Quantum Mechanics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology – Bombay

Lecture - 50
Stern-Gerlach Experiment- I

Okay so today I am just going to take you on a warm-up where we have already browsed to Stern-Gerlach but from them I try to give a representation for the matrices corresponding to spin operators okay using the Stern-Gerlach experiment data, this is what we are going to do.

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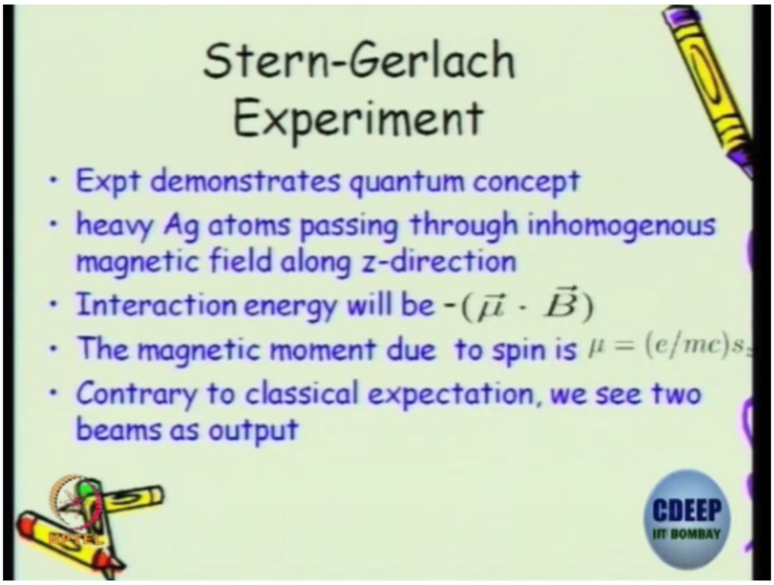

This is one of the experiment which tells us that you will have a quantum concept. So if you take heavy silver atoms passing through an inhomogeneous magnetic field along z-direction, you do see that contrary to your classical physics you see two splitting, two beams splitting and you can put in just like the way we write it for the orbital angular momentum we can put the interaction energy with spin component of the magnetic moment as mu dot B.

And the magnetic moment is proportional to the suppose I take the magnetic field along z-direction then the magnetic moment is proportional to the z component of the spin operator. These things we have done earlier. So contrary to the classical expectation we see two beams, this is why I said it is a quantum result.

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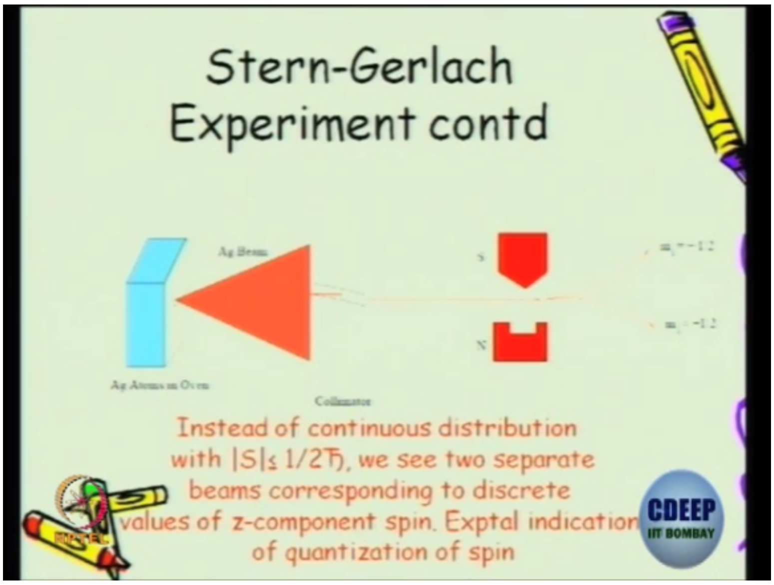

What exactly if you see pictorially you send the beam of silver atoms collimate them and pass it through an inhomogenous magnetic field. Classical you would have expected suppose the magnitude of the spin is let say +1/2. When you could have tried to get a continuous spectrum with mod S to be<or=1/2. That is what classically you would expect but quantum mechanically you see that you have only two distinct spectrum which is discrete.

