Quantum Mechanics
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Lecture - 44
Harmonic Oscillator - I

Okay, so today I am going to do, we have been trained on this Dirac operators, bra-ket notation in the first half of the semester. So today I am going to show you how it is elegant to work with this ladder operator and get whatever solutions which we got earlier. You remember how we got the solutions? We tried to write the Schrodinger equation in position representation which gives you the wave function.

We look at the differential equation then we see that it is corresponding to what special function, hermite polynomials and then we try to say what is the solutions and we also saw naturally we were getting the odd functions, even functions because it is a symmetric potential V of X = V of – X and we did these all these things in a rigger explicitly by working out the differential equation.

Now what we are going to do is we are going to use this power of operators and try to solve in a much elegant fashion. Is the motivation fine okay?

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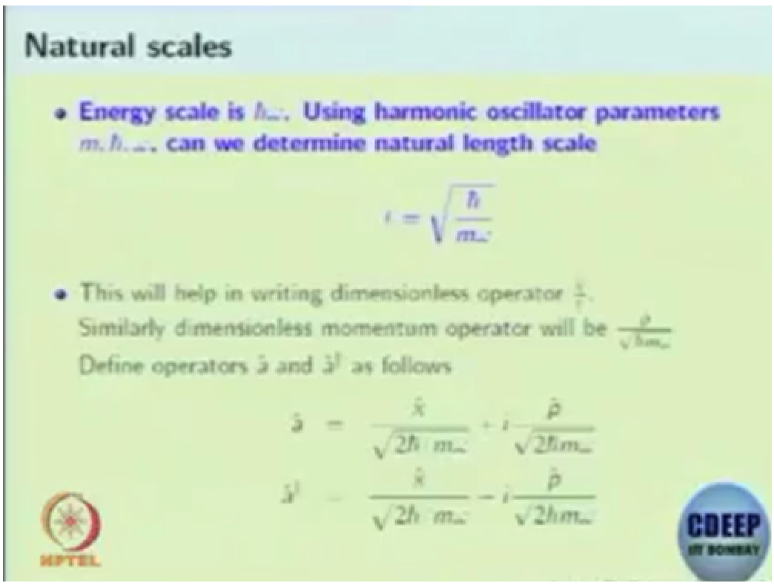

Okay first of all you need to see what are the natural scales in a given problem okay. So one is your familiar one which is your energy scale. H cross omega is in joule units of, it is an energy scale and in the harmonic oscillator parameters which are there are mass, frequency and H the Planck's constant. See these are the 3 constants which you have with which you can play around and get dimension full quantities and can we determine a length, natural length using these 3 parameters.

Sure all of you have done this dimensional analysis right. So can you try to figure out how to play around with m h cross and omega? So that you get the dimension of that as L.

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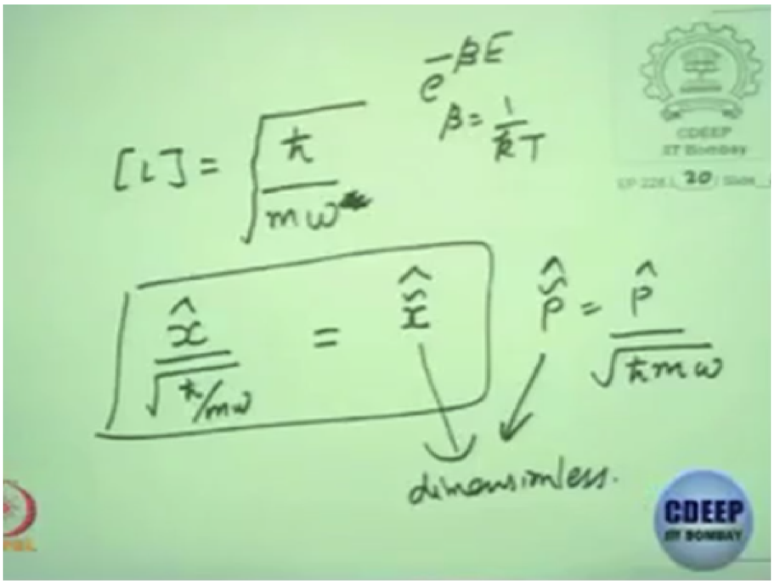

h bar by m omega square, square root of that is that correct, m omega squared or m omega. So the dimensional analysis if you do h cross omega is energy, h will be energy by omega and then omega is 1/time used at msm and you should be able to show that the dimension of this piece sn okay. Why are we doing this, we would like to write, suppose we write a position operator that has length dimension.

