Quantum Mechanics
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Lecture - 47
Ladder Operators - II

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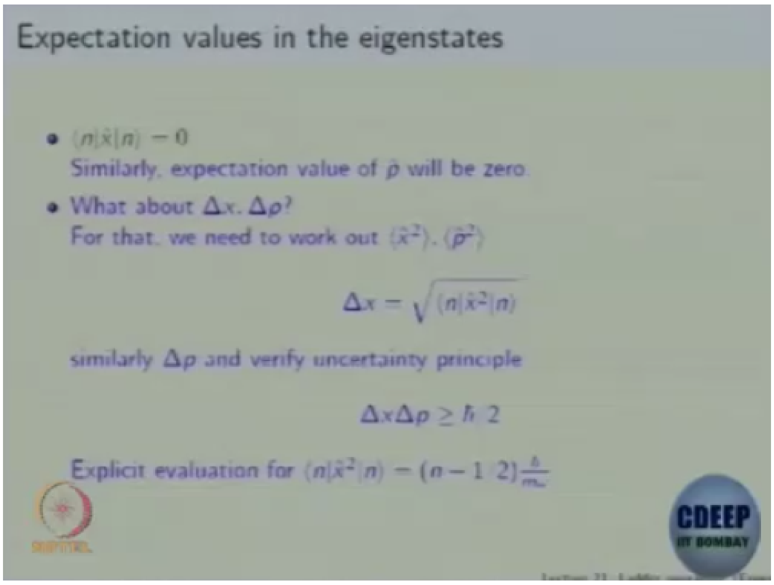

So explicit evaluation this we did already and similarly you can do this also, p squared and you can verify that they satisfy the product turns out to be >/= h cross/omega. What happens for n = 0, h cross/2, you get that equality. What are those states which satisfy the equality called? They are called the minimal uncertainty states okay. What is this uncertainty? I am sure you all know pictorially what is uncertainty.

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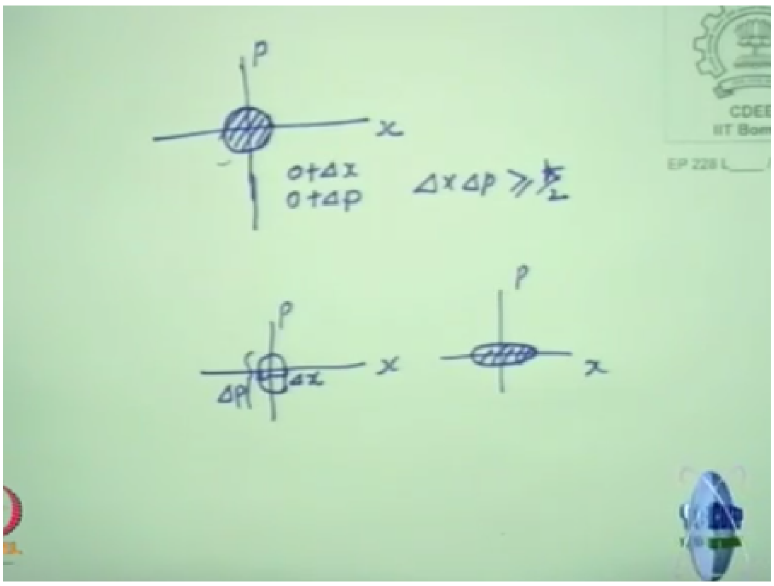

So if you take in classical mechanics as momentum and x what is this space called face space right. Typically, in quantum mechanics there is, suppose I want to say what is the, suppose I want to pin point origin in face pace that is okay in classical mechanics because you can simultaneously determine x and p, but in quantum mechanics you have some kind of haziness okay, that area should be >/= h cross/2.

I cannot say I am exactly at 0 at origin, but I will have some haziness where I will say it is origin + delta x and similarly origin for momentum + delta p such that delta x delta p should be >/= 1/2 h cross/2. When it becomes equal, this area will be a minimal area right with both delta x and delta p having equal legs if you take, it is a circle, it could also be an ellipses. You can have situations where you know you can have it something like this.

