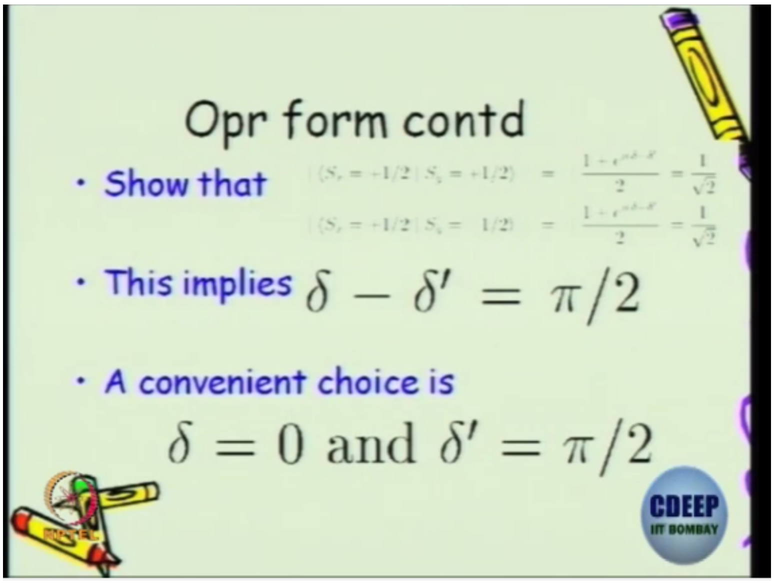


Show that the norm or the it is not norm the inner product of two states will be this and the mod of that should be 1/root 2 and similarly the inner product with the Sy=-1/2 and Sx=+1/2 is this and now you fix your choice. What is the possibility which will fix this choice? Delta-delta prime has to be pi/2. Is that correct? So that is it. Once I get delta-delta prime to be pi/2, I can now make a choice.

Let us take the Sx eigenstate to have no phase factor, put delta to be 0 then because of this property I have to make the relative phase for the Sy operator to have delta prime to be pi/2. There is no option, yeah +or- i pi/2 does not really matter.

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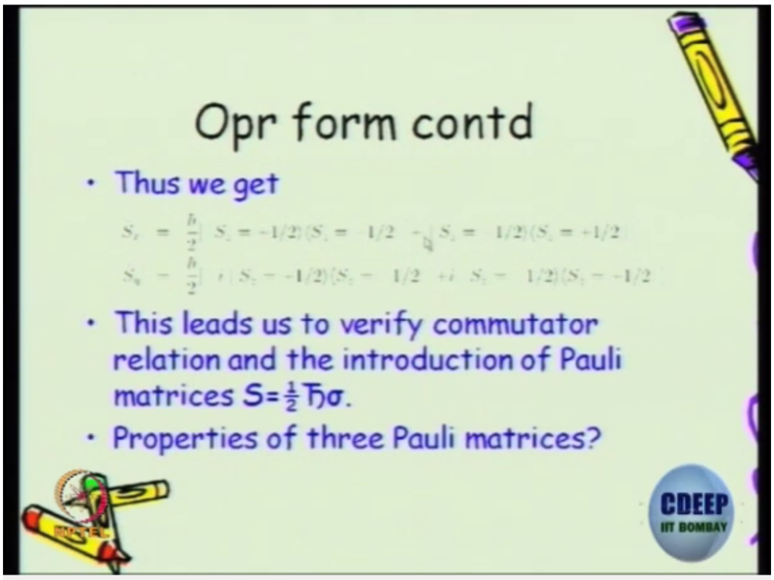


So with this if you try to write the matrix operator for Sx and Sy, so there is no phase factor here but here you will have a –i and +i okay but this as you are pointing it out, it could have been a +i and a –i also, it does not violate any of these properties but only thing is you have to make sure that your Sx Sy commutator is Sz and then one of the signs will be conveniently doing okay. So write the matrix form for the Sx and Sy and figure out what is sigma x and sigma y. Can you write it down?