I can make a dimensionless by introducing and call it as if you want as some dimension I do not know whether we need to use a tilde just to say that this is the dimension of this is dimensionless okay. Many times it is convenient to work with dimensionless quantity okay. If you see your Boltzmann distribution you write e to the –beta E right. Beta should have dimensions of energy, inverse of energy right.

So that the exponential thing becomes dimensionless. So beta is 1/kT and you make this, you can call this beta as some dimension full quantity with inverse energy so that E/that is dimensionless. So in a similar way you can treat this to be a, this one to be dimensionless okay. Can we do a similar thing for momentum? Can you find a unit using h cross m omega which can give you momentum?

Someone p tilde hat will be p hat/root h bar m omega, so this is also dimensionless. This is the first motivation if you see any books, they start dividing by these factors, it is nicer to work with dimensionless operators using the allowed parameters which are there in your given system in the harmonic oscillator you can play around with m and omega and h cross of course is there in any quantum physics.

It is a universal Plank's constant. Okay so we can determine this natural length scale I said is h cross m omega, and you can write a dimensionless operator as hat position operator divided by this length scale similarly you can write momentum operator okay. So this is going to be an algebra of linear operators okay. So this is a theoretical idea okay. What is the idea? We introduced 2 operators once I say introduce A and A dagger.

And if I say they are different you know that they are their own correspond observables. This is a mathematical way of doing it, we are trying to find sum expectation values or anything, it is not connected to any observables in the lab. Observables are connected only to the position and momentum, but this is written for convenience. We have written this operator for convenience.

The two operators, the complex conjugate of A is A dagger. You can see that +i, i is the imaginary i becomes a –i you know just to make my math simpler I have introduced your square root 2 here. Okay, so this x operator divided by h cross by m omega is dimensionless. P cross by h cross m omega is dimensionless that is the square root 2 which is added just to make things simpler as we will see okay.

So these are the 2 operators which we are introducing which are complex operators and note that they are not Hermitian operators.

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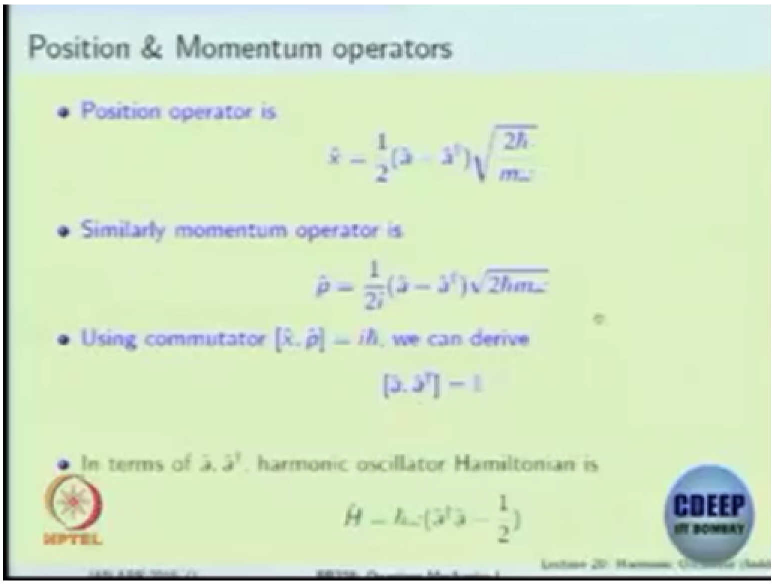

And you can reconstruct your position operator by reworking back rewriting the linear combinations of and this is Hermitian, it is very clear right. When you do dagger on this, this is a Hermitian operator. What will be for p, momentum operator will involve an i factor because then you do the dagger it becomes a negative sign, i denominator will pick up a negative sign so that p becomes Hermitian okay.

Please check these steps from the previous slide that you get position operator can be written in terms of the 2 operators. These operators are called ladder operators we will come to seeing why it is called ladder operators. What is a ladder? Ladder is climbing up or climbing down on a ladder. So this operators are supposed to do by climbing up on energy eigen states or climbing down on energy eigen states.

So that is why it is called as a ladder operators okay. We will see that when we do the math. That they are called A and A daggers are called the ladder operators. So we are always going to use the positive momentum commutator using this can we get what is commutator between A and A dagger okay. So take the commutator x and p, substitute here left hand side, do the right hand side and see whether we get a commutator of A with A dagger okay. Can you try to work it out?

