Quantum Mechanics
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Lecture - 48
Tutorial 8 - Part I

Let us start with tutorial 8 in which we will deal with ladder operator. You all know what is the ladder operator, a and a dagger is your creation and annihilation operator which we are going to use and solve few examples. So let us get start with the first problem. First problem is a simple exercise wherein you operate multiple ladder operators on state ground state, ket 0 is the ground state.

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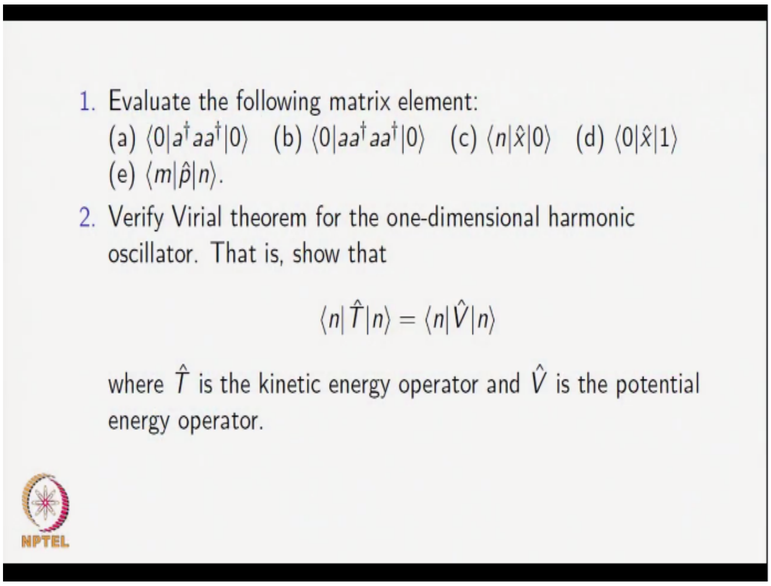

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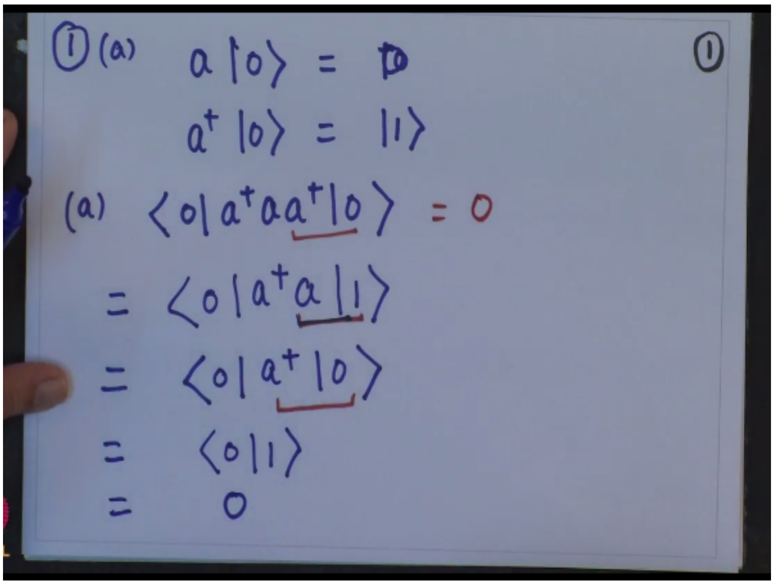

So you all know that first problem a part. Before we go to this a part, we know that when you operate a creation operator or an annihilation operator on the ground state, what you obtain? So when you operate the annihilation operator you get 0 okay. This we know and when you operate a creation operator on the ground state it will go to the state 1 okay. This you have to keep in mind while doing the examples.

So first problem, there are 3 ladder operators given to you, a dagger a a dagger 0, so ladder operator will either take you above in steps of 1 or it will take you down in the lower excited states or it can take you from lower excited state to the higher excited state. So a will take you down in the ladder and a dagger will take you up in the ladder. So this a dagger when operated on state 0 will be ket 0 a dagger a, it will go to the next state.

