Quantum Mechanics
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Lecture - 49
Tutorial 8 - Part II

Let us get started with few more problems on translation operator for a finite dimension that is the finite distance special finite distance.

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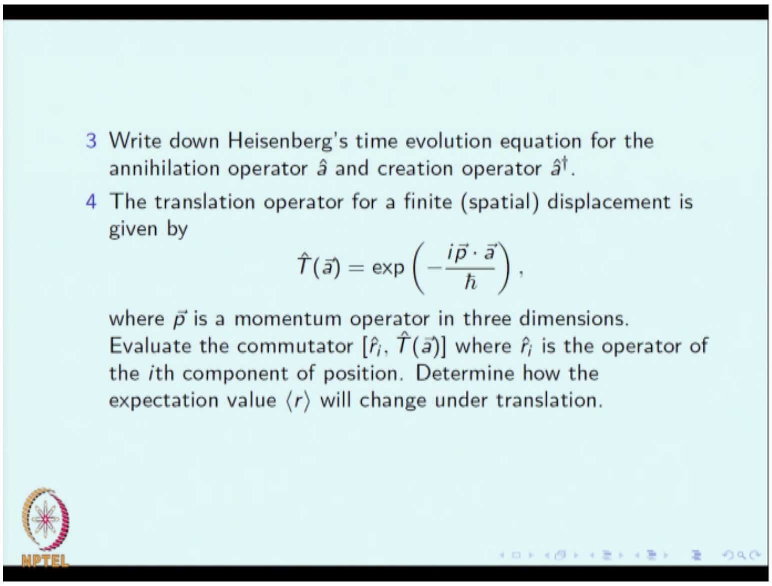

And this problem 4 will involve a problem on the commutator of the position operator and the translation operator with a finite displacement.

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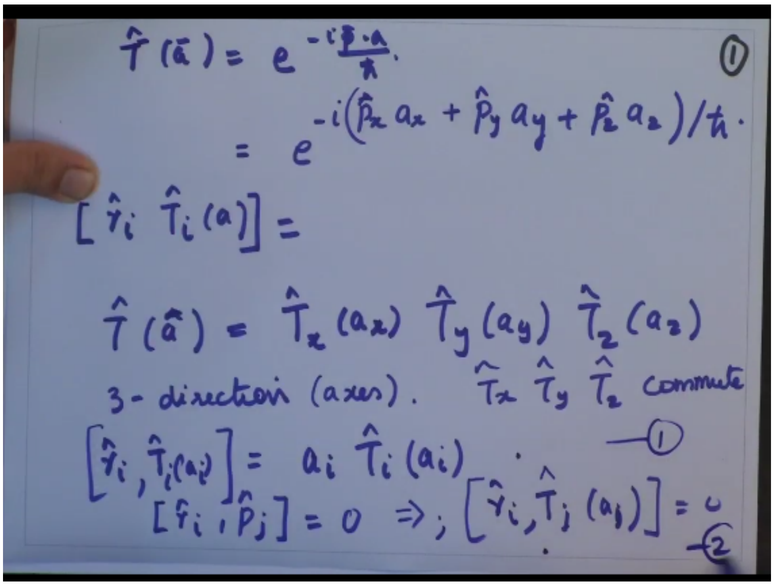

So in the question we have T a is exponential -ip dot a/h cross where p is the momentum operator in three dimension. So we can write this as e raise to -ipx ax okay. I can write this hat as the operator px, ax is a position fixed position, py ay+pz az/h cross. So we are given that p is a momentum operator in three dimension and ri is a position operator in some certain direction. So rx or r1 will give me x direction. Then, I can have y direction and z direction.

So let us start by writing a specific case like you can you have to first evaluate ri Ti a okay and then we generalize the result. We will generalize the result later. So as we know that the momentum operator px, py and pz commute with each other. So will be Tx, Ty and Tz. So this T cap a I can write it as T cap x ax sorry there will not be a cap Ty ay and Tz az. So these translation operators will also commute with each other.