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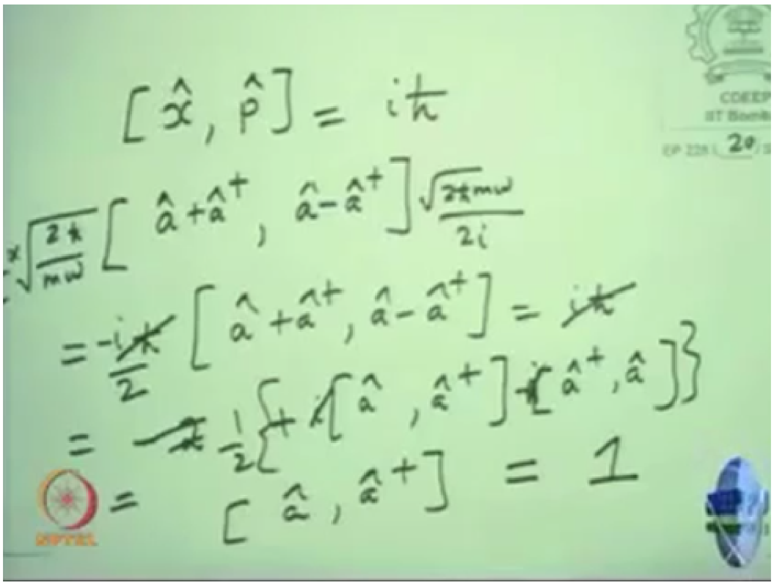

ih cross substitute here A + A dagger and A – A dagger, remove all these factors out. I just substituted the x and the p, simplify this it becomes h cross m omega cancels -ih cross, ih cross, a with a is 0 a dagger with a dagger is 0. So what will the final answer be. What happens someone? Already a minus sign -ia dagger with a. There is an extra 2, yes, good. So there is another 2 from, thank you.

So there will be a by 2 and this turns out to be, this is same as this one with an opposite sign. So it becomes, there is a 1/2 here. So that gives you a dagger, 2 cancels, I also cancels right. Why did I keep the i, I got cancelled I thought and this is this 1 okay? If you remember a and a dagger were dimensionless so I would expect this to be right hand side should be some number and we do get it to be 1 okay.

So this is to play around with commutator brackets. I assume that you all have been trained in the first half that you can do these things easily okay. So aa dagger turns out to be identity, call this to be an identity operator. By the way we know trace ab is same as trace ba what will happen here. So it has to be an infinite dimensional vector spaces. You cannot have a finite dimensional vector space like angular momentum.

Angular momentum if you know that we will write it in such a way that these mattresses are always traceless. Here I cannot do. Trace of identity is never traced in this right. Can you make trace of identity to be 0? Impossible. So this xp I told you earlier, xp = ih cross implies that we are in an infinite dimensional vector space. The same thing holds for this; it is the same system okay.

So now the next exercise for you. You had the Hamiltonian which is p squared/2m + 1/2 m omega squared, h squared, replace x squared by the square of this, replace p squared by the square of this and aa dagger is not same as a dagger a, why? Commutator is nonzero. So you can use this commutator to write aa dagger as a dagger a +1, you can use that okay. So do that and check out whether you get Hamiltonian to be a dagger a + 1/2.

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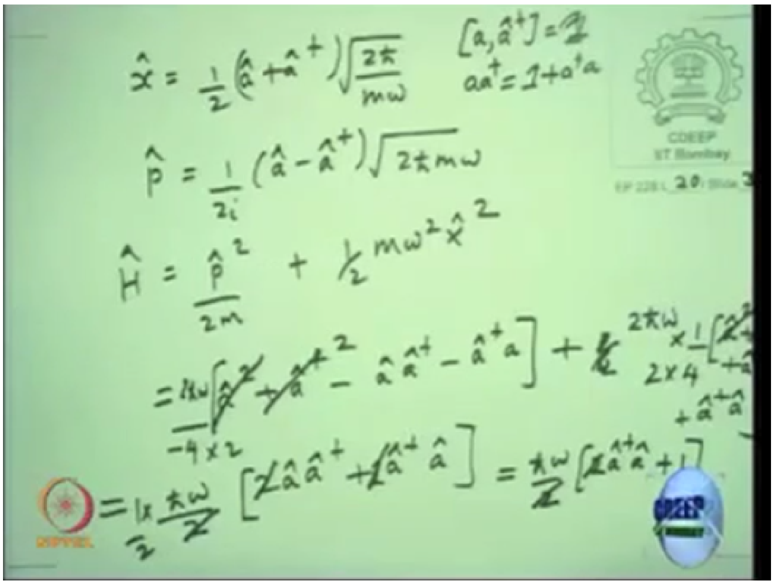