Then, so first we have operated this, you get this. So a dagger on 0 is 1, now I will operate 1a on 1 so it will lower the state, it will go back to 0 okay, you are going below the ladder. Then, when you operate a dagger on 0, you will obtain state 1, you will go to the state up. Now this is these states are orthogonal so you obtain 0 and 1 state which is 0. So the first problem you obtain value=0.

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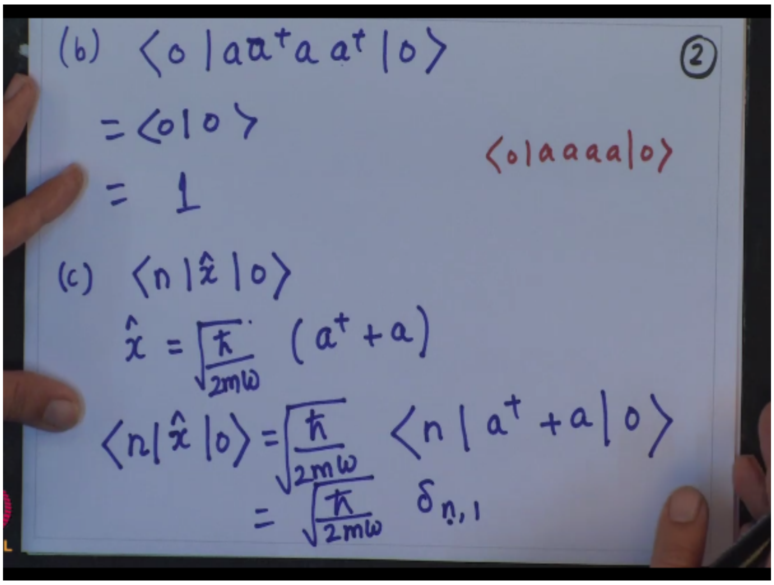

In the second problem, you have 4 operators, 4 ladder operators, b you can similarly obtain a a dagger a a dagger okay. So this will give me when a dagger is operated on 0, I will get a 1 state. Then, again I will come back to the 0 state, then again 1 state and then again I will come back to the 0 state. So if you solve it the way we did for the first problem, we will have 1 okay because we will have 0 0 that is 1.

So what do you observe? You observe that when there are odd number of ladder operators, you obtain a 0 and even number of ladder operators you get a 1. This is not a thumb rule but by observation you can see that even number of operators even number of ladder operators will give you 0 and odd number would give you 1. Again you have noticed here that there are two creation and two annihilation operators.

If I would have had this a a a a 4 operators again I would have obtained 0. So you should have even number of odd and sorry equal number of creation and annihilation operator to get a nonzero result. Let us go to the third problem. First problem, third part that is n x cap 0. Now you know you have to remember the definition of x that is a position operator and p momentum operator for the next problem.

So position operator can be written as square root of h cross/2m omega*a dagger+a okay, a+a dagger you can write it this way and we will also require the momentum operator for the next the last problem. So here I will have n is the nth state n ket and I have a constant term so x0 will be h cross/2m omega n a dagger+a on 0. This you can easily see when you operate a on 0 you will be you will get 0 okay. This term will not contribute.

This term will give you the next excited state. So your answer would be delta n, 1 because now if n is 1 then you will have a non-zero result, you will get 1. If n is any other number except 1, then you will have a 0 result okay. So similarly you can do problem number d but there you have initial and the final state are given to you which are 0 and 1.

