Quantum Mechanics

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Lecture - 45
Harmonic Oscillator - II

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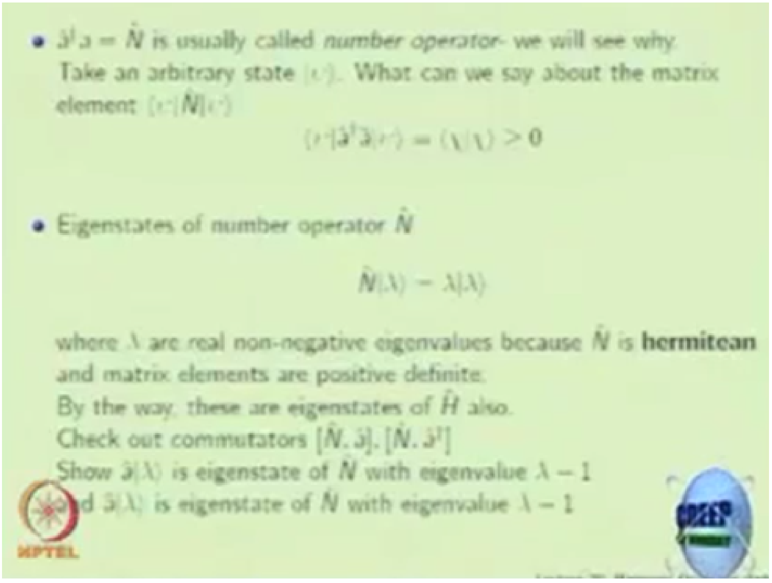

We need to find how these ladder operators a operator and a dagger, how they act on eigen states of the number operator. That is the first thing we will do. As a curiosity test, let us do that. So to do that we may need to know the commutators between the number operator and the ladder operators. Let us first do this commutators.

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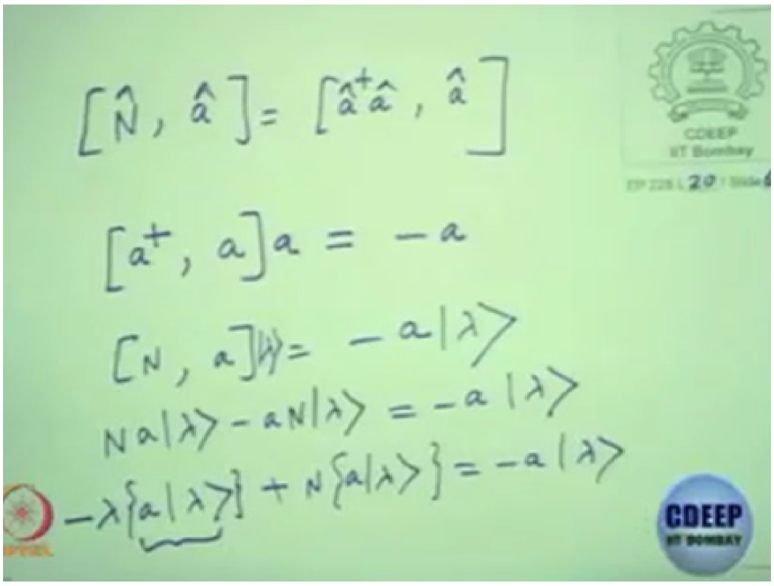

Number operator with a operator is a dagger a, a from now on I will not use a hat assumption is now we are doing from the rest of these lectures it is all only operators, so I am not really going to use the hat, but you will assume that these are all operators okay. So what will this be a dagger with a which side the other a should go? This side, very important, why? Order matters.

a and a dagger do not commute, what will this be? a with a dagger was 1, a dagger with a is -1, so you have a –a. So commutator of a number operator with a operator is –a. suppose I operate it on a lambda state, which is an eigen state of the number operator can do this. What will this be, Na on lambda – aN on lambda left hand side = -a on lambda, what is N on lambda? Lambda times lambda, it is an eigen state of the N operator.