We call the magnetic spin quantum number to be -1/2 and +1/2. So the experimental indication is that the spin is not a continuous variable just like your orbital angular momentum in hydrogen atom problem if you try to find what is the orbital angular momentum, it is discretized right. Lz operator operating on a hydrogen atom wave function gives you mh cross where m is an integer right.

So that is the quantum concept. The same quantum concept applies even to why silver atoms, silver atoms are almost like L=0 states. So it would not have orbital angular momentum. So that is why we can pinpoint that this is corresponding to the spin angular momentum. Otherwise, we have to worry about whether there is also the orbital angular momentum. So this is an indication that we have quantization of spin operator, spin angular momentum.

Okay so you can do one thing here, you can put a stopper to prevent this beam to further go away, you can just absorb it and you can filter only the magnetic spin quantum number as -1/2 to keep propagating. So you can put a stopper here.

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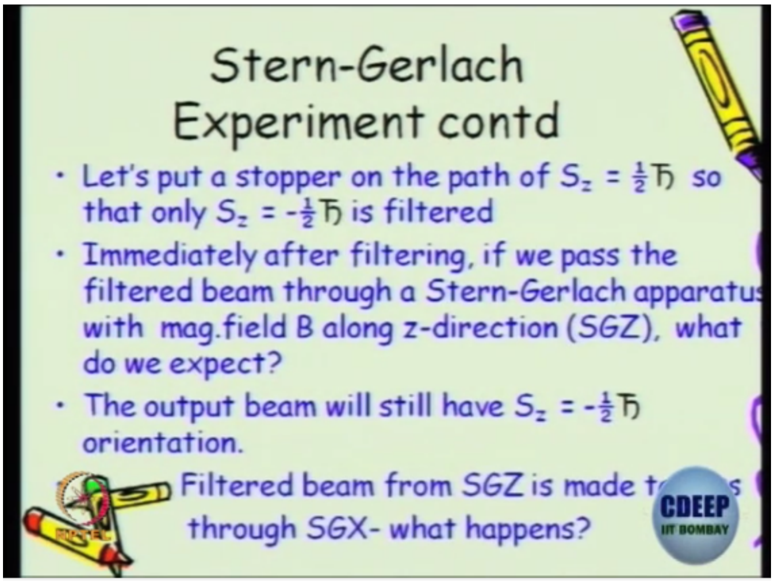

This is what I am saying here. Let us put a stopper on the path of Sz=+1/2 h cross so that only Sz=-1/2 h cross is filtered. You can do this kind of filtering of one direction, the beam coming in one direction. So suppose immediately after filtering, we again do the same Stern-Gerlach apparatus with the z-direction magnetic field what will you expect? Will you get two beams or one beam?

Suppose I put a stopper here and stop this beam. I take this beam and pass it through the Stern-Gerlach apparatus which is similar to this. What do you expect? Only the same MS=-1/2 only you will expect. Why? Because that is the only one which will be, there is no other components which will come in. What is the other way of saying in quantum mechanics? These two are eigenstates of the Sz operator.

And we are trying to block one of the eigenstates of Sz operator; it is like measuring the other Sz operator eigenvalue. Once I measure that and I keep passing it to the same system, it remains in the same state; it does not go to the other state. It is the math qualitative way of saying clear to you. So when you put a stopper and pass it through the other and filter this and pass this filtered beam through again the Stern-Gerlach apparatus with magnetic field along z-direction.

I am going to use a shorthand notation SGZ which means the Stern-Gerlach apparatus has magnetic field along z-direction okay. What do we expect is the question? I have already told you the answer. The output beam will still be having Sz quantum number to be -1/2 h cross orientation. Clear, now change in the direction of the magnetic field to be along x-direction.

You can do that. In the lab, I can take this beam which has orientation as -1/2 h cross when the magnetic field along z-direction is passed.

I take that and I pass it through a Stern-Gerlach apparatus which has magnetic field along x-direction. That is why I am denoting it as SGX. What is your expectation? Experimentally, we see again two beams and the intensities of both the beams are equal. This is what we see. Quantum mechanically we will say that the probability of splitting into two beams are equal okay and we write this Sz=-1/2 h cross state to be an equal superposition of the two eigenstates of SGX. Is that correct?