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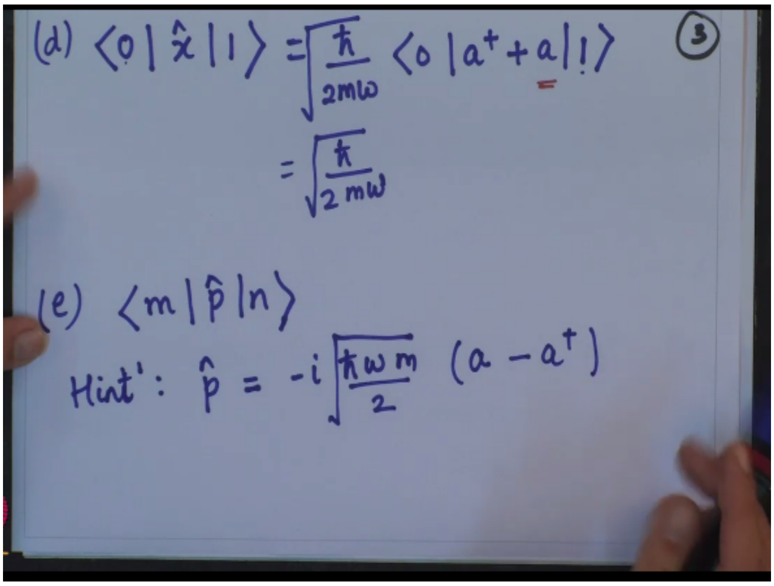

So the initial state is 1, final state is 0, 0 is the ket 0 x1, so this you will use the previous definition of x operator position operator and you have h cross square/2m omega times you have now 0 and 1 okay. So now you can guess from here, here you have 0 a dagger+a and here you have a 1. Since you have a 1 this term second term will contribute and here there is a dagger it will go to the next state that is 2.

So only this term would contribute for this problem, so you have h cross/2m omega, this will be in the square root and you have your this will become 0, so this will be 1. So delta 0 0 will be 1, so you are left with h cross square/2m omega okay. In the d part, we have got this because now you have a definite initial and the final state. Here you have 0 and 1. In the previous problem, we had n and 0.

So in e part, I left I will leave this for you to solve. It is a simple exercise. To evaluate this, I will give you a hint that is p is nothing but -ih cross omega m/2 in the square root and you have a-a dagger. So using this hint please evaluate this matrix element okay. So you will have some delta mn term, some mn or mn+1, n-1 depending upon what your which operator you are operating, creation operator or annihilation operator okay.

I think this is very clear, it is very new set, it is not at all difficult to solve, so with this I think we can go to the next problem wherein we have to verify Virial theorem for one-dimensional harmonic oscillator. So here you have to show that the when the T is the kinetic energy operator and V is the potential energy operator, you have to show that these two operator the matrix element of T with N and n is being operated.

The eigenstate are n and n and the same matrix element but with a potential energy sandwiched between the eigenstate should give you the same result. This is nothing but the Virial theorem. Now for that you have to start with the definition of the kinetic energy operator.

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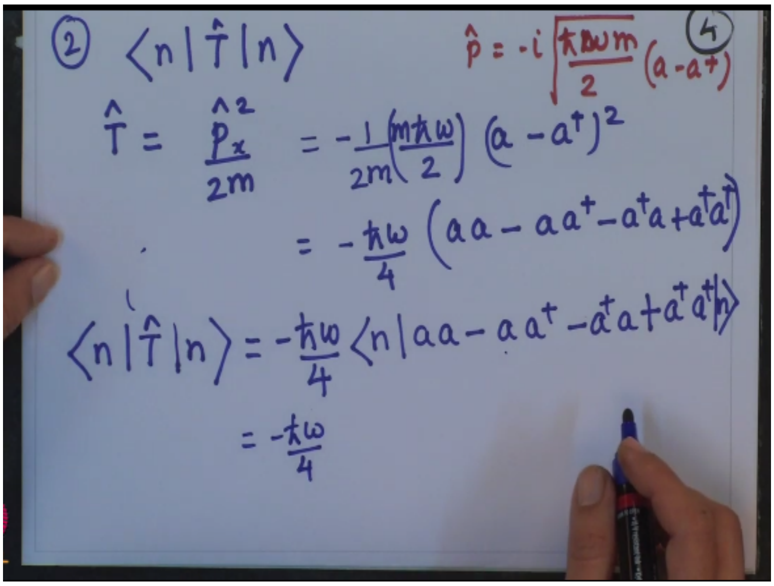

So second problem we have to evaluate first what is okay. For that you need to know the operator T which is kinetic energy operator which is p square/2m. We have done in many problems the kinetic energy term for harmonic oscillator is p square/2m and we are talking about one-dimensional harmonic oscillator. So you just have a px term, so I can write here px term okay.

