Quantum Mechanics

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Lecture - 34
Solutions to Other Coupled Potential Energies - I

Okay. So today I though a quick reference which one can ask for other coupled potential energies, okay.

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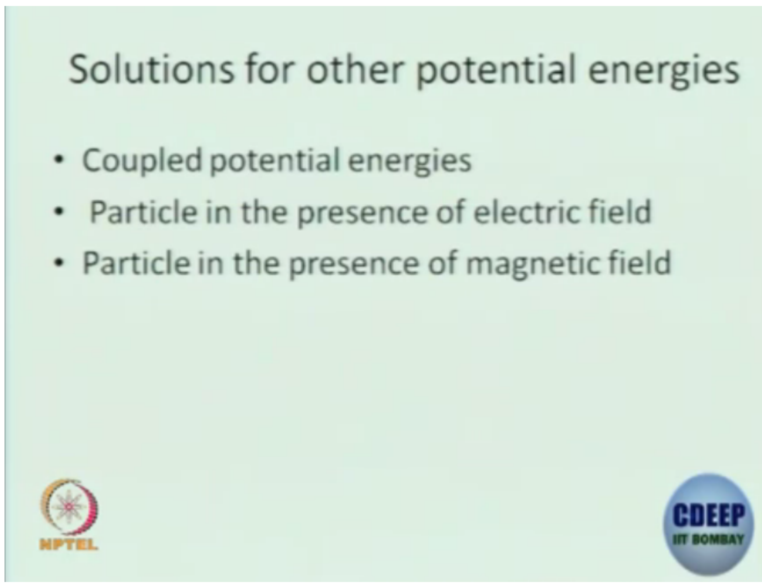

So the couple of them as you can have two coupled harmonic oscillators, you can have a harmonic oscillator in the presence of an electric field, you could have a particle which is oscillating in a harmonic oscillator potential and is also subjected to a magnetic field, so these are some of these things where you are already learned tools of harmonic oscillator will actually help you to use that they data and solve them.

You do not to redo these calculations. So these are the simple tricks which you can employ and get results directly, okay.

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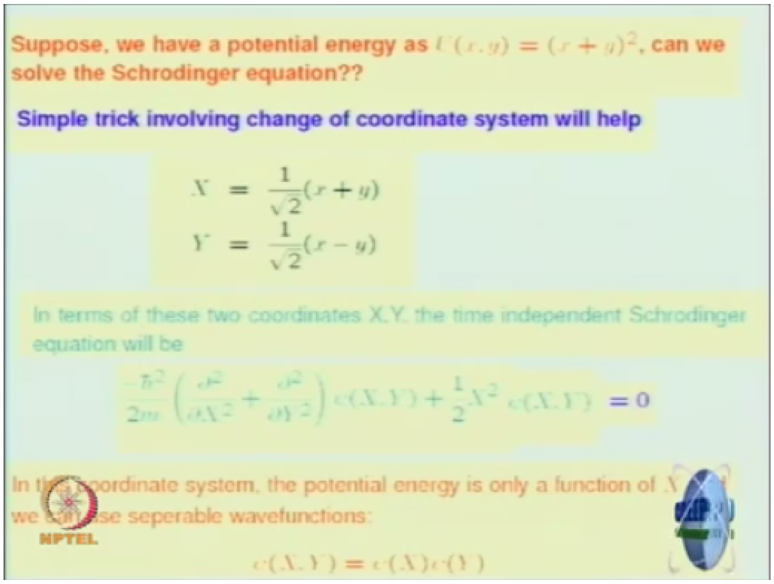

So suppose I have given you a potential energy which is x+y the whole squared. You can take it to the motion in two dimension, suppose I have a potential energy which is x+y the whole squared. Can you solve the Schrodinger equation? How will you solve it? You can try to rotate your axis coordinate axis, make 45 degree rotation such that there is a potential in one of that direction and 0 potential in the other direction and you can try to solve this. Okay.