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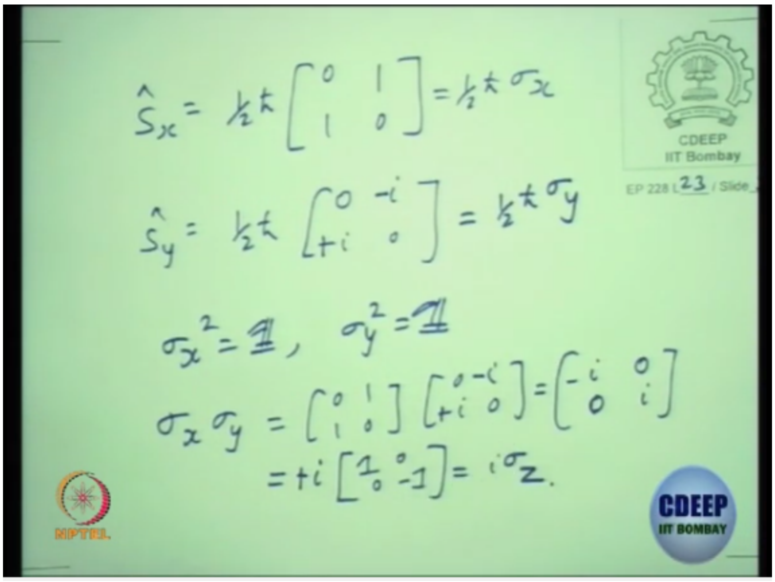


Matrix form for the Sx operator if you try to write this will be 1/2 h cross sigma x, note that each of them is Hermitian, this i blindly if you write i and i it would not be Hermitian sigma y. They are traceless, eigenvalues are +or-1 and they have the properties that sigma x squared is identity, sigma y squared is identity. You can also check what is sigma x sigma y. What is Sigma x sigma y? i times sigma z, very good, so you get this -i and if I do that that is 0.

Then i 0 with an i and that you can write it as i you can take it out - i you can take it out and 1, so there is a slight notation here if I put this to be –i and +i as sigma y and then you get i times sigma z. So this is one unique convention which is taking care of this and it will take care of your algebra Sx Sy commutator should be ih cross Sz okay. So 0 -i i 0 is what we will call it as a sigma y. So the sigma x sigma y sigma z in any textbook if you start seeing, they call the Pauli matrices.

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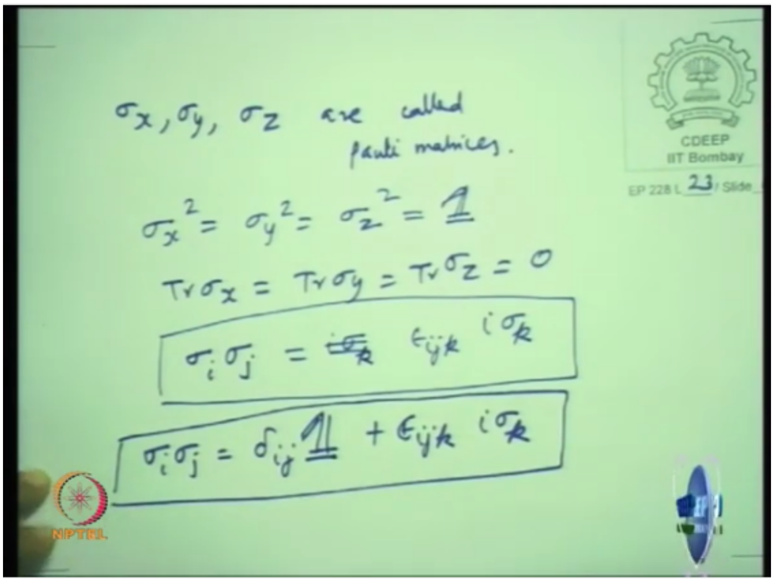


Sigma x sigma y sigma z are called Pauli matrices okay. What are the properties? Sigma x squared sigma y squared sigma z squared is identity. Trace of sigma x trace of sigma y trace of sigma z is 0. Sigma i sigma j is i times sigma k with an epsilon ijk i times sigma k. Is there an i or it is not there? It is there right. This is a compact way of writing what I wrote for sigma x sigma y is i times sigma z you can write sigma i sigma j as epsilon ijk i times sigma.