If you have something like this we say something is more certain than the other one what is more certain, delta x is < delta p right. So this is delta x, this one is delta p. You could have another case where you can have like this. The uncertainty region the hazy region has the delta x more than delta p if you want to precisely say that there are very little error in position suppose I say the position has very little error or I can measure the particle-like behavior which uncertainty is correct?

The one with delta x < delta p is correct. If I say that the wave, it has a single de Broglie wave with almost the same wave length, then it is this which has delta p which is more. So you need to see that the area of this will be satisfying this and the equality sign can also be done if both delta x and delta p are equal then this is what we will call it as a circle region okay, but if you have delta x and delta p to be unequal then these are different kinds of, we call this as squeeze okay.

A lot of relevance when you do optics, there are these squeezing states where you want to reduce the noise okay. So you try to do the noise reduction in the electromagnetic wave where there is lot of connections to the harmonic oscillators. Let me not get into too many details, but somebody is interested can look at that the minimal uncertainty states are satisfied by some states and you can have squeezing done to reduce the noise either in the amplitude or in the phase of the electromagnetic waves and this is really useful okay.

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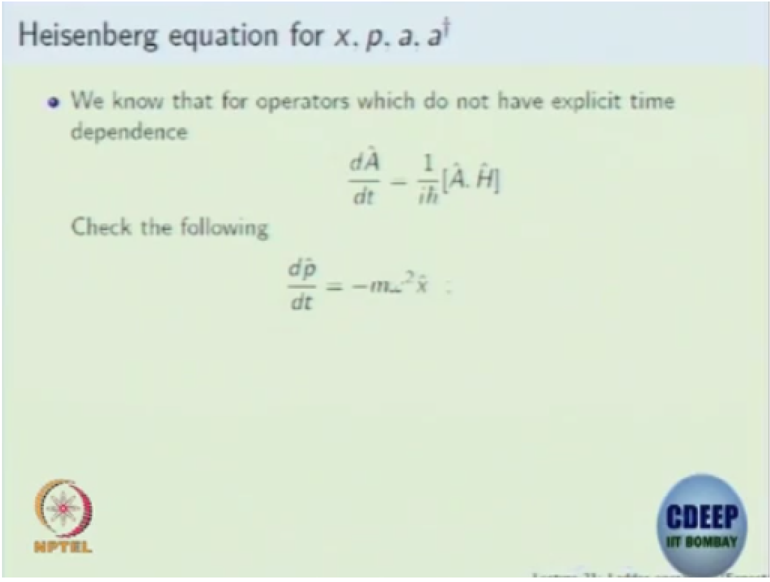

So then comes the writing the evolution equation for operators in the Heisenberg picture, which we have seen any operator if it is explicitly time-independent, there is no explicit time dependence for such cases you can write the time evolution of the operator involving for a system given by a Hamiltonian H, which is given by this, this we all know right. You have also done an exam problem in this.

This is for you to check for the harmonic oscillator, this is your familiar classical equations which is reproduced by the operator equation in Heisenberg picture. If I ask you to do for Schrodinger picture then what will we stay, Ehrenfest theorem tells you that it should be obeyed by the expectation values okay, because the states are the ones which evolves in time thing. Since I am writing it in Heisenberg picture you can write operator equation.

And thus operator equation will exactly resemble your familiar harmonic oscillator equation, is that right, but please check this. I have given it here for you to check, substitute here with the p operator and check it. Similarly, you can also show that dx/dt proportional to velocity and you can write it in terms of this. This is Ehrenfest theorem, but you see that this is 2 coupled equations, these 2 are 2 coupled equation. p related to x, x related to p.

But if you write x + ip and x-ip which is your ladder operators interestingly the equation becomes like a decoupled equation or uncoupled equation for both of them (()) (07:06) okay. If we get an uncoupled equation you can actually solve this, idea is to solve this. if you solve this you can try to put the initial time as 0 and please verify this is just a first-order equation. There will be a log function here and there will be an e to the i omega t.

So you can write a of t as a of 0 times exponential of this. I hope I have not made a mistake in this sign, please check it and tell me.