Because we see equal intensity of once you put it through the SGX, we see again splitting.

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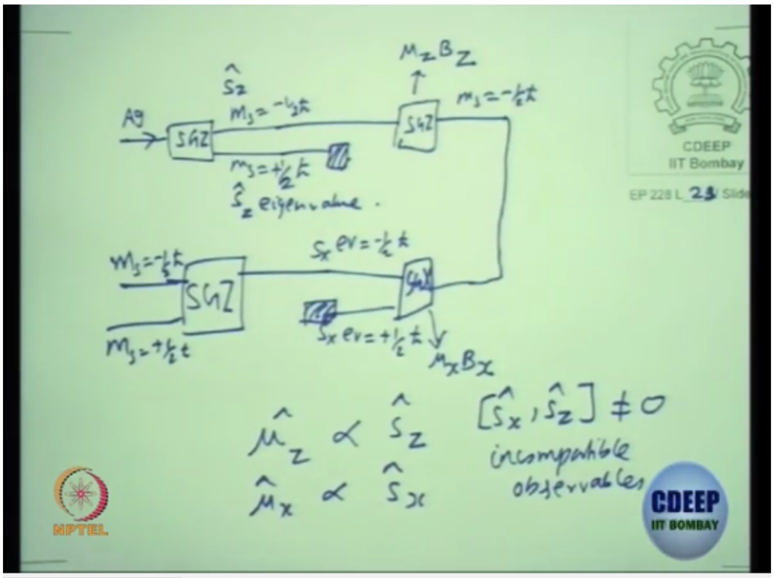

See we have an SGZ, we send in the beam of Ag atoms okay. What happens is you had two beams, one with ms=-1/2 h cross and ms=+1/2 h cross. I put a stopper here and make it to get absorbed. I get it absorbed. I make this beam, if I put another SGZ, it still remains as ms=-1/2 right and I take that beam. I put it through SGX, what happens? You see two beams okay. You see two beams with SX eigenvalue to be -1/2 h cross and SX eigenvalue to be +1/2 h cross.

There are two beams with equal eigenvalue. So this one can be said as Sz eigenvalue, this ms denotes your Sz eigenvalue okay. Now if I take suppose I put a stopper here and make this beam go through SGZ or what is your expectation. Already, I have filtered out the +1/2 h cross but in this series after I do the SGX the magic is we get both the components. Why? Why is that? Louder, they are incompatible observables.

If suppose the operator corresponding to magnetic field along z-direction commutes with the operator corresponding to magnetic field along x-direction, what is the operator is the interaction Hamiltonian. Here this will be mu z Bz. Here it will be mu x Bx and mu z is proportional to Sz operator. B is an external field. Similarly, mu x is proportional to Sx operator.

Experimentally, if you had seen the same single beam with ms=-1/2 h cross happening then I could have concluded. Suppose I did not see this beam, then you could have concluded that Sx and Sz are compatible operators because experimentally I see two beams in this chain of Stern-Gerlach apparatus. I know that Sx Sz is not 0. These are the inferences we get from experiment.

So this is incompatible okay. So this is what we see experimentally, we have seen the theory in the last set of lectures. Now if you want to corroborate the set of lectures with the experiment, this is what we see. So Sx and Sz are incompatible. Same thing you could do a Stern-Gerlach with y also. Let me not get to doing too many things but if you do it, you will start seeing that the particular one eigenstate of Sx operator when it passes through the Stern-Gerlach of y will give you equal intensities of two beams okay.

So the probability for it to split into the two quantum states of mu z or mu y operator will be equal. Is that clear? So this is what I was saying. Let us put a stopper on the path or one of the eigenvalue measurement so that that is blocked and the other eigenvalue keeps propagating the state corresponding to the other eigenvalue. Immediately, after filtering if you again put it along SGZ, you will expect to get the same eigenvalue.

Because it remains in the same eigenstate, nothing happens but if you put it along SGX then you will start seeing linear superposition okay.