So this is a trick of making a suitable coordinate choice which helps you. This we have done many times, right. We do take a suitable coordinate choice depending on the situation. So simple trick involves change coordinate system, write capital X as x+y and then the other coordinate which is orthogonal to this coordinate; so if this is rotated by 45 degrees to acquire rotated to be 45 degrees, you know what I mean and you clockwise in the two orthogonal coordinates have to be of this side. Then what we do?

Write the Schrodinger equation in two dimensions instead of you using small x and small y rewrite it in terms of capital X and capital Y. I am sure you have done all these kinds of variables many times in your JEE and your MATH course. So in this capital X, capital Y the coordinate change if you do in your Schrodinger equation this will be your Schrodinger equation with the potential this one will have 1/2x squared.

You can still treat it to be a one-dimensional problem but then if the motion is in two dimensions it behaves like a free particle in the other direction, so you have to keep track of that. So in this coordinate the potential energy is only a function of X and we can use separable functions Psi of x and Psi of y exactly like what you did in your, so use this separable functions; this is the standard one I was telling you in two-dimensional harmonic oscillator or two-dimensional particle in a box.

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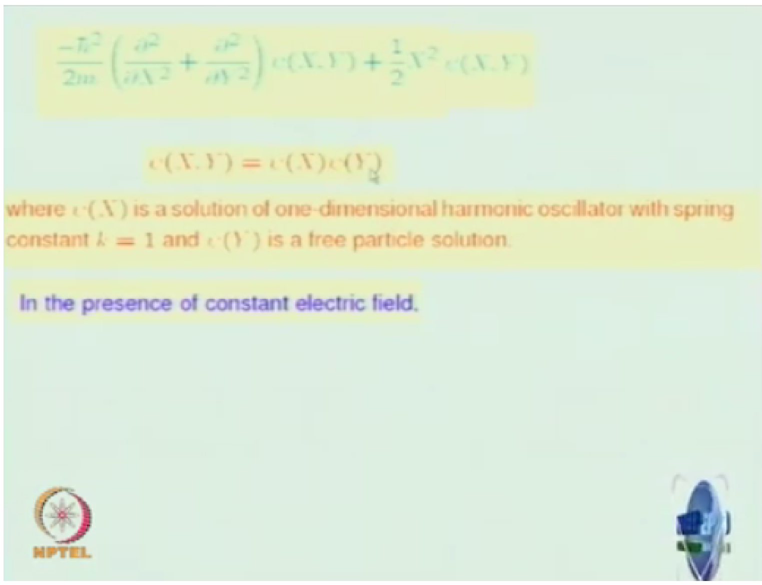

And then as he already pointed it out Psi of x will behave like a solution to a harmonic oscillator with spring constant like k=1. And Psi of y is like a free particle. So when you write the wave function like suppose I want to write the ground state wave function let us say; ground state in the capital X coordinate and a free particle with some momentum in the capital Y. How will you write this wave function, solution someone?

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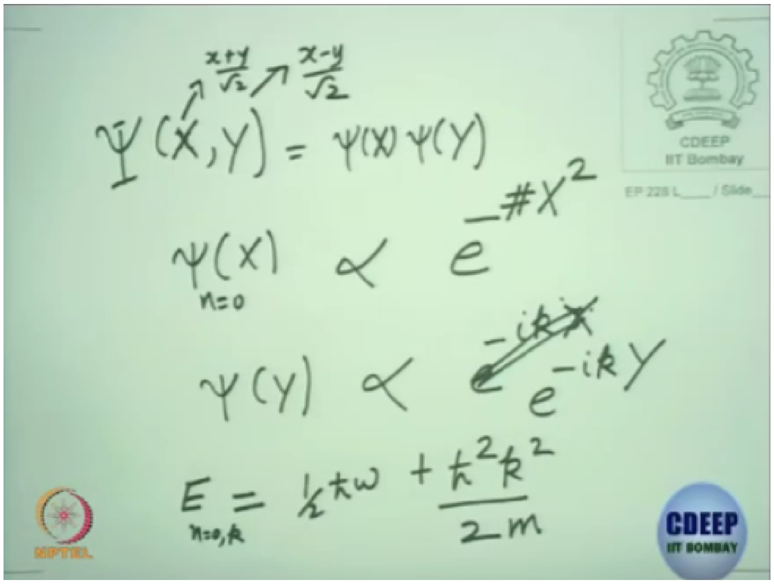

