Quantum Mechanics
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Lecture - 46
Ladder Operators - I

So today I am going to continue last lecture I introduced for you ladder operators, raising ladder and lowering ladder operators right. So today I am going to try and I am going to do more on ladder operators using them how to find expectation values it becomes much easier, what will be the Heisenberg equation for the ladder operators or Heisenberg picture. How the evolution of these ladder operators happen?

And then to make things close to classical physics where you know how the harmonic oscillator if you try to write the position x of t you write it as a cos omega t+b sin omega t kind of expression. You need to find the expectation value to also have similar behavior and that will happen only if we start looking at specific set of states which is called coherent states okay. So this is why this coherent states is important okay. So this is the theme for today based on using your ladder operators okay.

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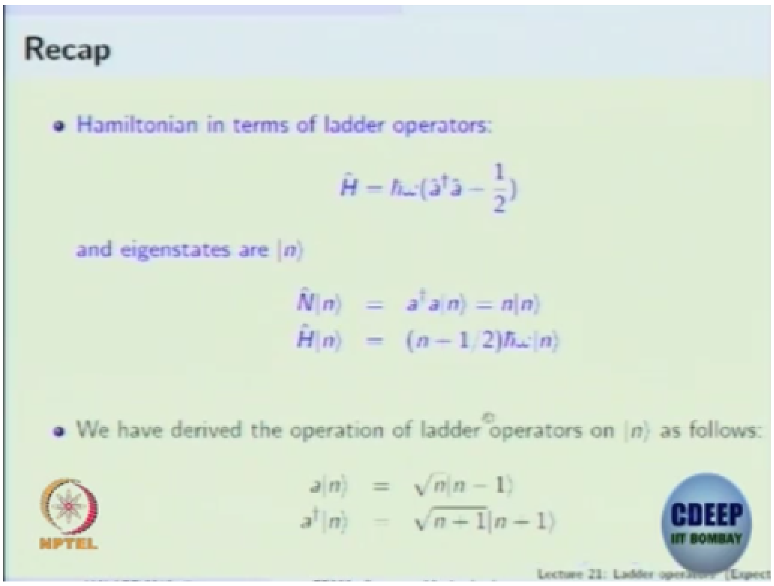

So let us just recap what we did in the last lecture Hamiltonian in terms of ladder operators right. It is a dagger a +1/2 * h cross omega, dimensionally correct, a dagger a is the number operator which is dimensionless and their eigen states which are denoted by the ket n, the number operator on the ket n nothing but a dagger a and it is an eigen state with eigen value n and n’s are integers, this also they have argued last time.

And the corresponding energy of the same state it is the simultaneous eigen states of number operator on Hamiltonian because it is just proportional to it (()) (02:31) so that is a shift n+1/2 h cross omega, is this clear? And we also derived that these are not eigen states of the ladder operators and the definition of a ladder operator a is lowering operator which takes n to n-1 and this coefficient has to be strictly square root of n, this also we derived.

We wrote d lambda and c lambda and we tried to fix these coefficients. Similarly, a dagger raises the state, by raises I mean it takes you from the lower energy state to a higher energy state with 1 step higher with corresponding coefficient which is n+1.

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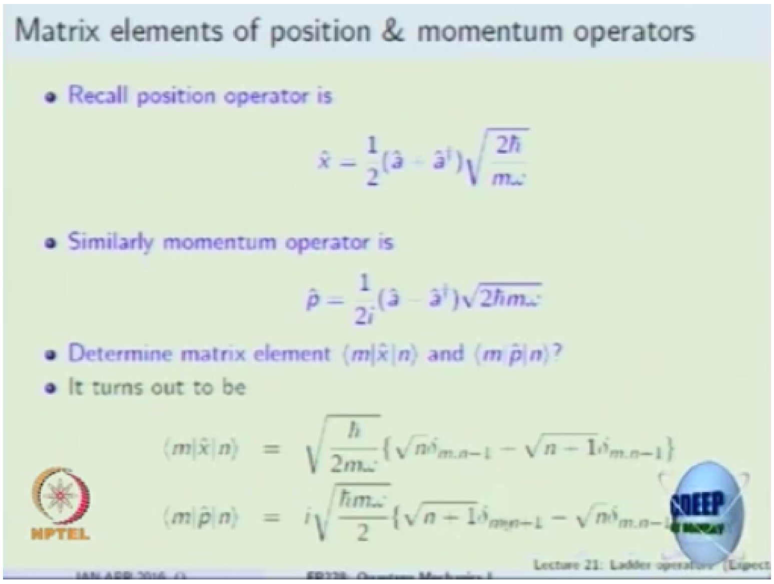