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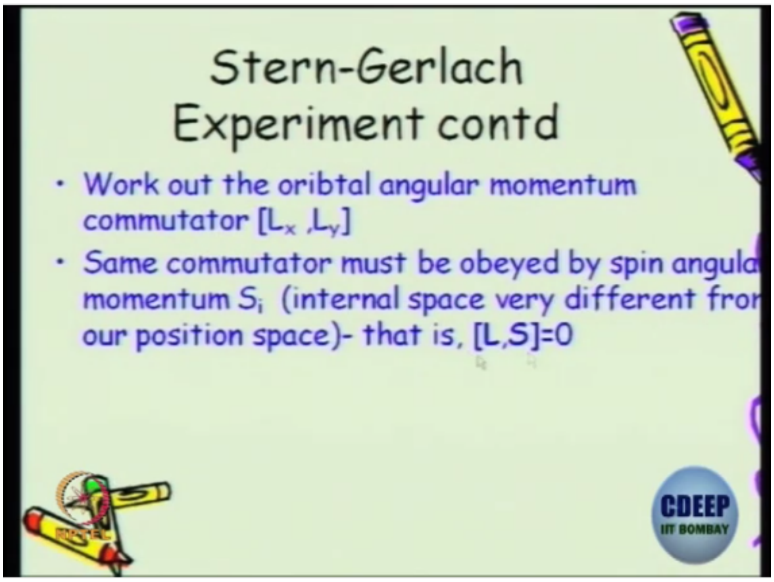

So this is what I am, is what experimentally seen. This is something which you have already worked out, orbital angular momentum Lx and Ly. We claim that the spin angular momentum is not very different from the orbital angular momentum as far as this commutator. Just like the way we do Lx Ly, we expect Sx, Sy to have the same commutator. The only thing is that the spin operator is in a very different space.

Nothing to do with your regular position space which means those operators do not talk to the orbital angular momentum operators. What do we mean by that? I can independently measure orbital angular momentum and the spin angular momentum. They are competitive because they are in two different spaces like if you had two harmonic oscillators, one harmonic oscillator having frequency omega 1, another harmonic oscillator having frequency omega 2.

And I say that these two harmonic oscillators are non-interacting, you can write ladder operators a1 and a1 dagger for the first harmonic oscillator. You can also write b1 and b1 dagger for the second harmonic oscillator but if I say they are non-interacting what does it mean?

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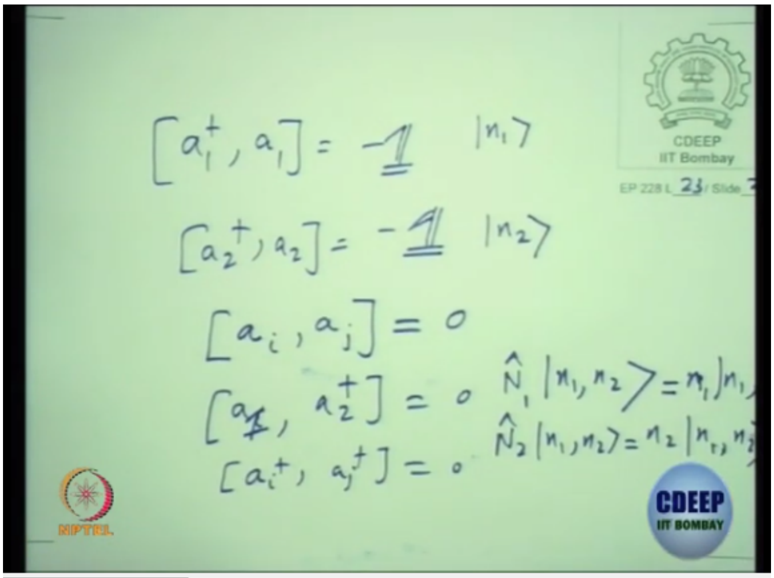

A1 dagger a1 commutator is -1, a2 dagger a2 commutator is -1 but if I say that these two harmonic oscillators are non-interacting, it means ai with aj is 0. You can also write ai or a1 with a2 dagger is 0 and so on right. You can say that similarly ai dagger with aj dagger is 0, that is trivial, the top and this one but this intermediate one tells you that these 2 harmonic oscillators are not interactive. So you can write a state to be a simultaneous eigenstate of both the harmonic oscillators.

If suppose the states here I denote it by n1, states here I denote it by n2. I could denote a state as n1, n2 if the number operator n1 operates on it, it will give you n1, n2. If the number operator for the second harmonic oscillator operates on this, it is n2, n1, you understand right. These are the shorthand notations of writing for a non-interactive.