Needless to say that it will be only the x component because we already know that it is a one-dimensional harmonic oscillator term. So this will be=1/2m will be as it is. What was operator p? It was i, let me write again here operator p was –ih cross omega m/root 2 times a-a dagger. This is what we had okay. So now p square will be i*i will give me - okay h cross omega m/2 okay.

Since it was in square root, you have this term and then you have a-a dagger square okay. So you will have, you have to be careful while multiplying these matrix elements, multiplying these operators creation and annihilation operators. The position of the operator is very crucial. So the first term would give me. So let me simplify this h cross square omega/4 a a-a a dagger-a dagger a+a dagger a dagger.

So this is the term which you will sandwich between the ket n, that is state n, the initial and the final state is n okay. So now my term for kinetic energy operator, the matrix element for the kinetic energy operator will be, this constant term I can take outside. So I will have a a-a a dagger-a dagger a+a dagger a dagger n okay.

A huge expression, again we have seen last time that I just gave a short in one sentence I just told that in the previous problem if you have equal number of creation and annihilation operator that term would contribute. Otherwise, the term with uneven number of creation and annihilation operator would give you a 0 result okay. So with that intuition, I would have so this these two terms would give me so a dagger when operated on n would give me n+1 and when you operate a on n, I will get n-1.

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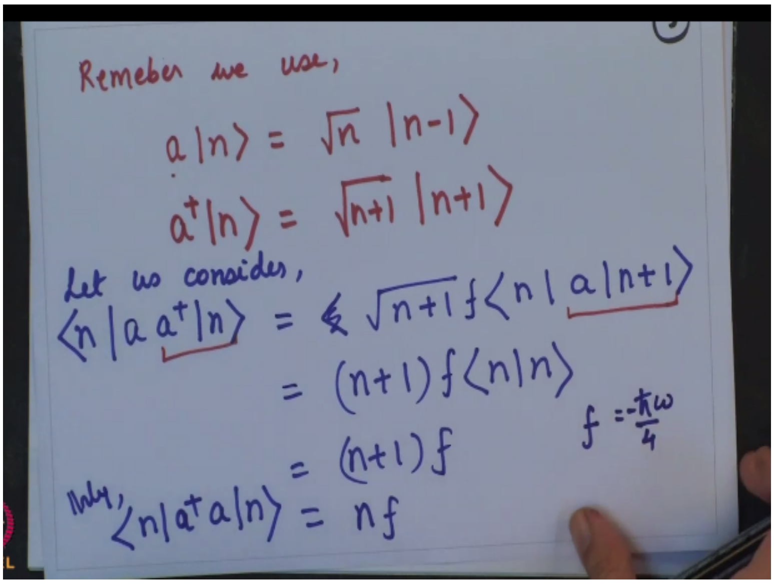

So remember here we are using one more thing which is just remember we use, what do we use? We use a on n when you operate a on n you obtain the operator a operated on the state n will take this state to a next lower state which is n-1 with this factor. You have already seen in your class and you have proved this in your lecture. So a dagger when operated on n would give square root of n+1.

So a dagger will take the eigenstate to the next state n+1 and you will obtain factor of a coefficient of square root of n+1. This you have to remember. So we have to use this. So when I use for a dagger okay let me do it separately so that you have you understand it better. So when I have a a dagger on n okay. Let us evaluate consider n and the n n is the matrix dimension and a a dagger is the operator which we have to evaluate.

So when a dagger is operated on n what do I obtain? Just a dagger on n would give me square root of n+1 and this will take to the next state n+1 and when again a is operated on n, a is operated on n+1 would give me square root of n+1 times the ket would go to the next lower state which is n, we will get back n so this will be n+1 n n. This is nothing but 1, so you have n+1 okay.

Similarly, you can obtain a dagger a n. So you can make a guess, here you will get n okay and I have dropped the factors okay h cross square upon h cross/2m omega, that we will put here whatever the factors were. For time being, I have just dropped the factors. So the factors will be there okay. So here what was our factor, it was for p it was h cross/4 I think right. We had this factor in h cross omega/4 okay.