So let me call this Psi of capital X capital Y to be Psi object value of capital X capital Y and Psi of capital X let us say for the ground state will be proportional to e to the power of let me not write those constants x squared some constant numbers and Psi of y will be proportional to e to the power –ik x sorry ky e to the power of –ik capital Y. What will be the energy now Energy for this case n=0 under specific k 1/2 h cross Omega for the y piece and then h cross squared k squared over 2m, right. Is it clear?

Energy is also sum up over the x coordinate energy. In the earlier case in the two-dimensional harmonic oscillator you had nx+1/2 h cross Omega + ny+1/2 h cross Omega. But here the capital X is like the harmonic oscillator. I am specifically technically taking the ground state so it will be 1/2 h cross Omega for the capital X coordinate and capital Y coordinate it will be like a free particle. So I did not do the calculations here.

I assumed the harmonic oscillator solution, looking at this given potential shows a different coordinate system where it behaves like harmonic oscillator in the capital X coordinate and free particle in the capital Y coordinate, okay. So this is why it is coupled x+y the whole squared is coupled. After you have got this wave function you could go back and write capital X and capital Y in terms for small x and small y and you can get it in your original coordinate system but the energy spectrum will not change.

Energy is an eigenvalue that will not change. You can go back and write this wave function in terms of, you can replace this by x+y/ root 2 and this one is x-y/ root 2 and then you can get back the wave function in the little x little y coordinate. This is one. Okay. So now we let us take the particle in a constant electric field.

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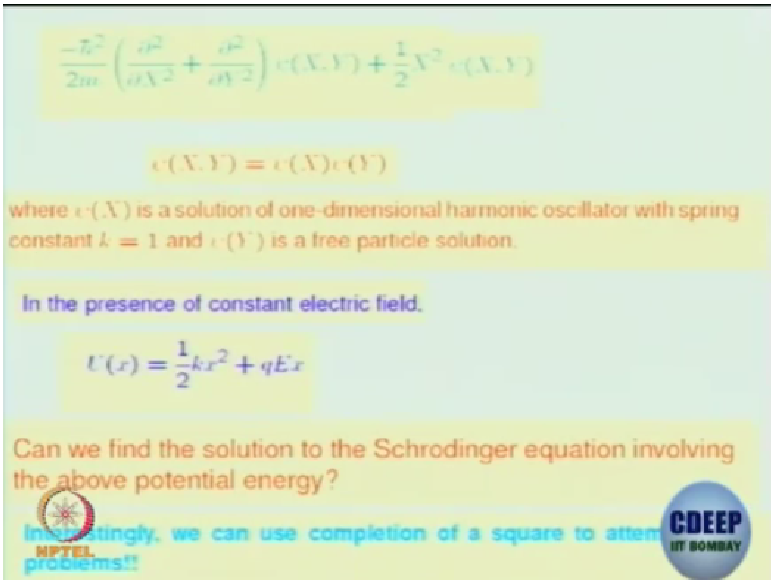

Okay. So let us take the particle in the harmonic oscillator potential to be subjected to an external electric field. So when you want to put the potential energy let us take it to be in one dimension you have to add a Ex the electric; do not confuse this with energy E is the electric field E times x as q is a charge of the particle. How will you solve this? So you try to see whether you can redefine again another coordinate such that it can become your familiar potential, so you complete the squares to attempt such problems.