And recall we rewrote the position operator in terms of the ladder operators. We wrote momentum operators in terms of the ladder operators. You can now use the earlier slide properties and try to work out a matrix element between 2 states with integer n and m. Whether this will be 0 or nonzero. Shall we work it out? It is really a power of these ladder operators helps you to do this very nicely. So let us do that.

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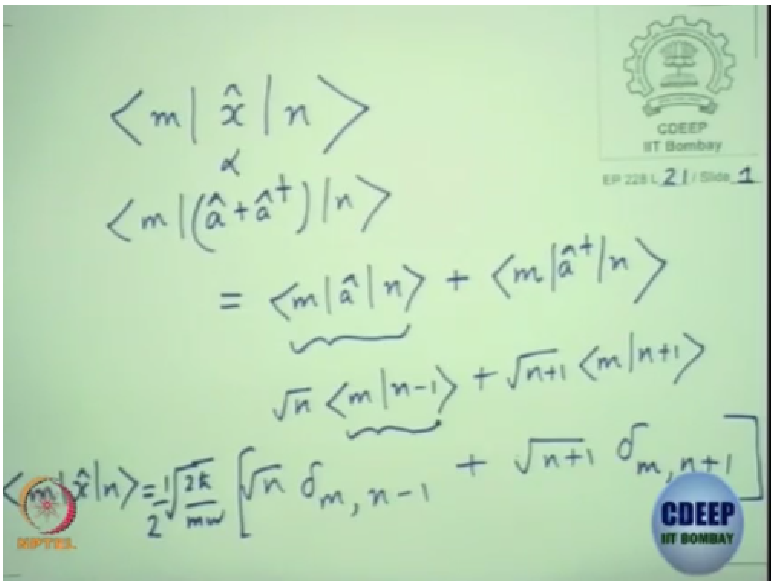

So you want to work out the matrix element of the x operator between m and x right. What is x apart from the factors we will put the factors later, this is proportional to a + a dagger okay. So this is nothing but m, a, n + m a dagger. What is this? a on n root n times n-1 you have m n-1 and a dagger on n, root n =1 m n+1, these are all what, they have to be orthonormal. So it will be root n delta, Kronecker delta + root n+1 Kronecker okay.

So this is what you will have apart from the factors if you want to write the factors, factors will have a root of 2h cross/m omega times 1/2 apart from that you will have this to be the matrix element of the x operator. Similarly, you can work out for the p operator. What is the modification? Let be a-a dagger and then the factors will have 1/2i and so on, okay. So this will be the matrix element which is written nicely when you write the x operator in terms of the ladder operators.

What will be the expectation value of x in suppose n = m, it is 0 right. If suppose I ask you what is the expectation value?

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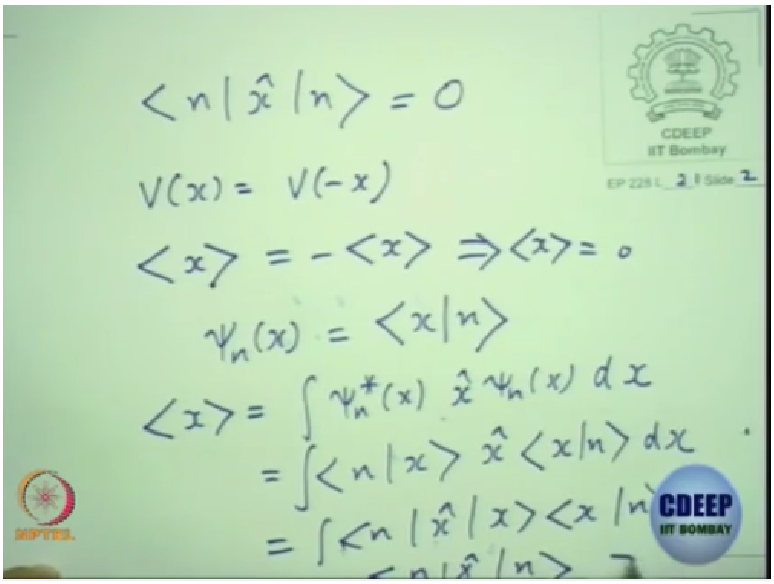

You can blindly say expectation value of x operator is 0 because a will take n to n -1, a dagger will take n to n+1, the inner product will always be 0. So here we did not do this explicit calculation of finding the expectation value which we did in the wave function formalism without even doing that we could say that the expectation value is 0 right. So that is the important interesting fact right.