So we will write lambda a on lambda + N on a of lambda = -a of lambda. Take this term to the right hand side and keep a of lambda to be some new state what do you get? What will you get? Take this term to this side, so you will get.

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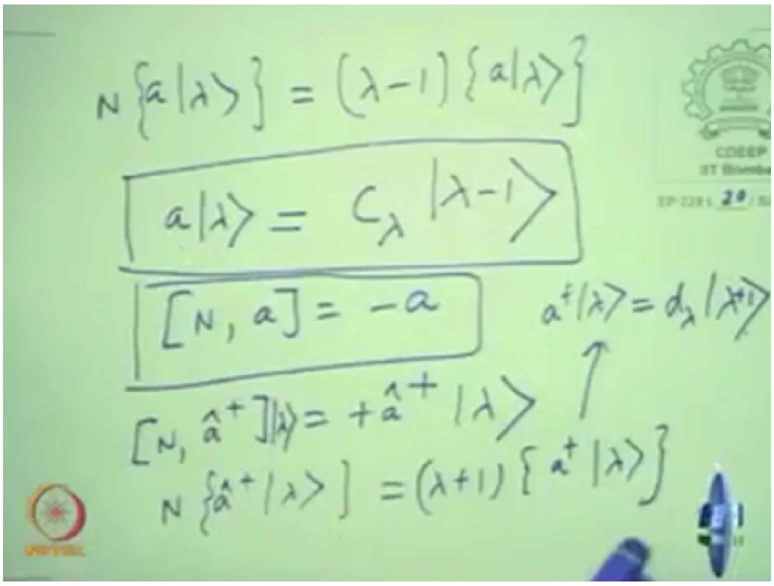

N a of lambda to be lambda -1 a of lambda. So this could be a new state but the new state is again an eigen state of the number operator with eigen value lambda -1, what is that tell me? What can you infer from here? I should be able to write a of lambda to be some coefficient times lambda -1. If I write a of lambda to reduce the state to lambda -1 then the number operator on that state will pull out the eigen value as lambda -1. You all agree?

What else is possible? anything else is possible? This is the only possibility, looking at this you can say okay, so this is one thing which we infer from that equation which is also equivalent to N with a is –a, from this we derived. So can you redo the same thing for N with a dagger what it is, it is + a dagger and then operate it on the state lambda on both sides and write N operator on a dagger lambda to be lambda +1 times a dagger on lambda okay.

What will this imply? This will imply a dagger on lambda should be some d lambda times lambda, something wrong, lambda +1. So clearly it is not an eigen state of way, it is not an eigen state of a dagger, which is expected because the commutator is nonzero. It cannot share the same simultaneous eigen states of hat and operator because the commutative of N with a is nonzero.

Commutator of N with a dagger is nonzero, but (()) (06:28) this you can also see whether a dagger a operating on a state lambda gives you the state back to lambda. You can see that. If you do a on lambda, it takes you to lambda -1, if you do a dagger here it will take you back to lambda, so it is an eigen state of lambda, very consistent. So we are just playing around with operators, algebra, for the particle in a box.

Particle in a box, I am not sure whether we can do this ladder operation because first of all you have to write your Hamiltonian and see whether you can rewrite the Hamiltonian in terms of ladder operators okay. So in the particle in a box or the step potentials you do not have this freedom because there was a quadratic potential here this freedom of writing in terms of ladder operators happen.

But there are many systems where you could do this, I will try to tell you later like for example if you want to write potential like a cosecant-squared potential, let us take cosecant squared x. Can you solve the solution if someone ask you? It is not very easy but you can construct a cook up a ladder operator a and a dagger, but then that leads you to a separate quantum mechanics problem which has been done.

Lot of these potentials have been done introducing a cooked up ladder operators. So this part is not, generally this ladder operator way of doing it has involving in many complicated systems. Here it turns out that both were quadratic and you could do it neatly there we will start introducing some w and then we try to construct an aa dagger. There is a systematic way of doing it and it is very powerful to get an operator method of solving the system.