Can I write further? If i and j are equal, sigma i sigma j is delta ij times identity if i and j are equal and if i and j are not equal which is epsilon ijk i times sigma k okay. So you can do various other things with these Pauli matrices.

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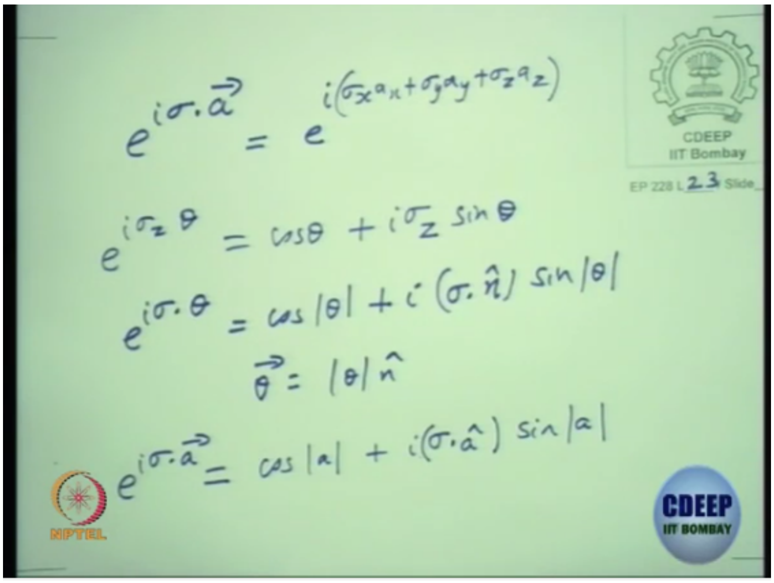


If I give e to the i a sigma dot a vector what is this? It is e to the i sigma x ax plus sigma y ay+sigma z az. We did this e to the power of i sigma z times some theta. What did we get here? Cos theta+i sigma z sin theta. You can argue that theta is the magnitude which is oriented along the z-direction to start with and what do you expect from there? Some theta vector if I had I would have written this as cos of mod theta+i sigma dot n hat sin of mod theta where theta vector is just by inspection.

You can also verify here using Pauli matrices properties, use the Pauli matrices properties and try and work it out what is e to the power i sigma dot a, it should be cos mod a+i times sigma dot unit vector of mod a only the direction sigma dot or sigma cross, sigma dot right. Please verify some of these things. I will put it in the assignment also but please verify that the Pauli matrices are very rich and it helps you to do many of these things. But why am I doing this exponentials, what is the reason? How did you write the translation operator?

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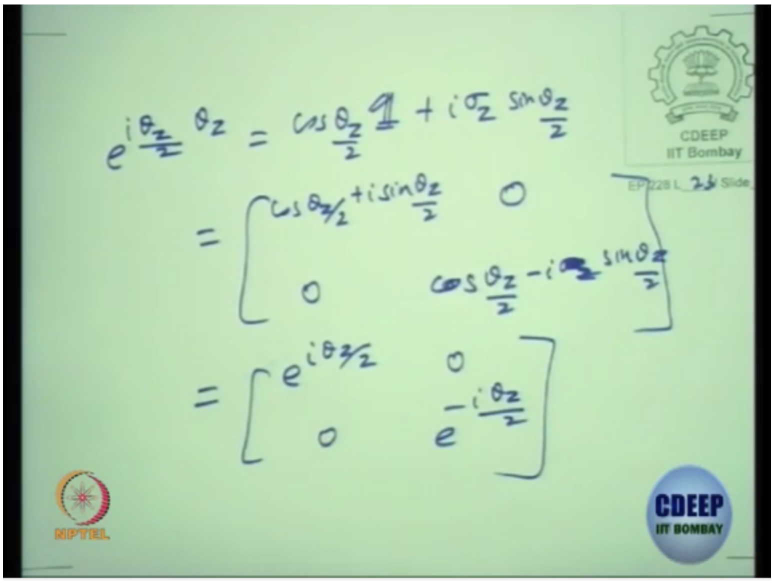


When I wrote the translation operator, which shifts by a vector unit, it had an exponential of i and then you had an operator which I said that operator is the linear momentum operator and how much steps you want to translate that is what you get is what you had. I also at some point try to motivate you to look at for very small rotation angle what should be the rotation operator. What did we do for the rotation operator?