So when we are taking the commutation relation, we have to use this hint. So these operators x, y, z are in 3 directions, 3 translation directions that is the 3 axes x, y and z and these Tx Ty and Tz commute with each other right. Now since we have this, let us evaluate ri T ai okay. So ri Tai you remember we have done such exercise in tutorial 5. In that, we have evaluated this quantity and this came out to be just try to recollect ai times T ai.

This is what we had obtained in tutorial 5 and remember we will use these relations while evaluating the general result. So ri commutator of ri and pi will be 0 okay. This we should recollect and we know that ri pi is nothing but ih cross. We have seen this commutator relation. These are the general relations which we have seen. In general, we have seen the expression of ri and j, ri and pj okay that is delta ij when i and j are equal the will not commute.

And when i and j are unequal they would commute in this particular case. So this is what we have here. So will be ri, Tj aj, this would imply this remember because Ti depends on pi so if I have a pj then I have a Tj which will be equal to 0 okay. So using 1 2 these two things.

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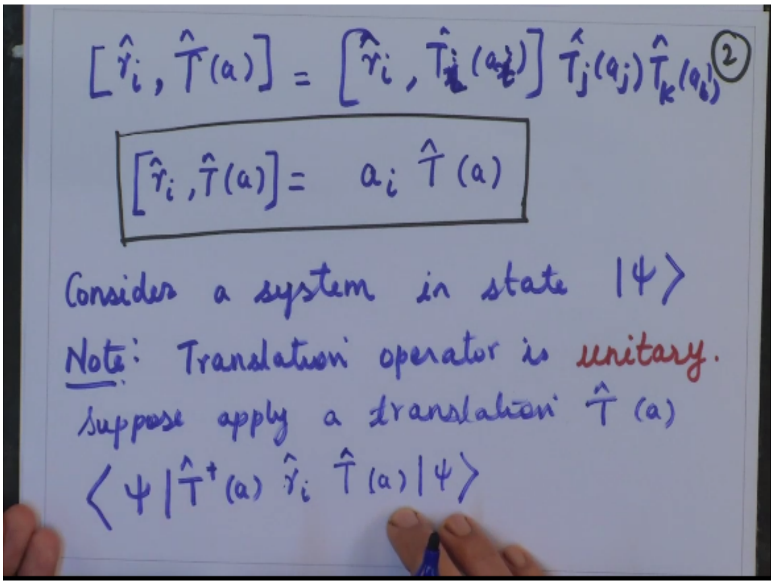

Using equation 1 and 2 what we obtain is let us right now ri this is position operator, T a okay. So ri, T a, T a can be written as Tx, Ty, Tz okay and ri will commute only with Ti, so I will have ri, Ti ai times that is Tx if I take and here if I take let me keep it i only okay. So I have a Tj aj Tk ak. Remember these are operators and we need to denote it by cap. So this is a ri Ti aj which is nothing but ai okay I will have ai and this quantity that is Ti ai Tj aj and Tk ak which will give me back this term okay.

In the question, we have this. In the question, we have that we have to determine how the expectation value of r will change under this translation. So let us now consider a system in state psi. So let psi be the state of the system and now we have to remember this. It is important to note here that translation operator is unitary okay. This is one thing you must remember.

And now when we suppose we apply a translation operator that is T a what do we obtain? So consider this T dagger a this is the operator T a ri. So we are finding out the expectation value of ri. This is what we are asked to find but on this state we are applying the translation operator. So T on psi on the right and T dagger on psi on the left because T will become T dagger.

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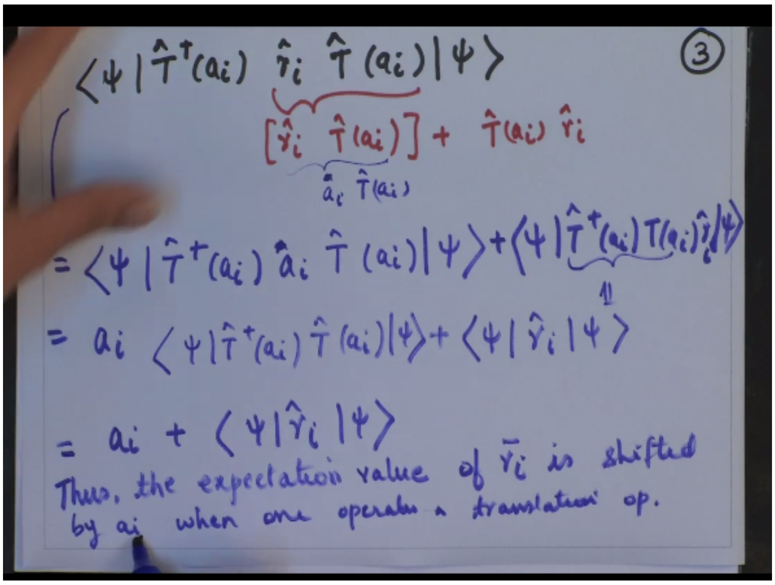