In fact you should see one article in resonance, I think it is 2003 or 2004 where I have written an article with one of the seminar student on super symmetry where we have shown or we have reviewed basically that a particle in a box can be mapped to a cosecant squared potential using this ladder operators in super symmetry context, it is very nice, it is just quantum mechanics problem, but you can get all the wave functions without solving the cosecant squared potential but using this ladder operator, take a look at it if you are interested okay.

So you can search for my name in resonance, in Indian academy, 2003-2004 is my memory. Now you can see why it is called ladder operators what is happening, this is what is taking you down from lambda to lambda -1 by one step down. This one raises it by one step, still lambda is non-negative real, okay but by doing this you can keep doing a many times and you cannot cross 0.

You cannot get negative pieces right, you all agree. So if you want to go in steps of 1, you have to at most hit a 0, what is the possibility, suppose lambda was 1.5 or something, if I do first a, it goes to 0.5. If I do another a it will go to -0.5, but -0.5 is not allowed because your expectation value of the number operator can be greater than or equal to 0. So lambda has to be necessarily an integer.

If suppose lambda was 5, I can go to 4, I can go to 3, I can utmost reach the 0. After I reach 0 I would claim that these coefficients are 0 okay. So it is basically working out the logic whether we can try to argue lambda even though it is real and nonnegative, can you have it to have some decimal points and not violating what you need for the matrix elements or expectation values of the number operator okay.

So just think over it, that is the ladder operation and finally where you can hit okay. Just kindling you to think about it okay. So just to check out these commentators which we have already done now okay and then using that commutator you can show a on lambda is an eigen state of hat on operator with eigenvalue lambda -1 everybody is with me and sorry this has to be a dagger. a dagger on lambda is an eigen state of again the number operator with eigen value lambda + 1.

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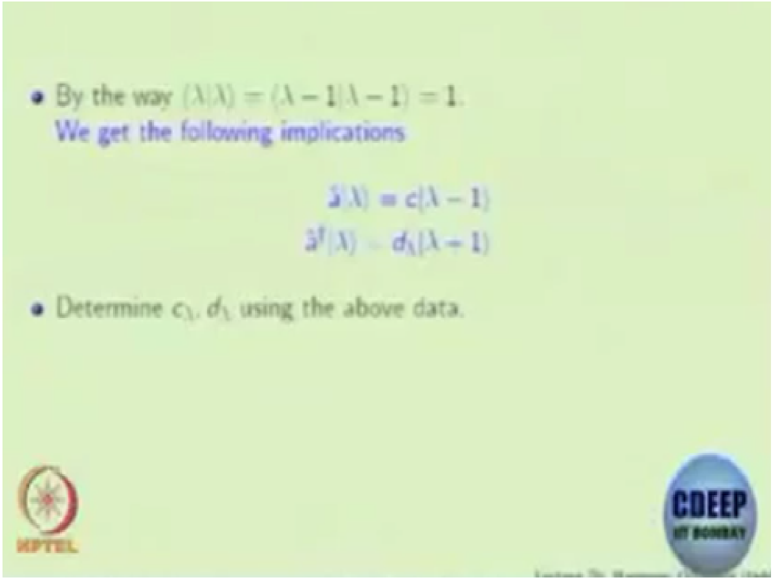

There are couple of things which you should keep in mind. These eigen states are all normalized. These are eigen states of the number operator. So we will have them to be normalized okay. So there should be a subscript lambda that is missing here, a on lambda is c lambda times lambda -1, a dagger on lambda is d lambda times lambda +1. So can you determine c lambda and d lambda using this above data. So can we work it out. Can you use this data and determine what is c lambda and what is d lambda?