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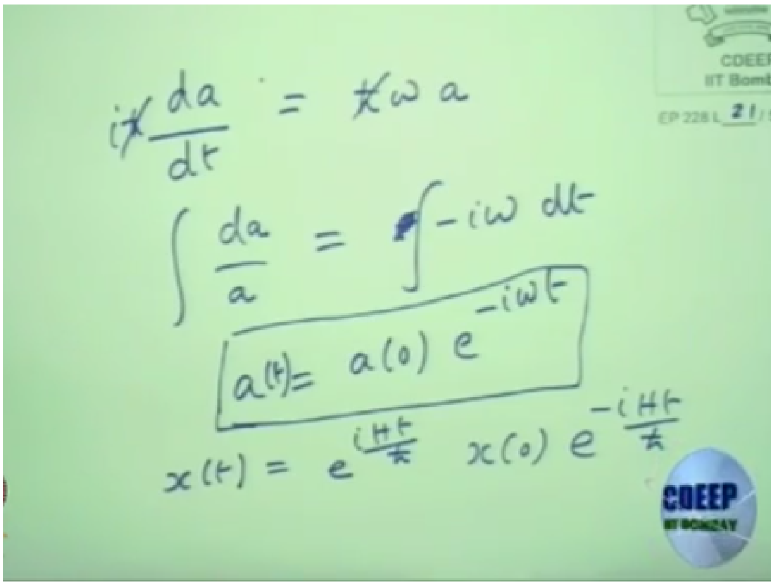

This is also something which you should see how beautiful it is that when I done the x and p to write x of t using that equation what will you do for finding x of t, you would have written e to the power of iHt/h cross, x of 0 right, so (()) (08:36) and doing this is much more harder, whereas I have written what is a of t, what is the time, after I get a of t, I can find what is x of t. Similarly, I can write for a dagger of t also.

Whereas here when you have to do it, this also can be done, it should be correct, both method should give you the same answer. It just becomes tedious whereas that one is because of decoupled equations you can get the solution at any time for the ladder operator and you can take a linear superposition a of t + a dagger of t will be proportional to x of t, okay. So that is why the a dagger of t is a dagger of 0, there will be a + psi omega t.

You can see the negative sign here and you can read off from these 2 solutions what is x of t by taking a of t + a dagger of t and similarly p of t.

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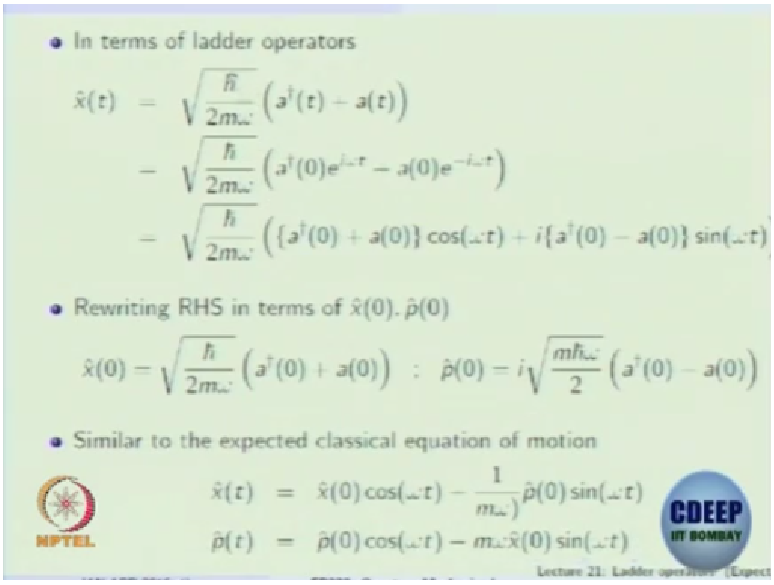

So how do you read off, this also I have put in here x of t is, maybe I have missed a half factor or this is correct? And then you substitute for time evolution of the a dagger operator and a operator, what else can we do further. You can write this exponential of i omega t as cos omega t + i sin omega t, do that. So you will have a dagger with a of 0 with cos omega t and you will have an a dagger with negative a of 0 with sin omega t.