If you remember I had asked you in harmonic oscillator V of x is V of –x, so if you want to find expectation value of x, it should satisfy that property x expectation is same as –x right and this implies expectation value is 0 right. Now we do it by the oscillator number eigen states and we see that explicitly it is happening and you can write psi n of x as x with n and you can find expectation value of x right.

You could try to rewrite in terms of psi and star, you can do this, you can rewrite this as n x, x operator, x n dx to give you an eigen value and then or you can even remove right. You can do this okay and even put it as n x operator x x n dx and then this becomes explicitly which is nothing but 0. I have used this identity, complete (()) (09:10) okay. So you can do various things by writing things in a operator fashion.

In fact, this in the position space you do not need to put it as operator, but you can try to write it as an operator on x and remove the unit operator and get. So we went through this in general for any operators I am just trying to say that how powerful is the ladder operator that you can directly get many of these expectations. What is the next one we could do? Uncertainties, what is uncertainty in x.

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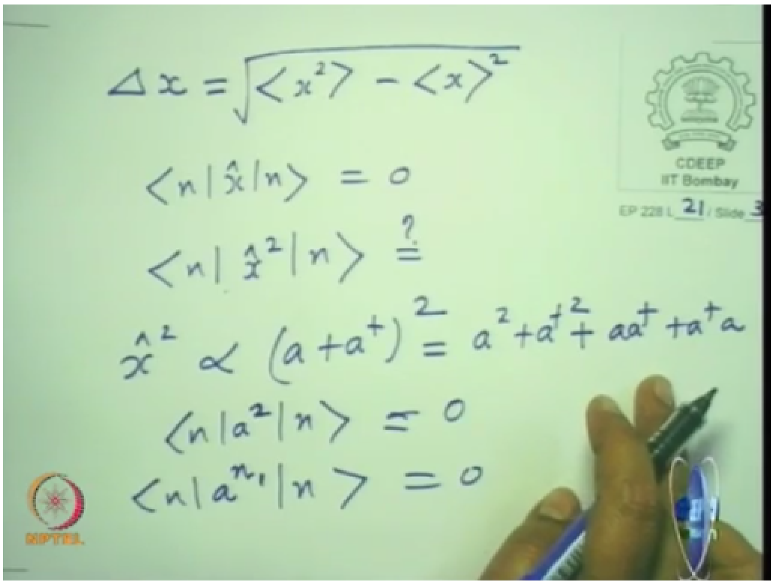

Delta x is expectation value of x squared - expectation value of x under square root. So in the number state n, x operator n is 0, what is n x operator squared on n. So rewrite this as x squared is proportional to a + a dagger the whole squared, I am not writing those constants, we will put it later, what is this? a squared + a dagger squared + a a dagger + a dagger a, do not write it as 2 a a dagger. aa dagger + a dagger a.

What is the matrix element of a squared somebody? a will reduce it, it is a lowering operator the n-1, another a will reduce it to n-2 and then the matrix element of n and n-2 is always 0, this is going to be 0, in fact matrix elements of any power of ladder operator sorry some number let me put it as this is always 0 you agree. The reason is that, it is going to lower the state and when you take the inner product they are orthogonal states it is going to be 0.

Is everybody with me, so the first term expectation value is 0, second term expectation value is 0. So the nonzero contribution can come only from if you have equal number of raising and lowering operators, that is the first thing you should note. If you have equal number of raising and lowering operators you get back to the same point, you raise n number of steps, lower n number of steps, get back to the same point then the matrix element will be nonzero.

Workout what is expectation value of x square which is nothing but expectation value of aa dagger + a dagger a up to the constants.