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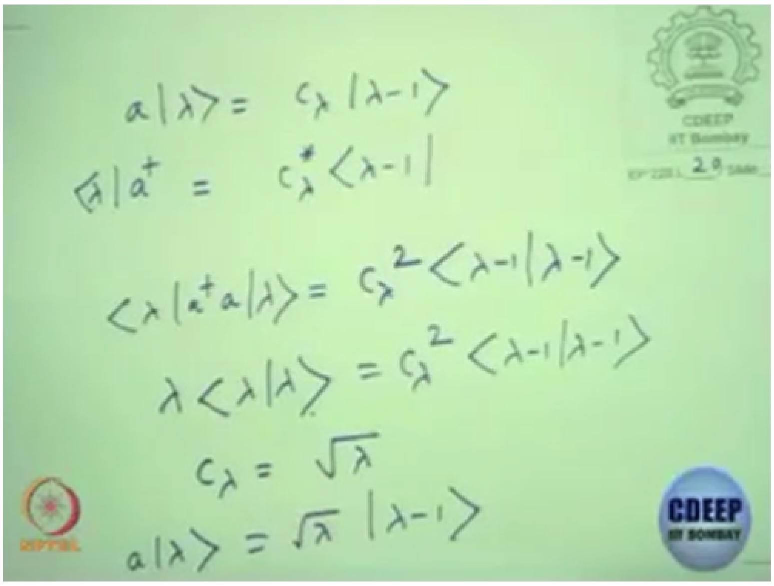

You have a on lambda, c lambda times lambda -1. So you can also write, c lambda squared lambda -1. What is this left hand side, what is left hand side, left hand side is lambda times, it is an eigen state of the number operator, with eigen value lambda which is c lambda squared lambda -1, lambda -1. So this implies c lambda up to some face factors is square root of lambda.

So what have we said, we got a on lambda, a square root of lambda times lambda -1. If lambda is 0 you get a lambda a 0 has to be 0 which means you do not go below lambda = c, do the same thing for d lambda.

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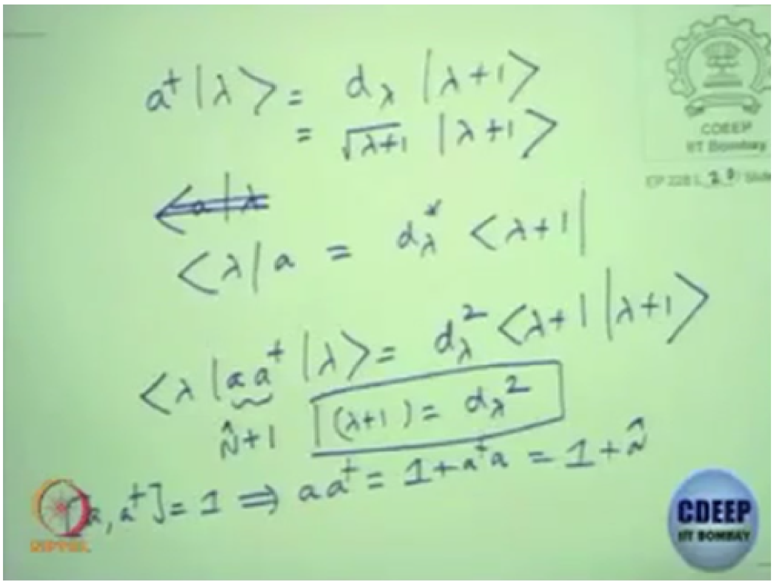

What is this? Yeah you can try to use the commutator and write it in terms of the, this can be written as n operator +1 right, is that right, using the commutator. If you use aa dagger to be 1, implies aa dagger is 1+ a dagger a which is 1+ number operator. Okay so this will give you lambda +1 okay. So you will get lambda +1 to be d lambda squared, so that tells you, clear. You can now check aa dagger or a dagger a on lambda is lambda times lambda whether that is also happening, you can see that that will also happen, you can check that out.