Psi T dagger ai ri T ai, so this we will evaluate. Here what I said is I will use ri T ai commutator+T ai ri. I will substitute for this here. We have seen what is the commutator of this; commutator of this was ai T ai correct. So commutator will give me this and the second term is this term okay. So what do I get next from here? This will be nothing but T cap ai I have got ri which I can actually take outside.

In the next step, I will do that. T cap ai this is the first term and the second term is T dagger ai and I have a T ai ri times psi okay. Now these two terms I will simplify, ai is a number, ai can be taken out. It is a number not an operator sorry. So ai I can take outside very well and I have a T dagger T which will give me just 1 okay. This is what I have correct and when ri is operated on a and this term will give me 1.

So let me write here okay, so this is the expectation value of ri on the state psi and when you operate the translation operator, it is getting shifted by amount ai. So in the end what we see is that there is a shift in the position by ai. So the position is shifted by ai, thus the expectation value of ri is shifted by ai when one operates a translation operator okay. So there is a shift by amount r ai okay. So this is the fourth problem we did. Let us go to the fifth problem.

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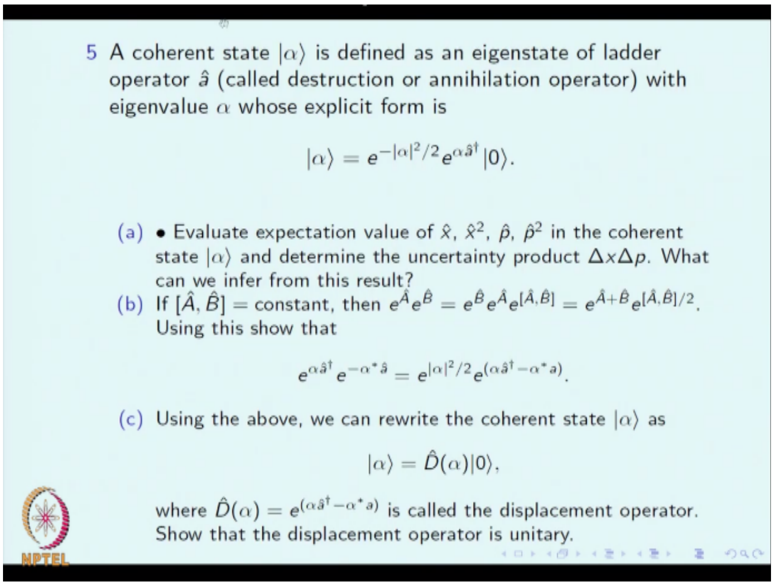

Fifth problem is dividing 3 parts and it is slightly lengthy but it is interesting to check these relations commutation relations and what will be the displacement operator and all things.