For a theta delta theta was very small, it was 1 some –i theta or +i theta and let us do it along z-direction I said. Then, what did we have Lz/h cross. I asked you to look at how the R vector changes under the infinity symbol rotation and fix what should be the rotation operator and you write this. This is for infinitesimal; you can keep continuing. If you keep rotating about the same axis, you can keep continuing exactly like the way we wrote here.

What will that be? Exponential of i Lz theta z/h cross will be for R of theta z. This is rotation in your physical position space but I have tried to tell you in the position space you have orbital angular momentum and Stern-Gerlach experiment tells you that you should also have spin angle of momentum, you can also have a rotation in the internal spin space okay. So that will be denoted by e to the power of i. Let us do Sz rotation by theta z/h cross and what is Sz?

Sigma z 1/2 h cross sigma z right, so that is a by 2 theta z. So this is an operator which will perform rotation in the internal space by an angle theta z and the by 2 comes because of the explicit representation of the spin operator, the eigenvalues are 1/2 +or- 1/2 h cross that is why this comes. So if I want to write what this is I have to write this as cos of theta z/2+i sigma z sin of theta z/2 because of this 2.

So all these Pauli matrices show up in rotation in the internal space and you have to familiar with it. What is the nature of these operators? Are they unitary? These operators are unitary, you can also say that R is they are real entries, then you can call them to be orthogonal. So these are in general unitary operators. What is the dimension of these matrices? You can write a matrix representation. This is a 2 x 2 matrix right. This is a 2 x 2.

The e to the by 2 so you can write this as cos theta/2+i sin theta z/2, sigma z is 1 and -1 right, 0 0 cos theta z/2 –i sigma z, sigma z is -1 times sin theta z/2. So what will this matrix be? e to the i theta z/2 0 0 right know. That is it. A lot of manipulation you can do with Pauli matrices and the core thing is that these are unitary operators which perform your rotation in the internal space and they are 2 x 2 matrices.

So these set of unitary matrices, they have some properties, group properties. We will come back to it at some point.

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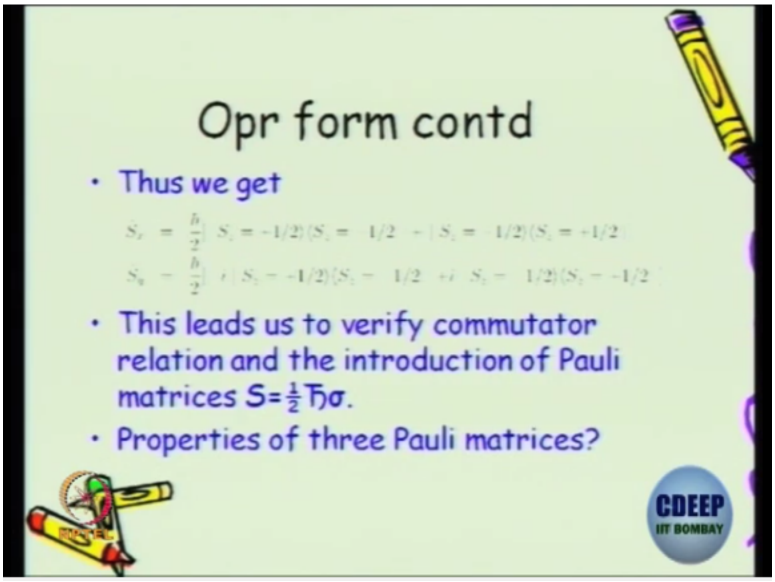


So for you to check Sx operator Sy operator and check your commutator relation and from there you put in all the properties of the Pauli matrices and we will use them in all our computation for a two-dimensional linear vector spaces which corresponds to this internal spin half space and you need to have this hands-on feel on playing with these Pauli matrices. So that is why I have put in this okay. So I stop here.