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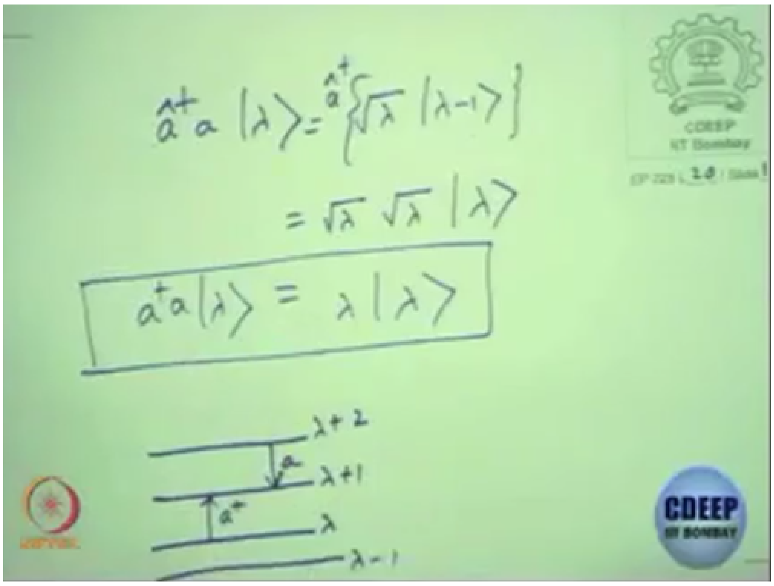

You agree, because on the lambda -1 state if a dagger operates it will give you square root lambda times it raises it back so you get, which is consistent, get back this, am I right, a on lambda is root lambda times lambda -1, a dagger on lambda -1 will be again root lambda times lambda, it raises it by 1 unit and you have that the number operators is an eigen state with eigen value lambda (()) (18:20) is consistent.

Okay so if you draw the lines you have lambda, you have lambda -1, you have lambda +1, lambda +2 and so on. So you can have lambda -1 and you can go below. This one is done by the a dagger operator, what about this one, it is an a operator, you can do first a dagger and then an a, it will get back to the same state and so on, you can play around even pictorially and show that this is a beautiful ladder operator. So what is the summary of what we have done today?

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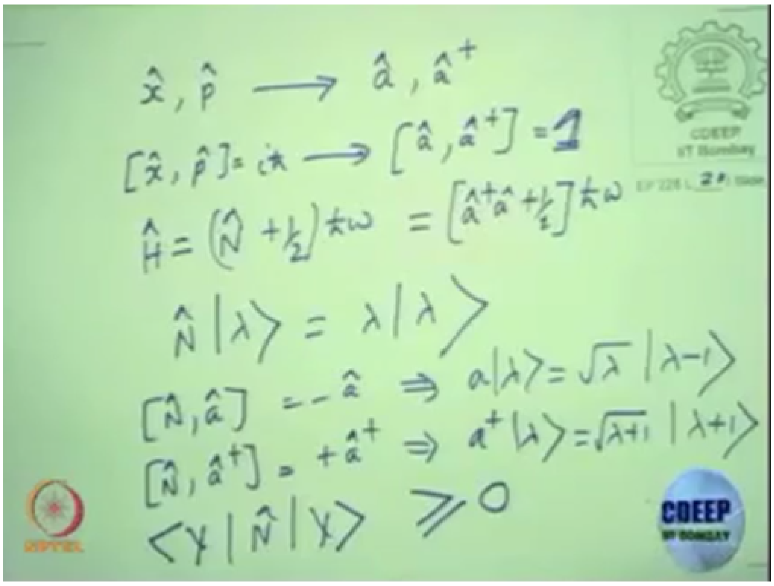

First we introduced from x and p in the harmonic oscillator. We went on to a and a dagger and using this algebra of x and p = ih cross we went on to write a a dagger commutator is 1, then we used that we can write our Hamiltonian as N operator + 1/2 times h cross omega where n operator is nothing but a dagger a + 1/2 times h cross omega. Then we said let us check the eigen states of the number operator which are eigen states of with eigen value lambda.