This term is similar to the first one at t = 0, so you can write it as x of 0, what about this term? This will be a p of, this will be proportion to the (()) (10:29). So without any efforts without doing our Heisenberg evolution for position operator for the harmonic oscillator Hamiltonian I used the time evolution for the ladder operators right and I can read off the time evolution of this position operators.

So use x of 0 to be this p of 0 to be this and you can read off that x of t is x of 0 cos omega t + sorry there is a bracket here, 1/m omega times p of 0 sin omega t. This exactly similar to your classical physics, classical oscillator x as the function of t which is oscillations is exactly like this, but if you find expectation value on the number eigen state on both sides it is going to be X operator on the number eigen state is 0, p operator on the number eigen state is 0. So expectation value of x of t does not give you this time evolution at all on a stationary state.

Similarly, I will leave it to you to redo the same exercise for p of t, please do it and verify that this is answer okay. I did this elaborately for x of t, I assume you will do again the p of t and please check that you get this answer.

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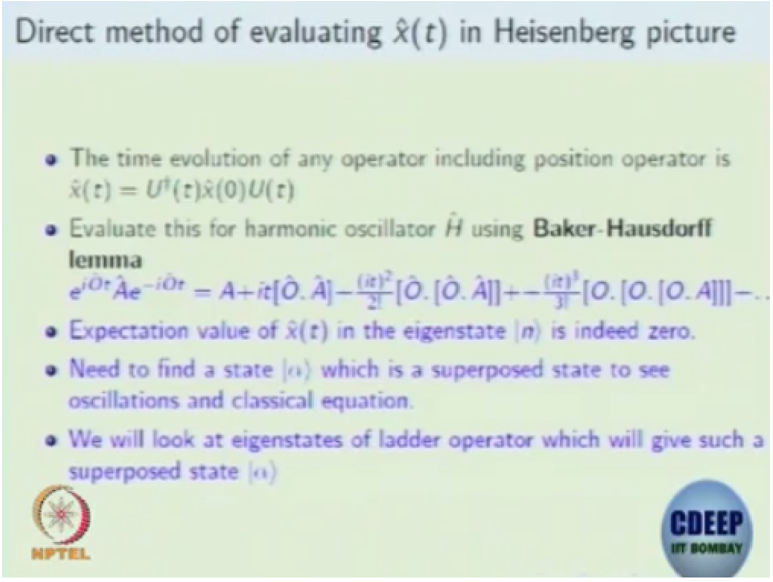

Just to give you a flavor of you can also do the x of t directly using this formal operator evolution, you can do this and verify whether both are correct. You have to use lemma which is called this Baker-Hausdorff lemma, I do not want you to memorize, you can use this and you write it in your formula sheet, but remember that when you have a U dagger operator and a U could be need not be for t, this parameter t is just for making contact with this.

You could put some other parameter there, some other constant. Instead of t you could have had some other A or B some constants, but whatever constant you put here please put it here also. So in this particular case I have used just to make it familiar here. You have a neat expansion. The first term which his t independent is the operator. Next term will be multiplied by it.

The third term will be it squared by 2 factorial and it involves a series of commutators. So it is an infinite series and cos omega t also has an infinite series right. You know how to write cos omega t as an infinite series, cos omega t is 1- omega t whole squared by 2 factorial, omega t whole power 4 x 4 factorial and so on. Sin omega t also has this. You can rewrite the (()) (13:50) solution in terms of that series.

You can compare order by order or looking at the series you can infer that they can be grouped as sin omega t and cos omega t. So this also I want you to do it as an exercise yourself and verify that using uncoupled ladder operators you could infer what is x of t, but you can also elaborately do without going into the ladder operators and you should get the same answer.

So this is just to stress that if you take the energy eigen state, number eigen state, the expectation values indeed 0. So in order to make contact with classical notions or classical equations we want to take a super posed state, we will call that super posed state as alpha and if we write expectation value of x of t on the superposed states I would like to get something which is similar to the x of 0 cos omega t + p of 0 sin omega t, I want to get that.