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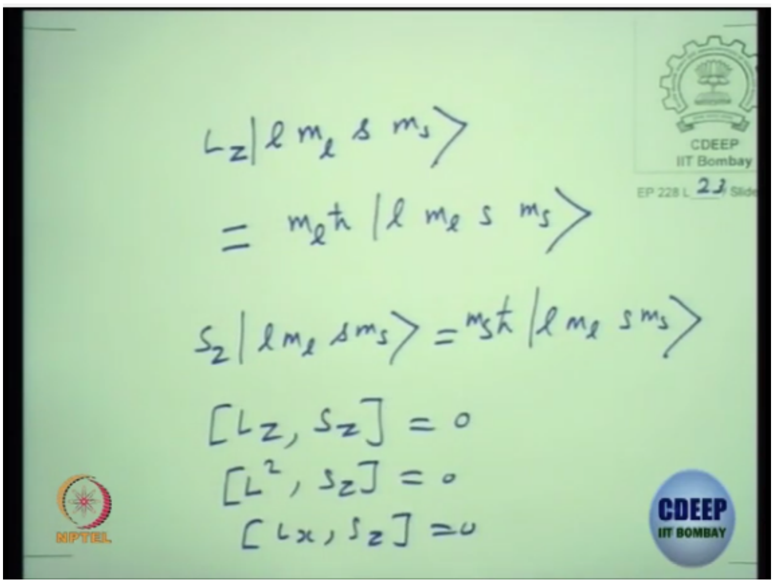

Similarly, for the spin and orbital angular momentum, I could write l ml s ms, if there is an Lz operator on this, it will be right. If there is an Sz operator on this, s ms right h cross, it does not touch the l, that is all I am trying to say fine. So these are the things which tells you that I can say that Lz with Sz is 0. I can even do further things that L squared with Sz is 0 and so on okay.

So you can show L squared with Lz is 0 so you can show L squared with Lx is 0 so you can show Lx with Sz is 0 and so on. So you can start arguing using properties of all your angular momentum. Algebra is which you learn for orbital angular momentum to infer that these two are two different spaces, either non-interacting or you know they are two different spaces; you can do this in the Dirac formalism on the states. Is this clear?

So there is an internal space with very different from our position space that is you can say all the components of Ly with all the components of S any of the components, the commutator bracket is 0, so you can write a simultaneous eigenstates of orbital angular momentum and the spin angular movement.

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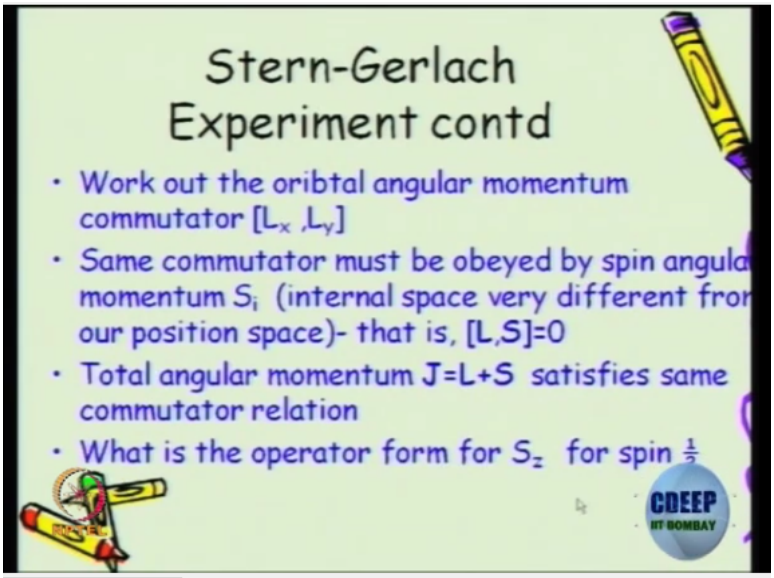

You can also write total angular momentum J to be a sum of L+S which will also satisfy. If Lx Ly is ih cross Lz, Sx Sy is ih cross Sz, Jx Jy will be ih cross Jz, x is just a property of the commutators which you do on linear operators. From the Stern-Gerlach experiment, we can try to write taking the states to be eigenstates of Sz. We can write what is a Sz operator.