And we have also used n with a to be –a to imply a on lambda, square root lambda times lambda, lambda -1 which his the lowering operator in steps of 1 unit and we also had n hat with a dagger to be + a dagger which implied a dagger on lambda to be a raising operator with a separate coefficient, different coefficient.

So using all these information if we keep going we have to make sure that any state which I write, a state chi with the number operator, some state chi I need this to be all rise greater than or equal to 0. This will force your zero this will force your lambda cannot become negative, can utmost become 0 and to go to the 0 state I need to use these ladder operators which goes in only in steps of 1. It is not going in steps of fractions right.

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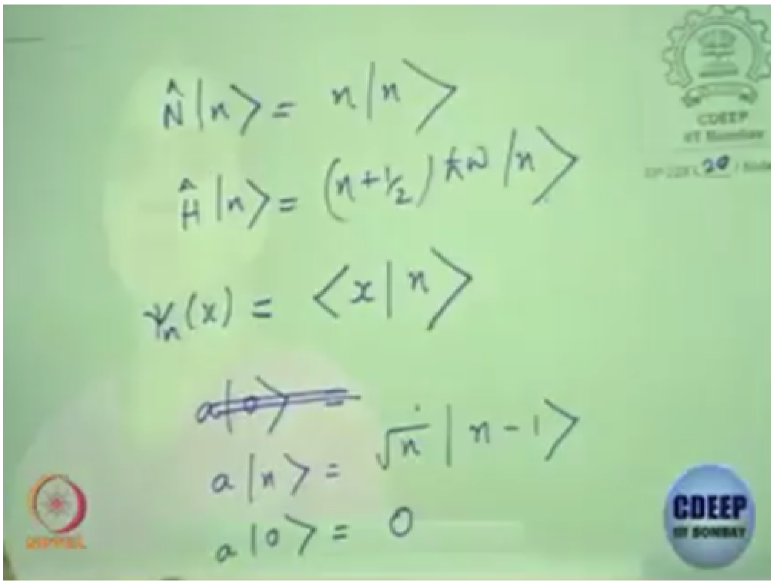

So you have only in steps of 1, you can go from let us say lambda to lambda -1 with an a operator and you can again use an a operator, go to lambda -2, but utmost you have to hit a value such that the number operator on that state let me call it as a ground state. On the ground state can be the lowest value has to be 0, you cannot go below that because your state has to satisfy on any state N operator on any state has to be greater than or equal to 0.

This forces you that you cannot go below the lowest state. Okay this is possible if you go through these, it is possible to reach a state which is 0, lambda will be 0 times 0 only if this is any integer, positive integer. Otherwise you cannot achieve this. Formally for any quantum system it has to be real non-negative that is all you have, but now with this construction you see that you cannot have yet to be any other you know real values it has to be only integers.

Okay so therefore lambda is an integer which we will denote it by some n and write our states, as n hat on n is n times n and Hamiltonian on n is n + 1/2 times h cross omega on n. So did we really struggle to get this eigen value? We went through the operator algebra, we did not solve any differential equations, but we ended up getting the familiar eigen values of your harmonic oscillator beautifully okay.

How do you find the wave functions? Tell me your harmonic oscillator wave functions you can take a x with n. What about a on 0, let us write what is a on n before we write this, a on n is square root n on n-1. So a on 0 will be a operator is x with upto a constant which is dimensionless + ip right.