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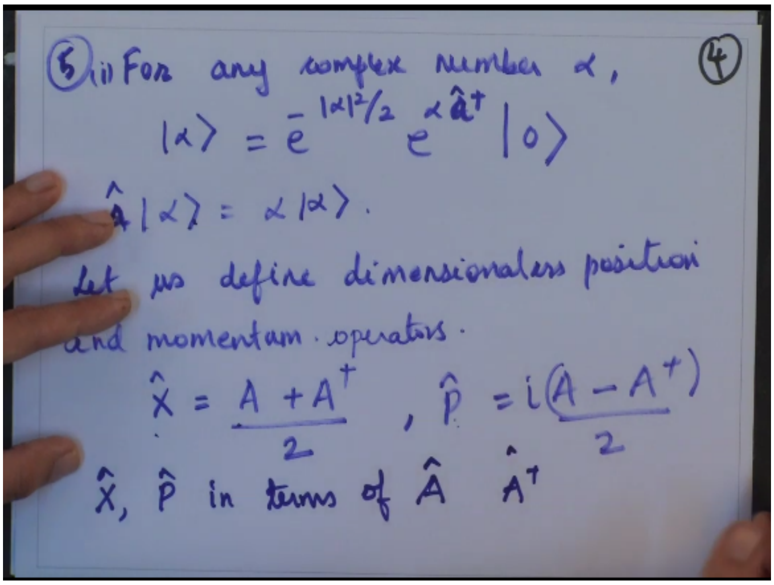

And this was discussed in the last part of the lecture and you are asked to prove some of the relations. So we have put some problems as the tutorial problems so that you actually sit and solve them. So first part, in the first part you are given a coherent state which is defined as the eigenstate of the ladder operator and you are given a relation. So for any operator or any complex number alpha you are given alpha is e raise to -alpha square/2 e raise to alpha a dagger so small a dagger okay and 0.

Just recollect that and this operator or the normalized eigen ket of a as eigenvalue a alpha. So when A operated A dagger or A, operator A is operated on alpha you get eigenvalue A alpha okay. This you have to remember. Now the next step would be that we have to define, we have to calculate X X square P P square in this coherent state basis. So for that you need to do some algebra.

Few hints I will give and you can maybe use those hints to work further. So let us define dimensionless position and momentum operators as A+A dagger/2 and p as A-A dagger/2 times i. So this is a position operator X and the momentum operator P which are defined in terms of the operator A which has eigenvalue alpha when you operate it on the eigen ket alpha okay. So now let us rewrite X and P in terms of A and A dagger okay A and A dagger. So what will be in terms of A and A dagger?

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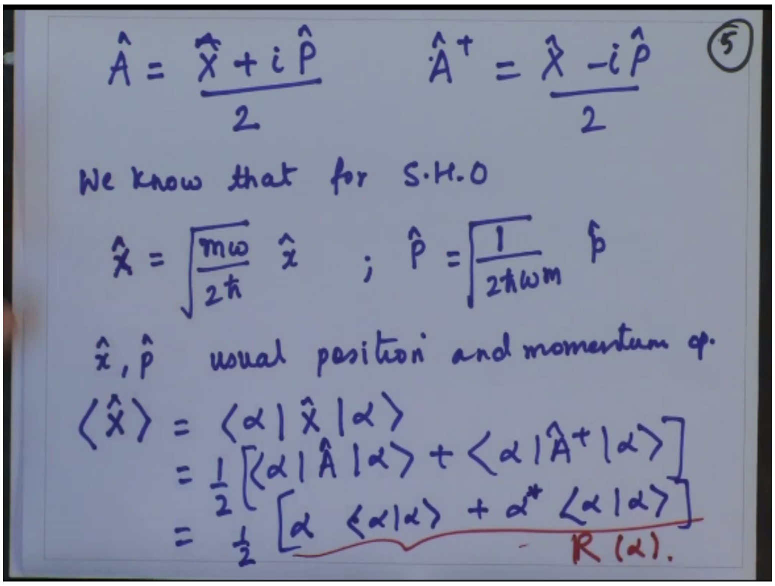

A will be=X+i P cap/2 and A dagger is simply the complex conjugate, you can see here correct. So we know that for simple harmonic oscillator, we have this relation capital X is nothing but m omega/2 h cross inside the square root x cap and p is nothing but 1/2 h cross omega m inside the square root p cap. So capital X and small x are related by this relation okay. We have defined capital X in terms of dimensionless variable.

So we have to now and remember x and p are our regular or usual position and momentum operators right. So it is now easy to evaluate what is X. This is X is what we are going to evaluate. X we had defined as A+A dagger/2. So first we will evaluate X, then we will evaluate X square, then P and then P square. So expectation value of X on ket alpha will be 1/2 I have A dagger A.