So here so this factor I am talking about which I had dropped. So let us rewrite this, what do I obtain here is this term gave me n+1.

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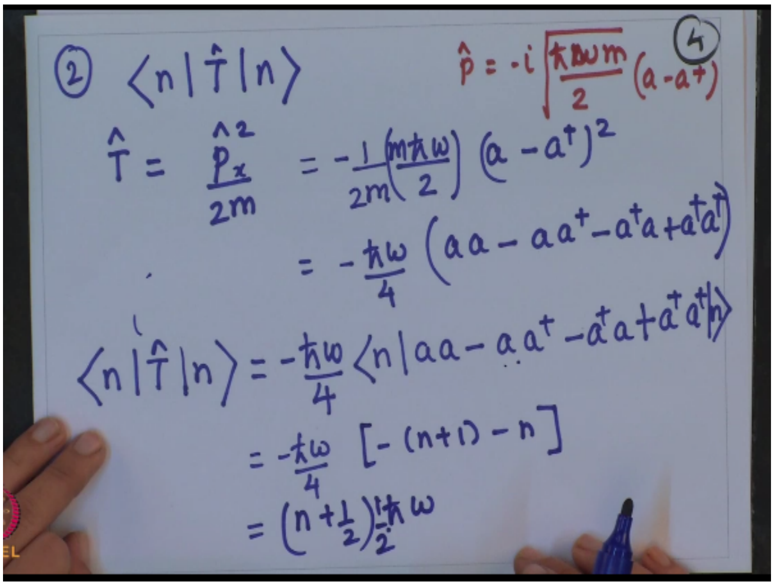

So I have –n+1-n, this is what I had and this I can rewrite as I will have 2n+1 okay. So I can write this as n+1/2. I just want to write it in some known form. So I am just writing it in this way okay. This is what I obtain for the kinetic energy operator right. Now it will be again not a difficult task to obtain the potential energy operator.

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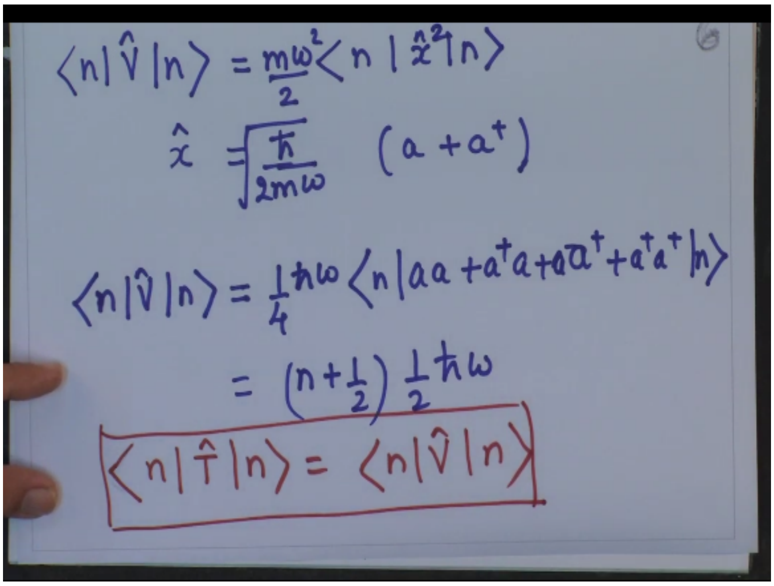

So potential energy operator would be n V n, this is what we have to evaluate. So this operator will be nothing but will be proportional to x square okay that is 1/2 kx square. So let me write it is proportional to n x n okay. That is m omega square/2 correct because 1/2 kx square I can write it as m omega square/2 times x square correct. This with the value of x operator we know. What is x operator?

We have seen before it is h cross/m omega 2m omega inside the square root a+a dagger. This we have seen and we know we will use this position operator and again find the value of. So the matrix element would look like I will try to write it in a simplified form. If in doubt please check once, h cross omega/4 after doing all this would give again a a+a dagger a+a a dagger+a dagger a dagger n.