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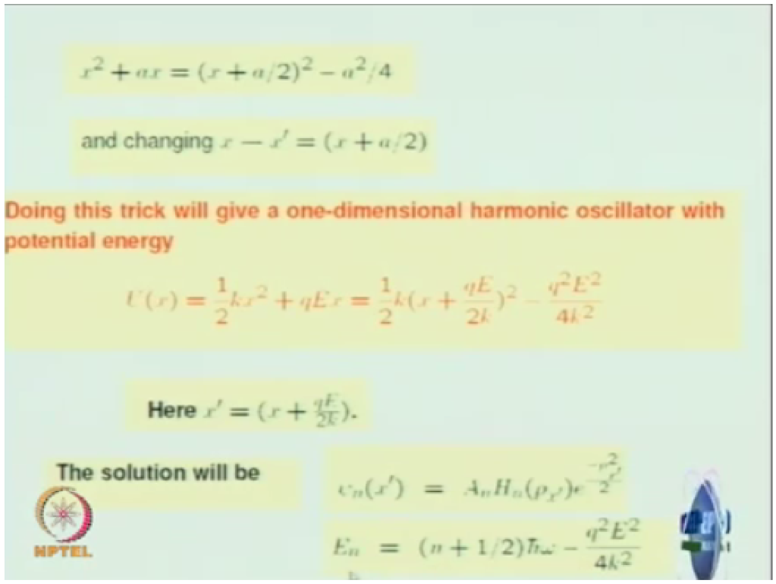

How do I do this? If you have x squared + Ax that same as shifting your x and then subtracting a constant, shift your x and subtract the constant. Then what do we have to do? We make a change of coordinate instead of x you call a new coordinate x prime which is a shifted coordinate. Take a shifted coordinate. Then what happens to your system? Your potential energy which was this becomes shifted, this A which I wrote as a compact notation is actually to be seen from this picture, how you change your coordinate.

And there is a 0 the potential energies shifted as just a constant. So what will be energy eigenvalues for this system? What will be the ground state energy? 1/2 h cross Omega - the constant shift in that energy which is q squared e square/4k squared. What happens to the wave function? Wave function will have a peak earlier at x=0 in the absence of electric field. The presence of electric field it will get shift, shifted by the qE/2.

You need to work out the solution? No need, if we know the harmonic oscillator solution you can say for this potential; you can do this trick of completing the squares and then you can read off that the wave function will get shifted; the peaks of the wave function wherever it goes it will shift by or the whole wave function is shifted actually. But the energy eigenvalues do not change. I am just asking that if you take the stationary state solutions just looking at the time independent solution, the solution will look like the same wave function plot but shifted to x Is that fine?

That phase factor time evaluation phase factor is there for all problems. So if I ask you giving you this problem and ask what is a first excited state energy? I will see all of you doing 10 pages of calculation in the mid sem. Do not do it. Just have to write a line that this is the trick assuming harmonic oscillator solution it is 3/2 h cross Omega- q squared E squared/ 4k squared. Use the power of many of these you know, simple tricks therefore you go into doing rigorous calculations, so no need to do a calculation.

Okay, so I have just said here the solution will be shifted by that X prime coordinate and you can rewrite it in terms of if you remember your earlier context how I wrote the roots you can rewrite it and you can say that the energies are Rn+1/2 h cross Omega for the excited states but it is shifted by a overall; every energy level is shifted by q squared E squared/4k squared in the negative edge.