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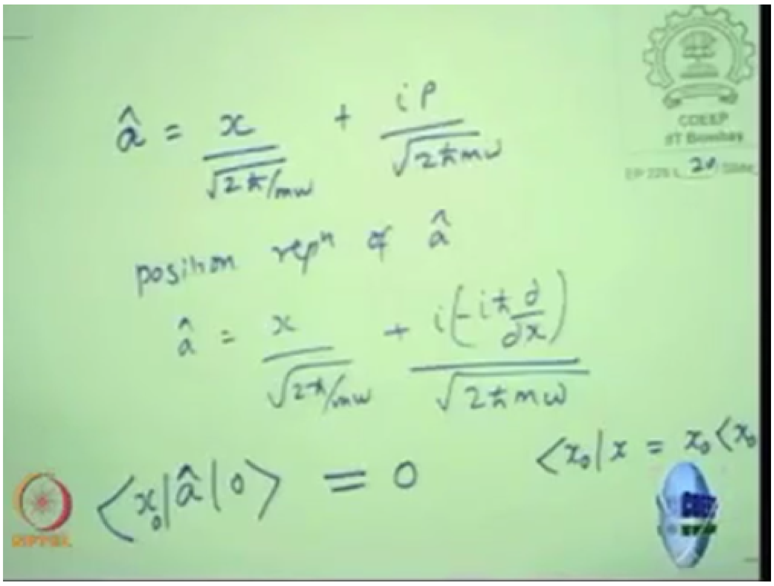

a operator is x/something, a position representation of a operator will be okay, so what we will do is. We will take a on 0 and take a position state, a on 0 is 0 and let us simplify this. So x on x is x right, you can put an x0 if you want and you can write x0 with x operator will be x0 x, a was p+ ix, did I wrote p+ix sorry if I have done it I am wrong. So what is the differential equation you will get out of this. So you will have an x0 divided by this and there will be a del/del X which will be del/del x0 on this state right. What is that equation?

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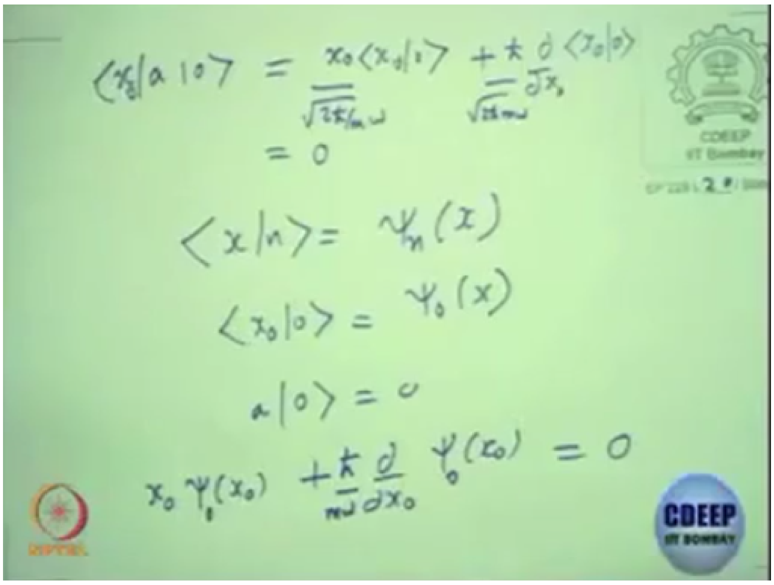

What is this (()) (27:32) x0 with 0, I have said that x with any n, I am going to call this as psi n of x so x0 with 0 will be psi 0 of x and I want this to be a on 0 is 0. So use that so this will be = 0. So it is essentially a first order differential equation psi of x0 with x0 upto those factors. What are those factors? 2h cross let me cancel. So you will have a h cross/m omega I think. Can you solve this equation? this is 0.

And are you getting your familiar ground state wave function? Please check it. So I am just showing you, I did not do much, I just used the ladder operators I extracted for you the eigen values, now I am also saying you can familiar position space wave functions can also be derived. At least let us derive the ground state wave function. Once you have the ground state wave function you can go to the next one, how do you go to the next one? You will have a squared, a dagger operating on that will give you the first excited state and so on.

So we can systematically go finding the solutions for the harmonic oscillator once you understand the power of the ladder operation. There are some curious questions I want to ask you, can you find the eigen states of the a operator and a dagger operator? Try it out. Find the eigen states of a operator. It will not have real eigenvalues, why? It is not hermitian, but can you find the eigen states of a operator and a dagger operator, that is the question for you, think about it.