This is what you obtain by taking the square of the position operator. Again, what do I obtain here is the same as the previous result. So you have to explicitly check this. It is similar but you can just go through once so that you are thorough with it. So what do we obtain in the final expression? We obtain that the matrix element of the kinetic energy operator and the potential energy operator turns out to be equal. This is nothing but the Virial theorem. So it was up till now things are very simple.

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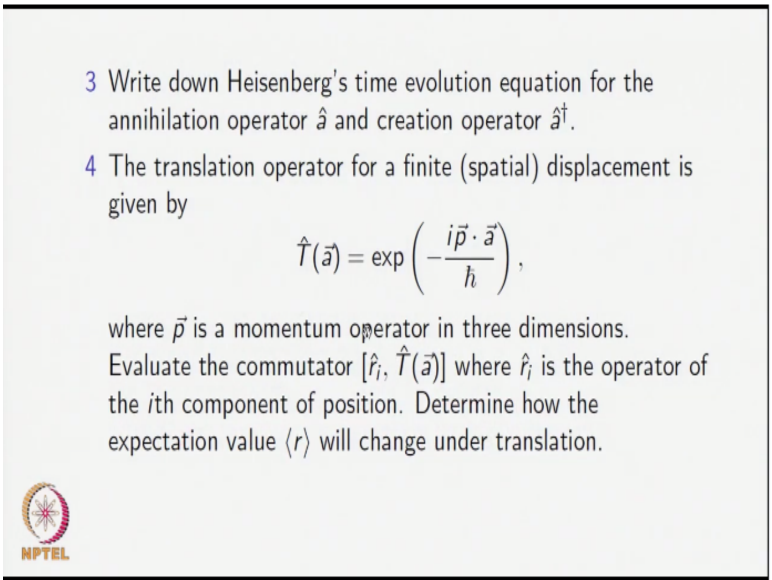

Let us go to the next problem. Write down the Heisenberg time evolution equation for annihilation operator a and creation operator a dagger. So for the ladder operators, we have to write the time evolution equation. In the third problem, let us start by assuming that the Hamiltonian operator is invariant in time, its time invariant.

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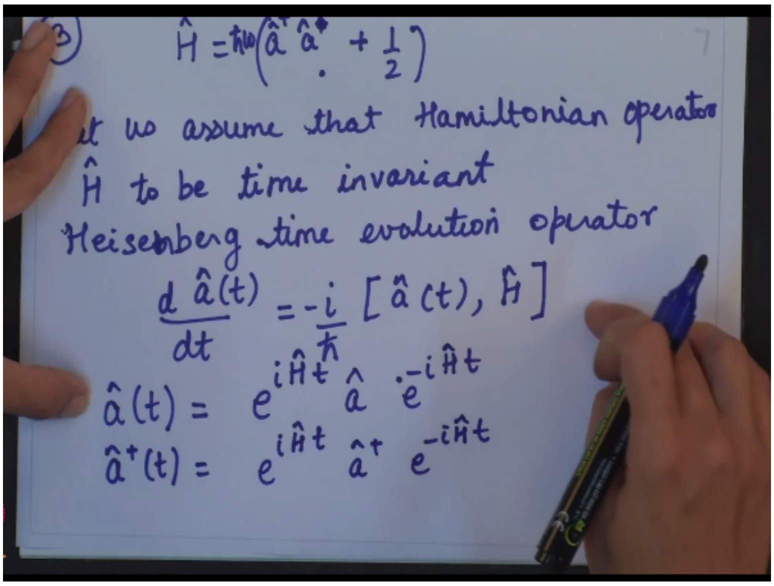

Because we know the Hamiltonian operator is written as a a dagger+1/2 h cross omega okay. So we know that the Hamiltonian operator, this is our Hamiltonian operator a a dagger with a h cross omega. So a dagger a+1/2*h cross omega is the Hamiltonian. We can see from here that it is time independent. The creation annihilation operator are time independent so our Hamiltonian is invariant in time.

So let us assume that Hamiltonian operator which is H to be time invariant okay. This can be seen from this operator definition. So this Hamiltonian operator we have written in terms of remember we have written in terms of the ladder operator, your usual notation we have always seen it to be p square/2m+V okay. So the Hamiltonian operator is time invariant and let us now write.