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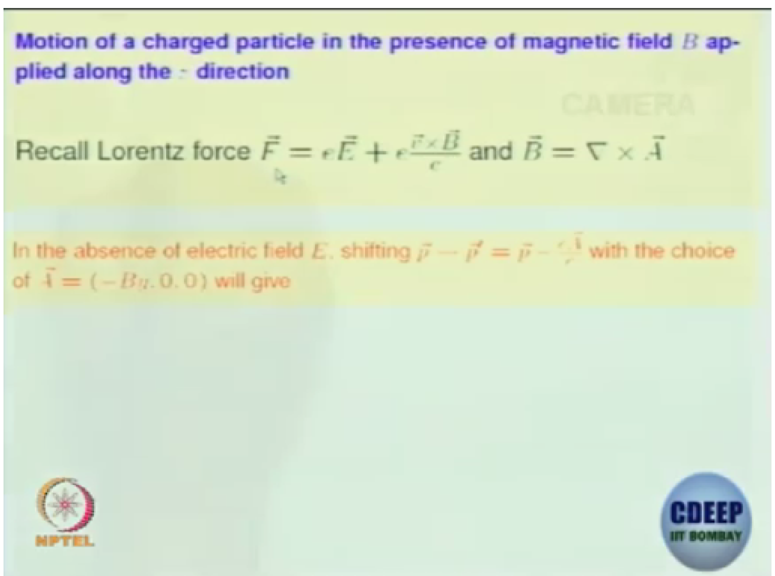

Okay. So classical mechanics and looking at Lorentz force I am assuming you all know. So Lorentz force suppose you want to look at charged particle in the presence of a magnetic field along the Z direction; this is the most general Lorentz force which includes electric field and the magnetic field but I can try to look at what is the force when you applying magnetic field in along Z direction, and B can be written as curl vector potential.

So we are going to look at absence of electric field. What we will do is naively we will replace the momentum in classic mechanics; this is what we do in classical mechanics. We replace the momentum by a shift; the reason why we do this is if we take the Lagrangian with shift and write the Euler Lagrangian equations you will get back in Lorentz force.

I am sure classical mechanics force you would have done this; if you have not done it right now you take this as a data, some point you have to write Euler Lagrangian equation for the shifted and you can show that you get the Lorentz force. Just to give you a flavor.

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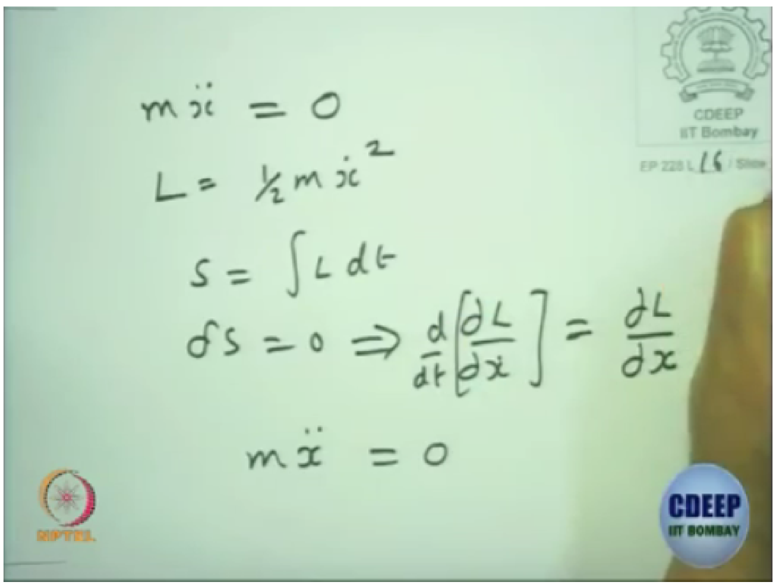

See for a free particle you want m x double dot to be equal to 0. How do you do this in classical mechanics? You take this to be 1/2 m x dot squared and then you have the principle of this action extermizing the action which is L dt delta S = 0 will give you del L/del X dot d/dt - or this is del L/ del X. You try to substitute here for this Lagrangian which is given you will end up getting del/del x dot will give you 2m x dot and d/dt will give you the half is already there.

So it m x double dot and del L/ del X for this is 0 so that is why you get this be sequential mode. Similarly, for a particle in; so similarly you can do for a particle in a magnetic field.