When you further simplify the first 2 terms will cancel right and you can show that it is h cross omega/2 * twice a a dagger + a dagger a. There is another one fourth, we have missed something? So there is a, I have put the 1/2, oh also there is a half okay good. Can you simplify this? use the fact that commutator of a with a dagger is 1 means aa dagger is 1 + a dagger a fine.

So if he use this for the first term, so let me cancel the 2, this can be simplified as h cross omega/2 there will be 2 a dagger a, because this a a dagger will have 1+ a dagger a and that 1 will also be added. So what happens finally the 2 can cancel with this but then you can put a half k. So I can remove this 2 but you put a half clear.

So the Hamiltonian finally turns out after the simplification turns out to be it has to be in units of h cross omega that is also we checked, dimensions of Hamiltonian x dimensionless operators a dagger a we all made dimensionless, a dagger a is a dimensionless operator, is it hermitian, a dagger a is hermitian, a is not hermitian, a dagger is not hermitian, they are still linear operators, but a dagger a is hermitian.

So everything is fitting in place. Left hand side if it has to be called as an observable is all correctly done, had you got a squared + a dagger squared here then also it is hermitian right. So it is only if you get some a squared separately then you know you has a mismatch but a squared is it an observable? is it hermitian, a squared alone? A squared alone is not hermitian okay. So these are ways in which you can make sure that your algebra is correct, dimensions are matching.

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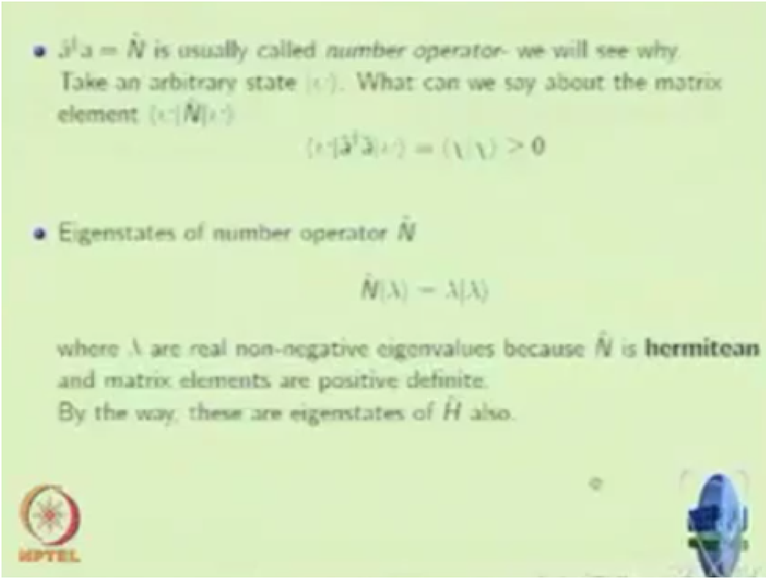

So in all the books you would have seen that a dagger a is called number operator okay. We will see why it is called as a number operator. So typically if you say it is a number operator it should extract the number of particles, some kind of an integer number okay. So then we call it to be a number operator. So we need to see why this a dagger a is called as a number operator, we will see at some point.

But right now we know a dagger a is a hermitian operator and if suppose I can find the eigen state of a dagger a will it also be an eigen state of Hamiltonian? It will be an eigen state of Hamiltonian because the number operator commutes with the Hamiltonian right. Number operator commutes with the Hamiltonian. So you can say that they share the same eigen states okay. Take an arbitrary state psi, what can we say about the matrix element.