So I will write A alpha+alpha A dagger alpha correct and when you operate this, this will give me an eigenvalue alpha, this will give me an eigenvalue alpha square alpha star. So I will have alpha+alpha star and these are normalized eigen ket. So these are of the form R, I can write it as real part of alpha okay.

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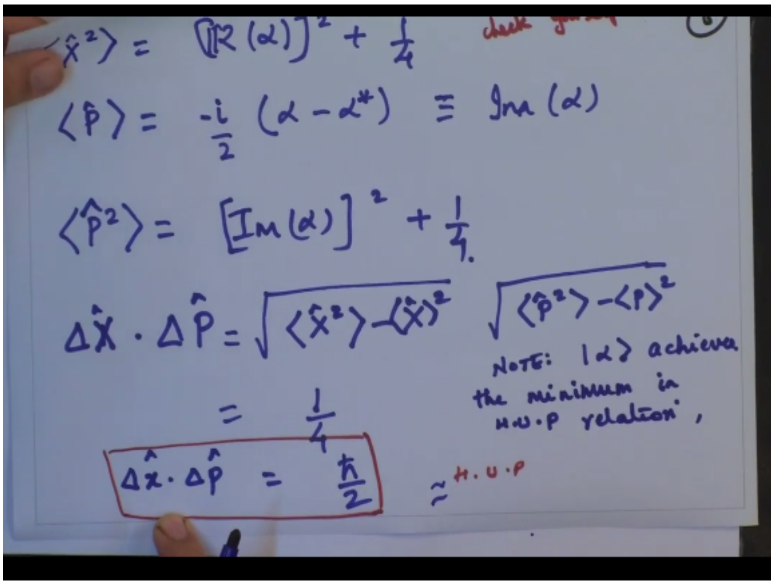

Now in the same manner you evaluate X square will be you will find when you solve this you will have the real part of alpha which we evaluated will be square+1/4. So please check this. Check yourself okay. Then, I will write what is my P. We have seen that operator P was i A-A dagger okay and I have skipped this but you must remember that when there is a hat it is an operator notation.

So I have i/2 and then I will have alpha-alpha star kind of a term which is equivalent to or which can be written as imaginary part of alpha okay. Similarly, you evaluate this and you will find that imaginary part of alpha square+1/4 is what you obtain. Now the next step would be to calculate delta X dot delta capital P okay. Delta X dot delta P will be square root of you know the definition, X square-expectation value of X square*similarly P-P square.

We have evaluated all these and when you substitute this, you will have something like this. Now you will substitute for capital X and capital P in terms of small x and small p. This is what we have right. So when I substitute for these, I get delta x dot delta p as in the operator notation let me be very explicit. So this relation is nothing but Heisenberg uncertainty principle, delta x dot delta p should be<or=h cross/2 okay right.

So this coherent state, note coherent state alpha actually achieves the minima in Heisenberg uncertainty principle relation which is equal to the value of the uncertainty principle because of capital X and capital P, we have seen that.

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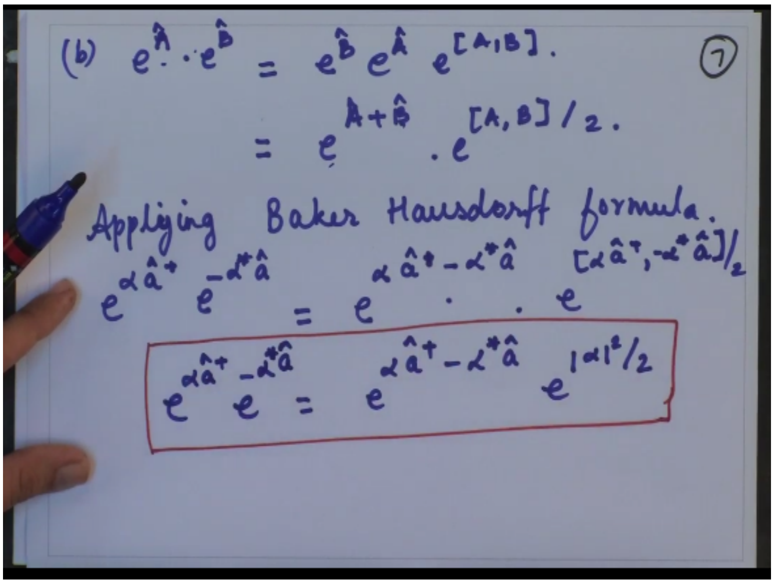