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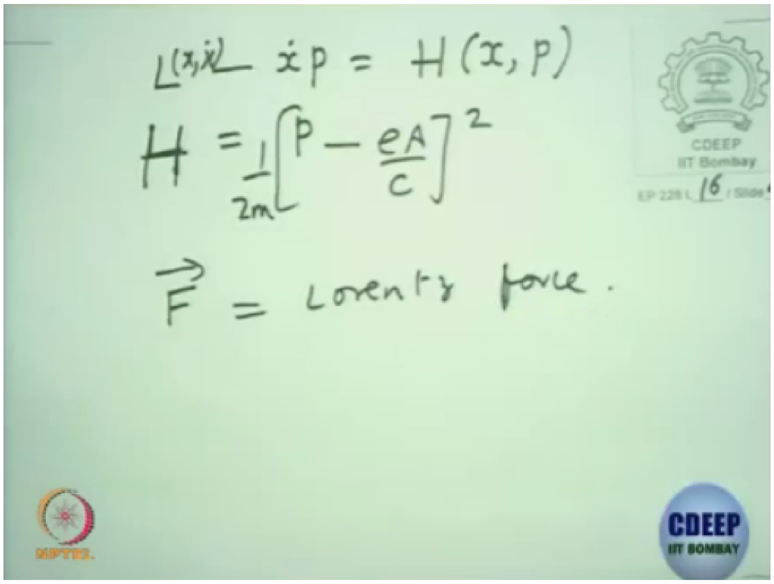

So you can try to write your Lagrangian as you will have a term where you can try to rewrite it in terms of x dot squared, so you can; basically Lagrangian is not the right one when you have to write the momentum it becomes the Hamiltonian. How is Hamiltonian related to Lagrangian? Qp dot or Z dot P, yeah. This is the Legendre transform which will give you a function of X and P and this is a function of X and X dot and you can write what is the Hamiltonian and look at the equations of motion there also.

So what I am saying is that for the particle in a magnetic field if you replace P/P – eA/C the whole squared/2m the free particle and if you try to rewrite your either the Euler-Lagrangian equation using the above or the Hamilton's equation and convert it into second order differential equation you will end up getting a force which is the Lorentz force. Okay. This I will leave it to you to check, fine.

So in the absence of electric field you can make a shifting to make contact with the Lorentz force and let us take a specific choice for the vector potential this is just in hands, white is in our hands you can try to get the same electric field and magnetic field but different is which are related by Gauge transformation, right. You can choose another choice another A such that the curl A will give you the same magnetic field.

And similarly the corresponding electric field if you want to find. So this is one choice, this is one choice which will give me the magnetic field along the Zh.

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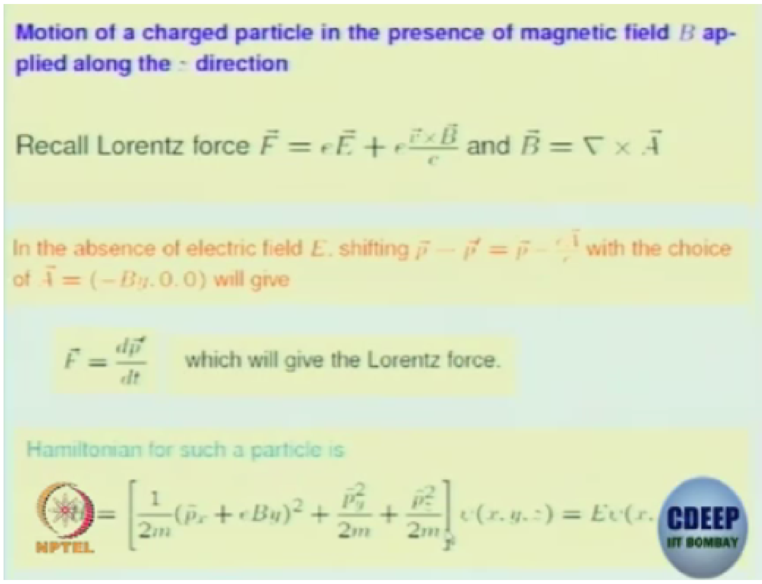

So force law which is dp prime/dt this is what I was saying. And you can write the Hamiltonian with this choice of A this one specific I get this as a Hamiltonian. Now look at this Hamiltonian and tell me what is happening. So let us just keep those C to be 1 just for; you can put the C also but let us just; many time what we do is we put m=1 c=1 so maybe I mechanically put it which is not write.