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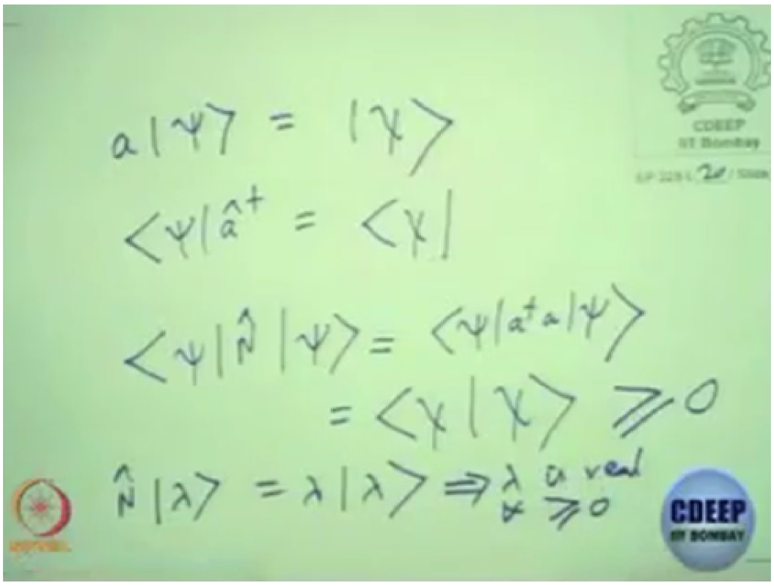

If you write psi with a as some chi then the adjoint action will be psi with a dagger will be chi. So psi with the number operator psi is nothing but psi a dagger a on psi which is nothing but chi with chi and this is a norm. norm can be positive or at most it can be 0 for a null vector. So this indirectly tell me that the expectation value of the number operator in any state cannot become negative.

Suppose i was an eigen state of the number operator, number operator is hermitian, eigen values have to be real. On top of it it has to be positive or at most 0, not negative. So if suppose I find eigen state lambda of a number operator is lambda on lambda then this implies lambda is real and greater than or equal to 0, both the conditions apply okay.

So what can we say about this matrix element as we have already worked it out now. We see that the matrix element has to be positive definite, eigen state of the number operator at N if suppose lambdas are called to be the eigen states with eigen value lambda, they have to be real non negative eigen values because N is hermitian and matrix elements are positive definite.

So this is just to say that the number operator commutes with the Hamiltonian, they will also be eigen states of the number operator, Hamiltonian and number operator will share the same eigen states and eigen value of the Hamiltonian will be h cross omega time lambda +1/2 right, is that right, okay. So when I write n hat on lambda to be lambda times lambda.

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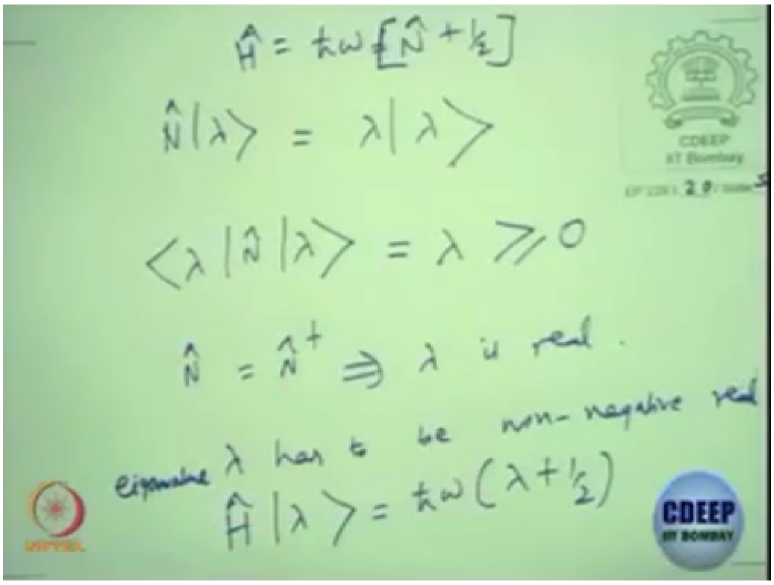

If I try to find the matrix element of the expectation value of a particle in this state this will give you lambda and this has to be greater than or equal to 0 and N hat is hermitian which implies lambda is real. So lambda has to be non-negative real okay, eigen value. Is this clear? Now on top of it we also say that Hamiltonian is h cross omega times hat N + 1/2 right. So therefore Hamiltonian on lambda will give you h cross omega lambda + 1/2. We will also tell you why lambda is number operator now.

Because since you know what were your eigen values of the harmonic oscillators involving integers which one non-negative right, which is consistent with all these, but why integers. This condition only tells me lambda has to be real. We need to figure out why lambda has to be integers, is that correct? I am not still proved for you that lambda is integers. I only said from this algebra introducing the ladder operators that lambda has to be nonnegative that is all I have said and real. We need to still prove the lambda is integer, we will do that.