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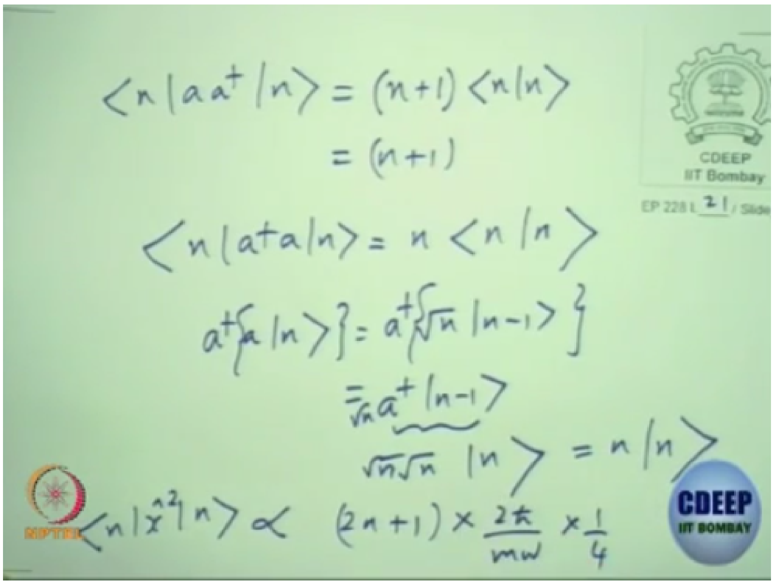

aa dagger on n is n + 1, n + 1 correct, is it right and n a dagger a on n will be, a will take square root n a dagger will give, it is already reduced n-1, that will give you a dagger on n-1 with the square root n. So this piece is another square root n times n. So expectation value of n x squared n is proposal to up to those factors you will have 2 n + 1 times what is those factors, it is 2h cross by m omega *1/4, is that right.

So this is what you get for the expectation value of a squared. This is also consistent with the fact that for symmetric potential v of x = v of –x even powers of x squared, even powers of x matrix elements, expectation value has to be nonzero, which is what we find. Odd powers of x, expectation value of x is 0, you can also show expectation value of x cube will be 0 why? When you take the cube the number of ladder operators which is raising will not be equal to the number of lowering operators when you take the cube.

Only even powers will allow at least some terms to have equal number of raising lowering operators sorry raising ladder operators or lowering ladder operators, that is why those matrix elements are going to be nonzero. So then you can determine what is the uncertainty.

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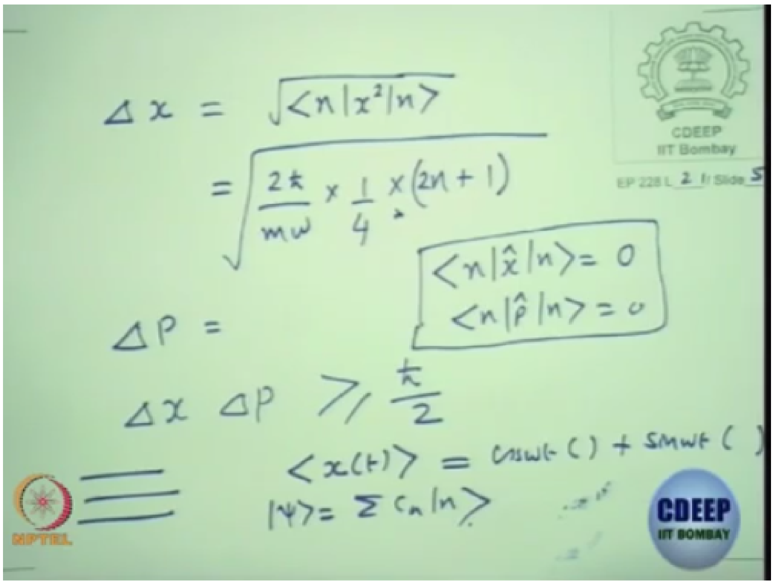

Delta x in this essentially it is other term is 0, right expectation value of x is 0. So this one will be, is that right. Similarly work out for delta p, not going to do this for you, please do this and what do you have to check delta x, delta p should be greater than or equal to h cross by 2. So please do this delta P and make sure that you are able to verify this uncertainty relation. So the only thing which is upsetting about this number operator states is that the expectation value of x operated is 0.