So what is the, we want to find the superposed state, before we try to find the superposed state we know the number eigen states are not simultaneous eigen states of a operator and a dagger a right. Why? a dagger a and a do not commute. Number operator does not commute with ladder operator, is that correct.

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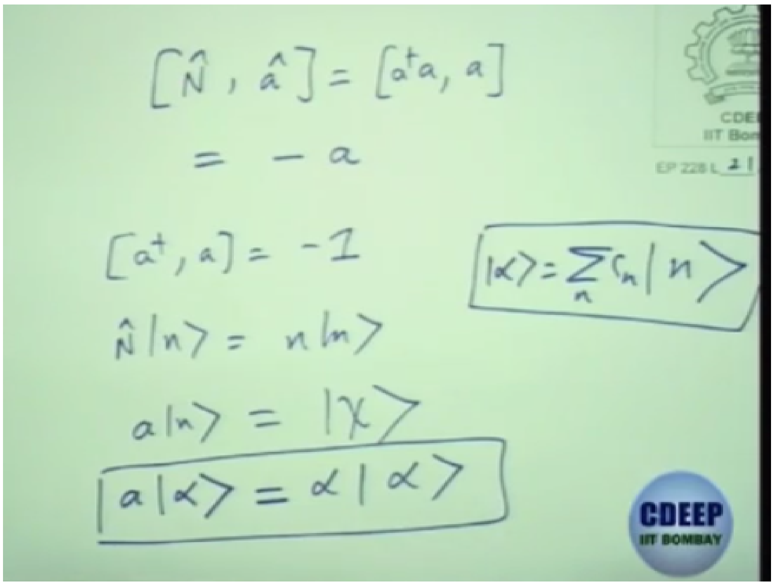

What is the commutator of number operator with a operator? It is a dagger a with a which is nothing but -a because a dagger with a is -1 correct, everybody is with me? So n hat on n if it is n times n, a on n will give you a new state. It is not a simultaneous eigen state of n operator, most probably that new state or most probably if I can find some state alpha to be eigen value times alpha that state is not same as n, but that state can be rewrite in as superposition of states.

I can always write that state as a superposed state, what remains then, what do we have to do? We have to find eigen state of the ladder operator. The motivation is we want to find the superposed states only if we find a superposed state, say expectation value of x of t will look like some constant times cos omega t + another constant times sin omega t which is classical equations.

Otherwise we cannot find expectation value of x of t which is showing these oscillations and to find that one of the states which you would like to take this an eigen state of these ladder operators, can we construct or can we find the eigen states of these ladder operators.

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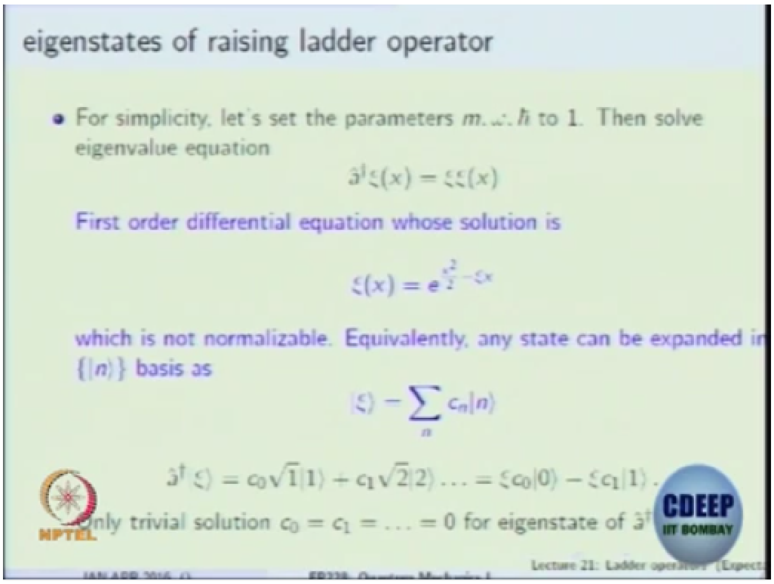

So eigen states of the raising ladder operators, so take the eigen value equation on the wave functions because a dagger is a first-order differential operator in the position space and I hope I will be able to solve it. So try to solve the first-order differential equation. This will involve a d/dx + nx on psi of x so that first order differential equation I am sure all of you can solve.