The second part or the b part, b part you are asked to calculate the exponential eB if commutator of A and B is equal to a constant then you have to show that e raise to alpha A dagger e raise to –alpha star A dagger is having some relation like e raise to alpha square/2 e raise to alpha A dagger-alpha star A. So we have this relation and we are going to use Baker–Hausdorff formula.

So what we have let me write here as B A e raise to A, B okay and you have this equal to e raise to A+B times e raise to A, B commutator of A, B/2. So using this relation and applying Baker–Hausdorff formula, we have e raise to alpha a dagger and another term is e raise to alpha star a okay. This expression I can rewrite using this result. I can rewrite this as e raise to I can write this as alpha a dagger okay.

This term -alpha star a, so I have used this times e raise to alpha a dagger-alpha star a/2 and this will be a dagger alpha a dagger-alpha star a and I can have alpha square outside and commutator of a dagger and a and a dagger is 1, so I have alpha square/2. So this is what we have proved. It is just one step okay. These relations will be useful in proving other results also. So now the c part is that we can rewrite the coherent state in terms of the displacement operator. Can we do that?

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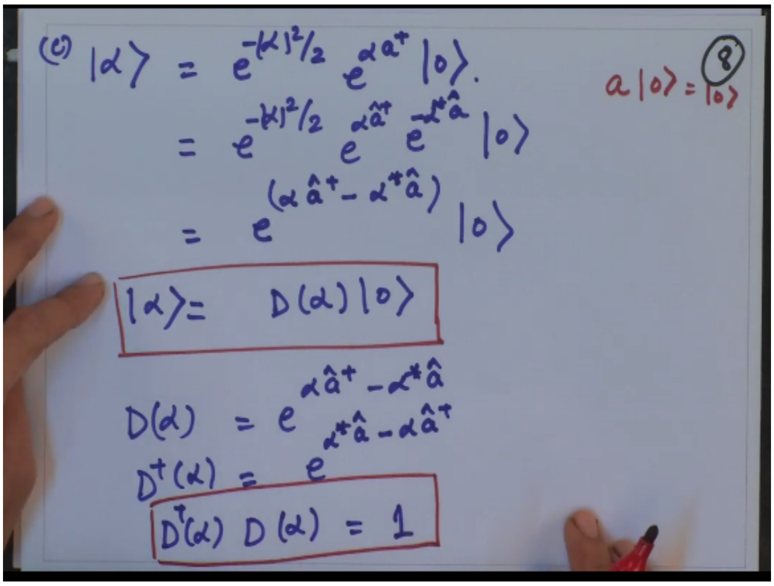

Now let us write the displacement operator we have to write in terms of the coherent state. So the coherent state given to us is e raise to -alpha square/2 e raise to alpha a dagger. This is what is given to us and remember we know that when you operate annihilation operator on any inner state 0, you will obtain 0 okay. It will not go further. This is the end of the ladder. So I can write this again as alpha square/2 times e raise to alpha a dagger.

I can write this as e star a, I can just multiply by some coefficient. It will not matter; we have used this okay. Now remember we had this relation over here okay. I will use this relation in this expression, so when I use this relation I will have this term times this and I have 1 exponential-alpha square/2 in the expression already. So I will just end up having e raise to a dagger-a star a on alpha.

And this is the definition of displacement operator and hence we can prove that this is the coherent state can be written in terms of displacement operator and you have to show in this part that the displacement operator is unitary. This is explicitly seen, it is very simple, e raise to alpha a dagger-alpha star a and D dagger alpha will be e raise to alpha star a-alpha a dagger okay.

So when I take D dagger alpha D alpha, so U dagger U is 1 that is it is unitary operator. It is very simple. You can see it without even solving okay. So these exercises and you had in your lecture few more exercises, try and solve it and see if you can do solve those exercises which were discussed.