Okay, look at this Hamiltonian. I substituted p prime as p-A/C and the capital A the vector potential vector I am choosing the Ax component to be –By so there could be a by C as it was pointing it out let us just lift it with putting C=1 so there is By shift here on the Bx. Incidentally it just correct, this gives you magnetic field along the Z direction, curl of A if you do, you should get Bz to be B. How do we separate the wave functions?

Definitely the Z coordinate is like a free particle. X and Y is coupled. Why? There is a white term here and the Py term here and there is also cross term Px with Y so you cannot separate the Xy wave function but you can separate the Z wave function. So first of all you have to stare at the differential equation and see how to first get a feel of how to write the solution. Then you can also make trial functions.

You can start trying various trial functions for the coupled XY components and see if I plug-in and I compare power of X, compare powers of Y f I get consistent solution then that is how the solution. This is what you have done in your earlier course. You have given a trial function and you compare the powers of X and find what is the energy eigenvalue so that are also can be done. But before we do that let us just state it and see how to separate the wave functions. And we know that.

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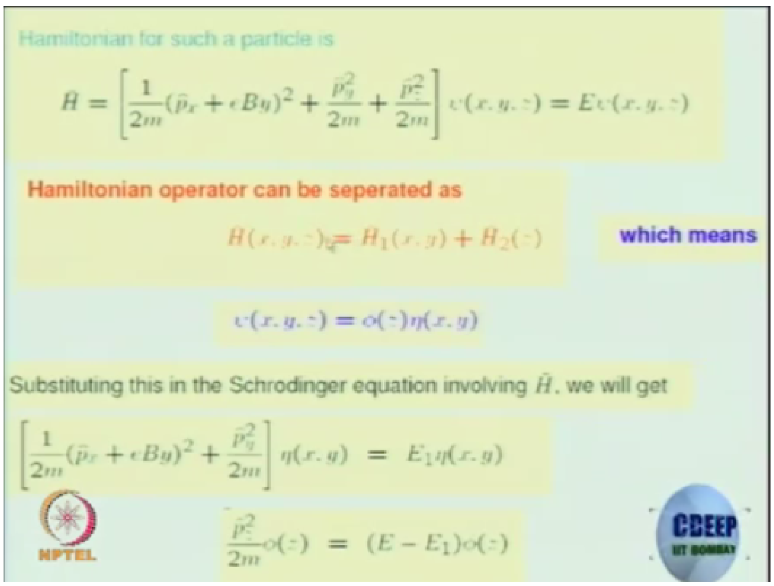

So we see that the Hamiltonian operator can be separated as a coupled one with X and Y coordinates and a Hamiltonian for the Z which is just Pz squared/2m which means I can write Psi of X Y Z as Phi of Z times C dot x. So separation of variables and you have E-E. I will solve the problem, not it. Why not it? I need to find what is the totally; I need to find what is E1; I need to find what is E dot XY; Phi of X I know. Why Phi of X I know?

It is just a free particle equation. We have been doing this even in one-dimension step potential energy.