But is this wave function allowed is this normalizable? this is not normalizable, it is not allowed that is what we see or equivalently I cannot find normalizable wave function solution for this eigen value equation, can we see from the Dirac formalism? Write the xi as a linear superposition of the number eigen states. I want to show that the only possible solution here for normalizable solution is all the c's are 0 should be trivial okay.

This is what null state, this is what we should, can we verify that here, try and do that, a dagger on psi will give you c0 root 1 times 1 + c1 root 2 times 2 and so on and equate it to eigen value times the initial state. Now you compare, it is a complete set of bases 0 to n, 0 to infinity infact, when you compare there is no 0 ket on this right hand side, there is no analog here. Only way you can make both match is put the c0 coefficient to be 0.

If you put c0 to be 0 then ket 1 is gone and then c1 has to be 0 and so on. What is the possibility? Only trivial solution c0 c1, so on up to all the coefficients are 0 for eigen state of a dagger. Let us redo the same thing for the a operator, a operator, the only difference is that the differential operator here will have a different sign, this +ip –ip or vice versa. Can you do it and tell me what you will get. Try to do that.

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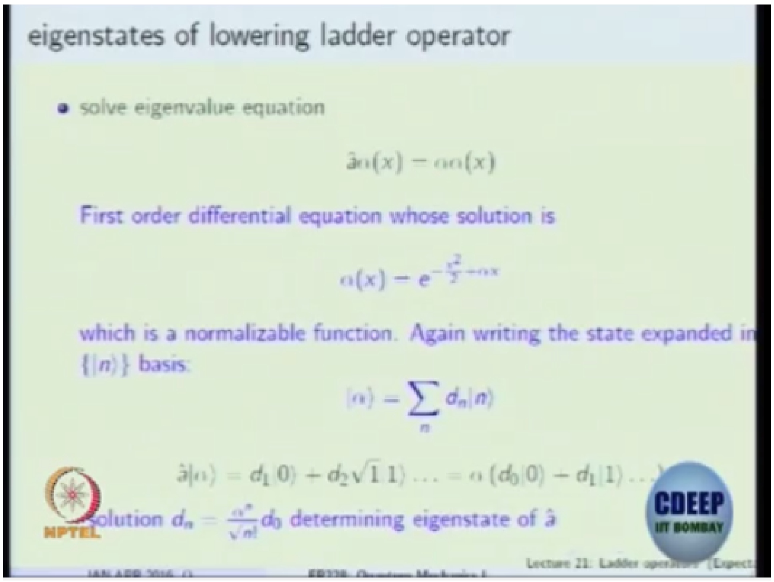

I am calling that as alpha of x, can you solve the first order differential equation again, and see whether you get a solution, is it normalizable or not normalizable? It is normalizable, so at least I have a hint that I can find a normalizable wave function for eigen states of the a operator. This wave function, I want to write the state alpha, so again write the state alpha in terms of the number operator eigen basis.

What is my aim, I want to determine all the dn's, I know it exist, but I want to know what exactly are these coefficients, how do you do that? how will you do this? a on alpha and then a on n and write out the eigen value equation. Please do it yourself and then compare the ket 0 with ket 0, ket 1 with ket 1 and relate the coefficients. Alpha times d0 will be d1, alpha times d1 will be d2 times root 1.

From there it can be determined all the coefficient in terms of d0, are you getting it, dn is proportional to d0 and the proportionality is alpha to the power of n/root n factorial where alpha is an eigen value of the state ket alpha and it is an eigen value equation for the lowering ladder operator. Okay so we have actually determined what is alpha and how do you fix d0, normalization.

Say that alpha is the normalized state and you fix by the normalization condition what should be d0. Can you fix d0 now using normalization condition?