Similarly, you can show p operator is also 0, which means there wold not be any evolution of these matrix elements with time right and you do not expect it also why? Because these are all stationary states and whenever you want to look at transition and oscillations you cannot do it by living in the stationary state. If you want to look for oscillations, want to see your classical oscillations you need to be in a superposed state.

I said this right, some point. You have to be in a superposed states of a number of states or 2 states then only you start seeing that the expectation value as the function of time is cos omega t times something + sin omega t times something. If the state psi is a superposition, if psi was only a particular eigen state then this x of t is also 0.

If you want to go and look close to classical physics, harmonic oscillator, then you write the position, solution for the position or momentum which is evolving in time, it should have this oscillation cos omega t and sin omega t, you cannot get it with the stationary states. Okay so let me just summarize here that if you want to determine these matrix elements which you have already done we have verified the first equation.

This will be on the module, you can verify the second equation and check whether what I have written here is correct okay.

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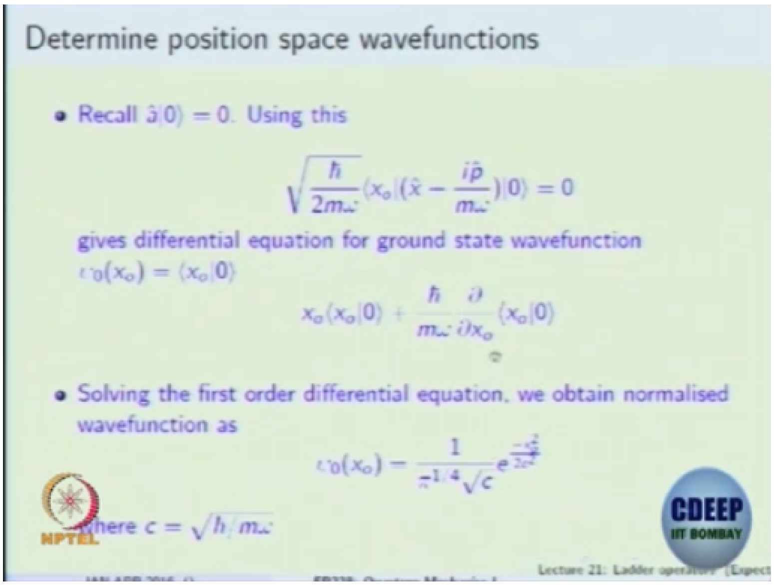

The operator a when it acts on 0 it is going to be 0. Sometimes this is little misleading why? It appears like a 0 appears like a null vector, is it a null vector? It is not why? The number the Hamiltonian when you operate it on 0, it is giving you nonzero energy + the state norm of the state is 1, okay. Even though I write it as 0 and it is the a operator on 0, is 0, this is not a null vector, do not get confused.

Null vector is one where any operator operating on a null vector will be 0 and the norm of that vector will be 0, norm of this 0 is also 1 okay. So please remember that just because I write a 0 here you should not interpret it as that it is a null vector. So you can call the 0 as a vacuum state.

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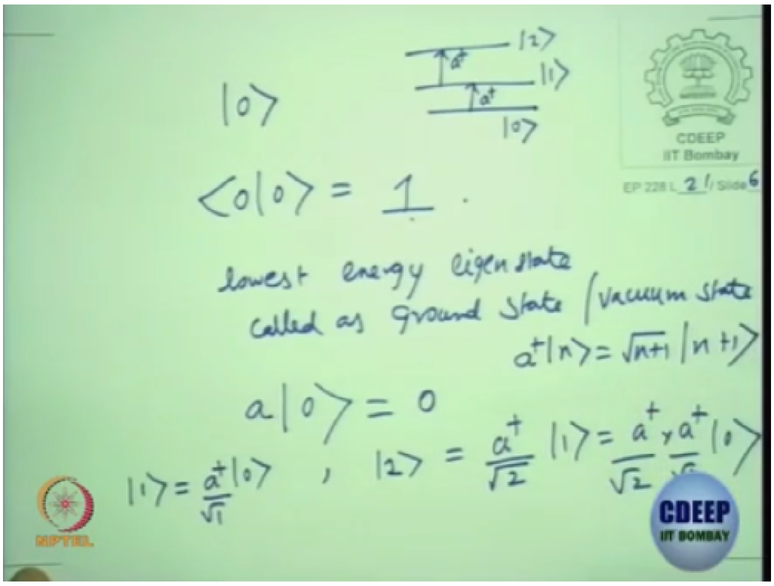