Let us write the Heisenberg time evolution operator. Let us write the time evolution operator equation. So what is the time evolution equation? Heisenberg time evolution equation for say operator a is –i/h cross. We have seen this definition for angular momentum okay. In the previous problem, the previous tutorial I think tutorial 6, we have seen. We have used this definition.

So we have this problem wherein we have taken the time evolution operator definition and what is operator a t. So when you go from Schrodinger picture to Heisenberg picture how the operator would evolve? It would be e raise to iHt a operator a which is time independent iHt. Similarly, what will be a dagger t? It would be e raise to iHt a dagger e raise to iHt okay. So we are going to use this definition in writing the Heisenberg time evolution operator.

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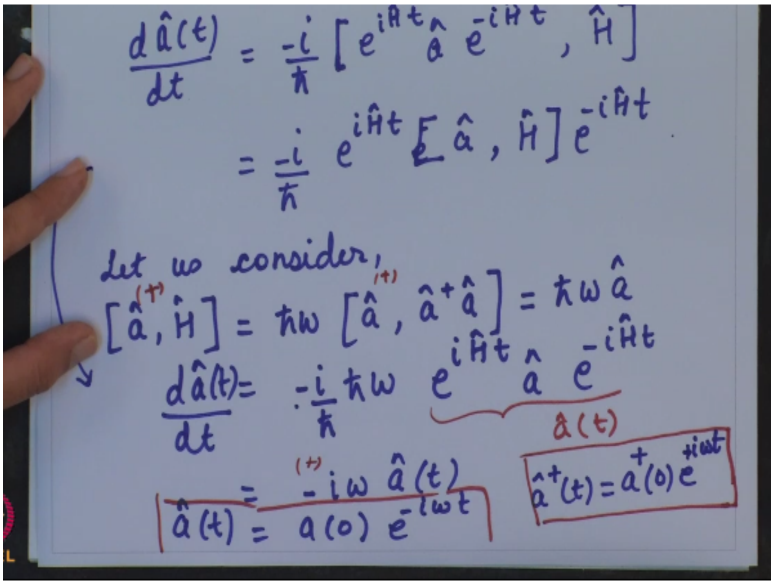

Heisenberg time evolution operator would then become d a t/dt will be –i/h cross and here I will have the definition which I had put. So e raise to iHt a e raise to –iHt. Now we have assumed that Hamiltonian is time invariant. So we can very well write this very easily write it as e raise to iHt e raise to -i sorry a because this is the operator on the left so this will come here H e raise to –iHt.

So this term we have got, now what we will do is let us now evaluate. Let us consider a, H. Now we know the definition of Hamiltonian operator which is h cross omega a dagger a+1/2. So h cross omega is common, so let me take it out and you have a a dagger a. The second term will not contribute. I will just not write it, so this you all know would give me commutator of a and a is 0.

So I will have commutator of a a dagger which is 1, so I will have just h cross omega a right. So what will happen to this? This would become, this would be d a of t/dt is –i/h cross this will be h cross omega e raise to iHt I have a e raise to –iHt. This is the definition of I can put this back. So I will have this as i, then I have omega a t. So a of t is a0 e raise to -i omega t. So we have obtained this.

Similarly, we can rewrite this equation for a dagger and we can obtain, let me write here for a dagger. So I have a dagger e raise to –i omega t. So this is the definition. So this will be a +sign okay. So be very careful with the signs okay. Be very careful when you are putting these signs. Here there will be change in sign because there will be a dagger remember. This will I will write in red so that so this becomes a dagger okay.

So I will have a dagger here and then a dagger will come out and I will have a dagger a which is +1. So I will have a +sign here okay. So hence this result. So we have seen now the time evolution operator for the creation and annihilation operator, we can write in terms of Heisenberg's time evolution operator. There are two more problems which we will continue. They will also have some interesting questions which we will discuss in the later part.

So keep solving such example. They will give you hands on the bra and ket notation as well as how to represent the position and momentum operator in terms of creation and annihilation operators. They will help you in further tutorials. Let us meet next time.