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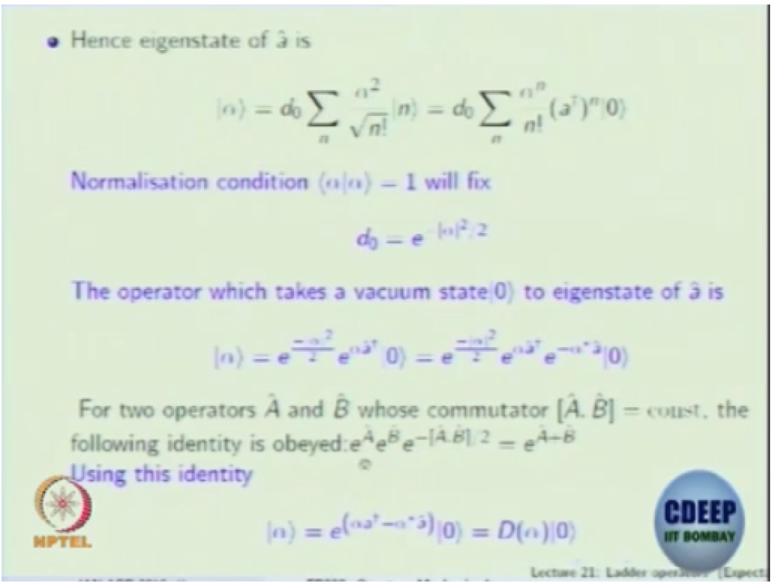

So alpha state in terms of d0 is written, I have rewritten it here and I have also told you that the ket n can be written as raising operator to the power of n/root n factorial, it is already a root n so together it becomes an n factorial. these are various ways of writing the state alpha which is an eigen state of the a operator with eigenvalue alpha. Please do the normalization condition, alpha in general could be complex, why?

Because a operator is a linear operator it is not a hermitian operator. If it was a hermitian operator I can assertively say alpha is real, but because a is not a hermitian operator right, a is not equal to a dagger. So alpha could be in general complex eigen value. So if you take the norm, you have to make sure that alpha, alpha star, please do that and mod alpha squared is alpha alpha star and some more exercises I here you can check it out.

Rewrite once I have d0 you can put the d0 here, you can write this also compactly. What is this expansion, this expansion is nothing but exponential of alpha a dagger. So I am now playing around only with operators, this is an allowed superposed state, which is an eigen state of the lowering ladder operator with eigenvalue alpha, alpha is in general complex and you can play around and write this as an exponential operator.

Can I also add some exponential operator of a here in front? Am I allowed? Before the ket 0 I can operate by any function of a operator, exponential of an a operator, what will it be 0 or 1? The first term will contribute; it will behave like an identity operator right.

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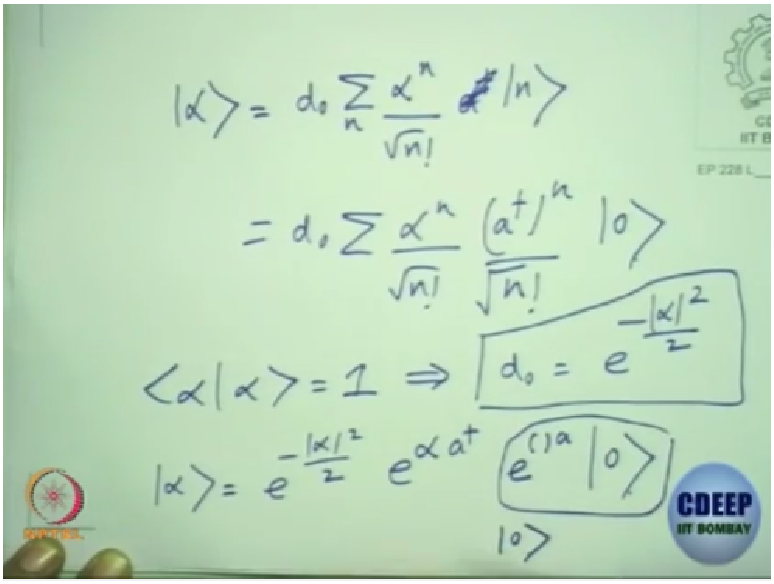

So I have written alpha to be d0 summation over n alpha to the power of n, a dagger sorry I can write this as n state which I can write it as do alpha to the power of n, root n factorial a dagger to the power of n by root n factorial on 0 and we have fixed alpha alpha to be 1, which implies you get d0 to be e to the power of –mod alpha square/2, this I want you to do the exercise, please check it out.