0 has the property that 0 0 is 1, it is the lowest energy state, energy eigen state called as ground state or vacuum state okay. So this condition clearly tells you that it is not a null state. Lowering operator on the ground state is root n times a lowered state that root n becomes 0, so you cannot lower further below the 0 state which means there is a lower bound which is 0, you cannot go below that, you can only keep going up by the ladder operation okay, is the picture clear?

You have some parallel processing going on or is it clear, okay. So this also we went through that using this take the inner product with an x0 on the state and write this to be a first order differential equations now. So the first order differential equation in the position representation for the momentum is del/del x0, h ih cross so you have this –ih cross. So there is i with a –i will give you a +1.

If there is any typo you should tell me. So what is this, x0 with 0 is the ground state wave function at a position which is x0 and this is also the ground state wave function. It is not difficult for you to solve this. Solve the first order differential equation and then you make sure that you want to normalize it and get the normalized wave function. See how simple it is. I am just trying to tell you here I am not looking at second order differential equations or anything.

I am just using the power of this ladder operator and I get a first order differential equation from which I find the ground state solution, not invoking any special functions or anything.

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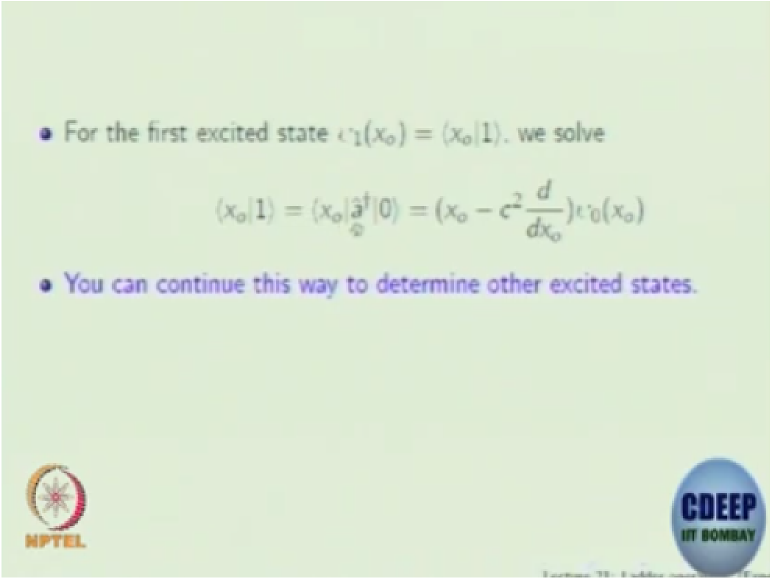

How do you find the first excited state? First excited state will be take the ket 1 find it is projection on x0 that will give you the position space wave function of the first excited state at the point x0. I can write that x0 with 1 as a dagger or 0, a dagger if I operate it on 0 it is root 1 times 1, which his exactly this, root 1 is just 1 and then what is a dagger, you can write it in terms of x-ip substitute them and you get again a first order differential equation.

You know psi 0 now you can differentiate and write the psi 1. So it boils down to not knowing any special functions, but you can systematically solve. If you want to find a ket n what do you have to do? I have to do a dagger to the power of n, but there will be a factor you have to divide it by root n factorial. So if I want to write ket 1, it is a dagger on 0/square root 1. If I want to write ket 2 it is a dagger/square root 2 on 1 which is a dagger/square root 2, a dagger/square root 1 on 0.

I have used this fact that whenever you have to do a dagger on n it is square root n+1 times n+1. So I am just bringing that factor below and writing things. So what does this tell us?