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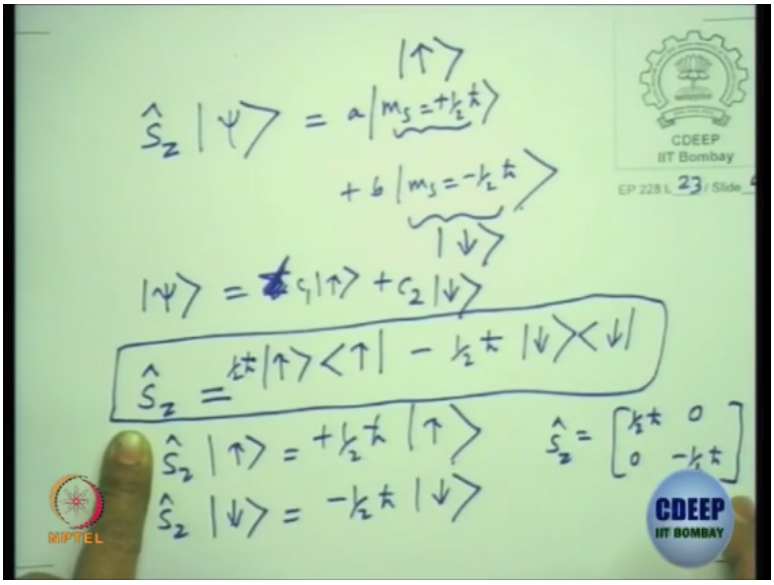

The Stern-Gerlach experiment gave you two beams, Sz operator on any arbitrary state if you make it go through this, it gave you linear combination of ms=+1/2 h cross+ms=-1/2 h cross right. So we are going to write an arbitrary state. Arbitrary state I could write it as some linear combination of two possibilities. This is what the experiment told us. Any arbitrary state, I can write I will denote this state by an upstate and denote this state by a down state.

This is another notation but it means the same, the magnetic quantum number if you try to measure or the eigenvalue if you try to measure for the Sz operator, it will be +1/2 h cross and the +1/2 h cross I can formally denote it like it points in the up direction and -1/2 h cross it points in the down direction. This just a symbolic way of denoting, so you can principle write it as a linear combination of up spin with some coefficient times c2 two down spin.

There is no need for a summation. How will you write the Sz operator? Sz operator Sz on up spin is +1/2 h cross up spin. Sz on down spin is -1/2 h cross down spin. Using these two, can we write the matrix operator? You can take the matrix element of this with up that will give you nonzero, with down it will be zero. So what will that be, with what coefficient, 1/2 h cross, then this one -1/2 h cross with down. Is that right?

Can we write it like this? Yes, or no? Done many of these two state linear vector spaces, if I give you these two data, you try to find the matrix elements in the two-dimensional linear vector space that will be 4 basis states for the operators. The off diagonal elements of the operators are all 0, you can write it this way. Can you also write what is the matrix form for it? Will be 1/2 h cross 0 0 -1/2 h cross. Is that right?

If I put the first state to be up up, this one to be down down okay. So Sz operator, so there are two, one is operator basis you can expand.

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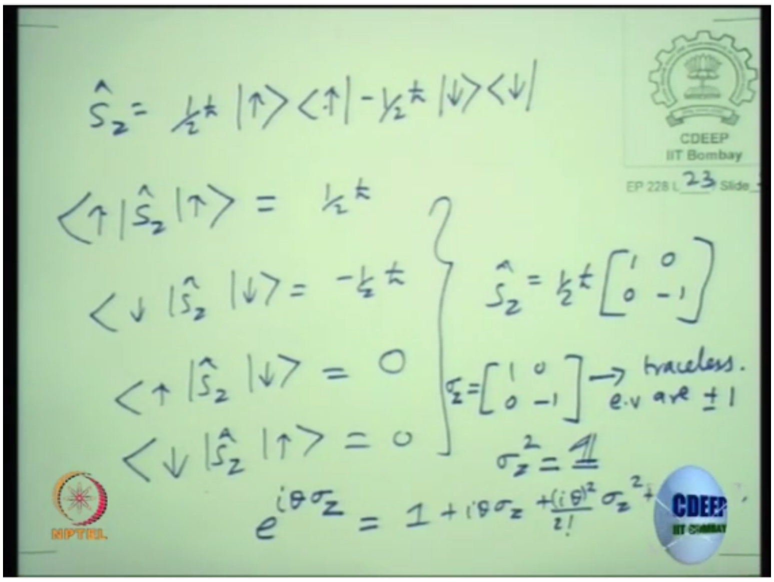