And you can write alpha to be a to the - mod alpha square/2, e to the alpha a dagger, I am just saying I can also put in something times a on 0, is that allowed? Why? e to the a is 1 + a + a squared/2 factorial and so on, a on 0 is always 0, e to the a on 0 will be, this will always be the same state, this will be like an identity operator okay. So I am just trying to introduce a convenient factor there, a convenient factor.

The reason for introducing this will be obvious from here which follows an identity. If you have 2 operators take A and B to be a a dagger, the commutator of a with a dagger is constant, its identity, commutator of LX with LY is it a constant or some new operator? It is a new operator. So there you cannot use this identity. You can use this identity only if the commutator of the 2 operators is a constant.

For x and p you can use, you can use it for a and a dagger, you can use for any other operators which have this property okay. What is the identity? Identity is product of 2 exponential operators times exponential of negative of the commutator/2, you can rewrite it as exponential of A + B. So I am not saying you should try to verify this, but you can verify and this is the identity which we are going to use.

So you have, you can check if you take this alpha star with the negative sign then you will get this as a commutator and you can rewrite the alpha state very compactly on the exponential, put a sum of 2 operator. This is not same as e to the A times e to the B, e to the A + B is not same as e to the A, e to the B, when will it be same? If the commutator is 0 okay. Otherwise e e to the A + B is very different from e to the A, e to the B.

The explicit relation is e to the A, e to the B, e to the - commutator of this relation is true only for operators which satisfies this condition. You cannot apply this for other operators. Incidentally what is the property of d operator? will it be Hermitian? will it be unitary? You know, I will leave it you to check all these things, it will be unitary right, very simple, yeah, the norm has to be satisfied and you can show that it is unitary okay.

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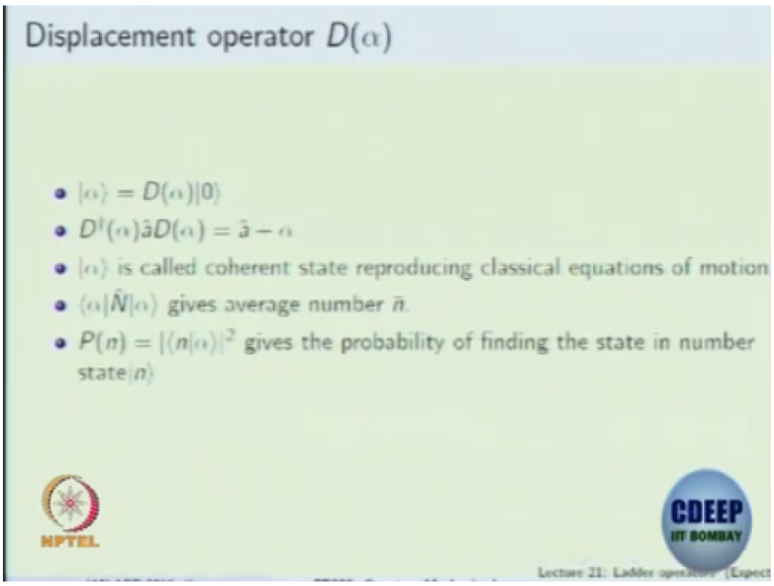

So these are the properties of the displacement operators which I will give it to you in your assignment also and incidentally this alpha is called as a coherent state which is an eigen state of the lowering operator and which will reproduce all your classical equations, oscillator equation by that I mean if you take expectation value of x of t in the coherent state you will reproduce your results which you have studied okay. I stop here yeah.