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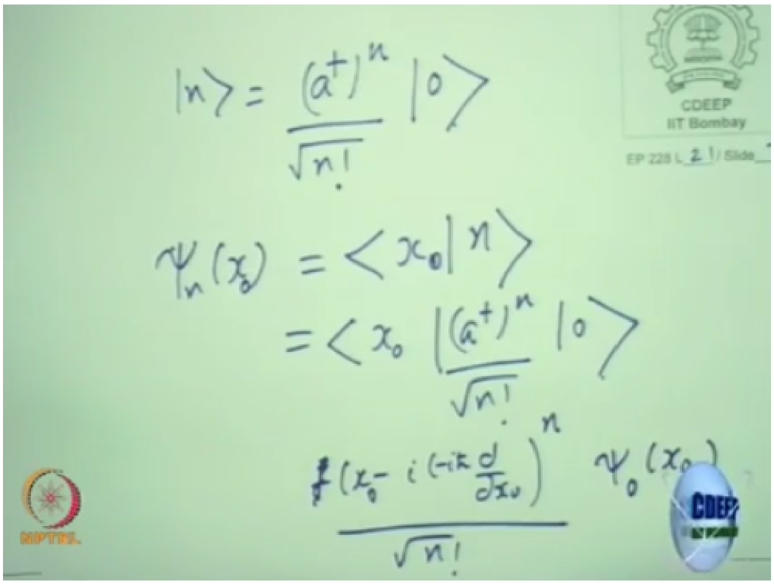

You can write any ket n as a dagger to the power of n/n factorial or n-1 factorial? n factorial on, so if you want to write psi n of x it is at a specific point x0, it is this I can also write it as x0, a dagger to the power of n on 0. You can now it will be involving only position and momentum, so try to write a dagger in terms of that to the power of n, you will have a series but you have to operate it on the ground state right.

You can write this as a, this is a function of x-ip, you can write it as a function of x-i times – ih cross del/del x0 some function form or you can even write it as x0 to the power of n/root n factorial on psi of x0 that is it. Is this difficult? You can work it out and find all the excited states. All the excited states can be found from the ground state by doing the a dagger operation.

And a dagger involves only these differential operators which has to operate on the functions whose function you know. I am just trying to show that the ladder operators when I started introducing it might have looked abstract but without going into the details or looking at second order differential equations using the power of operator algebra I have managed to get the energy eigen functions and also the corresponding energy eigenvalues as n+1/2 h cross omega.

So this is the tool which is really you know very powerful and it is applicable to many complex systems we try our best to map it to either a harmonic oscillator algebra. So when I say harmonic oscillator algebra I know that I can write an a operator and an a dagger operator and I can solve that problem okay. So this is why the power of operators and direct formalism is really useful okay.

So you can continue like this and determine all the other excited state as a check you can do the second excited state and see what you get.

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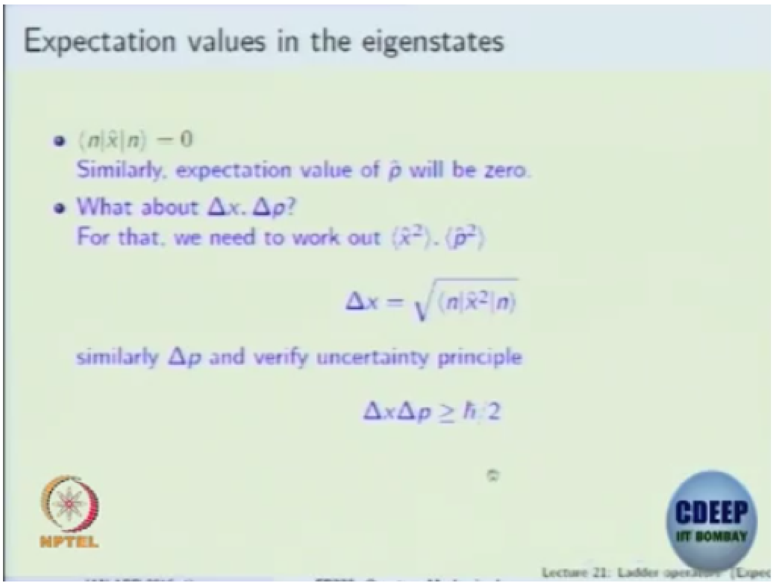

This we have already worked it out expectation value is 0 right and similarly expectation value of p will also be 0, what about standard deviations or uncertainties in x and p for that you need to work out expectation value of x squared and expectation value of p squared and this also you could. We did this for the specific state and you verify that after you do for p squared you verify that they satisfy the uncertainty. I will leave it to you to do this.