You can write this as 1/2 h cross up up -1/2 h cross down down right and you can try to find what is Sx up with up, that will turn out to be 1/2 h cross, down Sz with down will be -1/2 h cross. Up Sz with down will be 0, down Sz with up will be 0. So using these 3 with this operator, you can write Sz to be 1/2 h cross 1 0 0 -1. We will come back to these matrices. These matrices have some special properties 1 0 0 -1 matrix.

What are the properties? It is traceless, eigenvalues of this matrices are +or-1. What about square of this matrix? So let me call this matrix by a notation, we will come back to it. We call it as a sigma z, sigma z is traceless, eigenvalues are +or-1. We can also show that sigma z squared is identity okay. Also interestingly you can do what is e to the i theta sigma z. What will this be?

1+i theta sigma z+i theta squared/2 factorial sigma z squared and so on. Sigma z squared is identity, you can keep adding things. What will you get? Cos theta. So let us write that.

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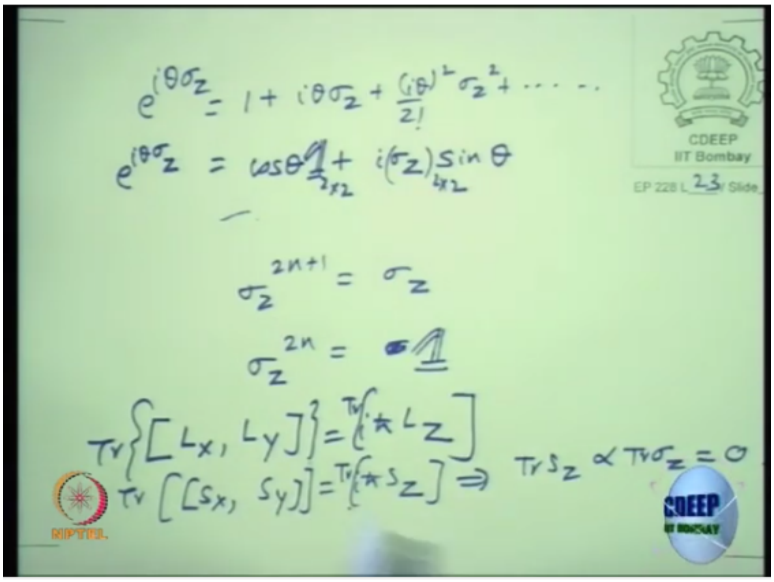

So this can be shown to be cos theta+i sigma z sin theta, it is still a matrix. You can put a 2 x 2 matrix here because sigma z is 2 x 2 matrix okay. That is a nice way in which you can write because these matrices are traceless. What is the determinant of it? Determinant is -1 right and the eigenvalues are +or-1 and it also has a nice property that even powers of these matrices are always identity.

So therefore sigma z to the power of 2n+1 is sigma z. Sigma z to the power of 2n is sigma sorry it is identity. We indirectly knew that it has to be a traceless matrices. How did we know that? Formally, for the orbital angular momentum when I write Lx Ly to be ih cross Lz. If I take the trace, these are all finite dimensional linear vector spaces on which it acts. This side is trace of a commutator of two operators is 0.

You have done this as an exercise which means trace of Lz has to be 0 which means Lz has to be a traceless operator. Whatever I did for orbital angular momentum, the claim is it should be true for spin angular momentum and sigma z is up to proportionality is proportional to this sigma z is proportional to Sz right. So if I take a trace here and a trace here, it implies trace of Sz which is proportional to trace of sigma z has to be 0.

So the same argument will go through for all the components whichever other matrices which we are going to find if I put this through the Stern-Gerlach apparatus with magnetic field along x-direction, you can show that trace of Sx has to be 0 and we can write a corresponding sigma x.