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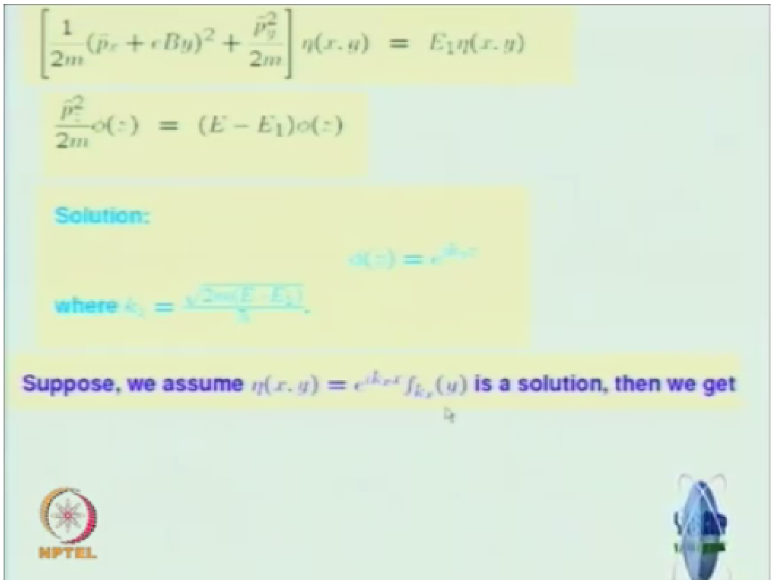

So solution to the Z is a free particle putting a K subscript Z where K subscript Z is root of 2m E-E1/h cross. Okay. So now comes we have no way in which we could try and fix E dot XY. We try to look at a trial solution, okay. So the just a trial solution this is not the derivation. If you want to just plug this solution into this equation and see what we get for harm in doing that. Only thing you can see here is that a pretend as if there is a free particle in the X direction.

But then the Y component is dictated by this kx in some fashion. It is not separate actually. This F of Y is not independent of this. There is a kx here which shows up here. So this is the way I am trying to look for a plausible solution to this equation or take it to the trial solution and will plug it to the above equation.

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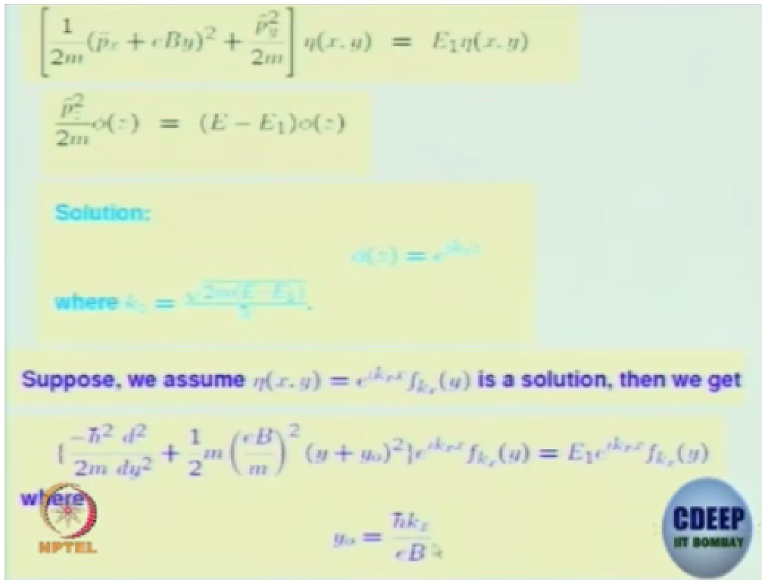

So let us plug in this and write out the top equation. Can you plug it in and check with whether you get this? So there will be a px squared px squared will give you a kx squared because you will have del/del x – h cross del/del x will give you a kx. So substitute this and check whether you get this. I have written it for some selfish reason in this fashion, looks like a harmonic oscillator shifted harmonic oscillator.

But the shift is dependent on the magnetic field and the kx which is the X component wave function. This is just a completely different, this is not; this if I have not given to you, you will not be able to reach this. But if suppose if I give you this trial function you will do by a different method that we will get this feel also, I just want you to get a feel of if you are given potential energies of different types, knowing the harmonic oscillator can you attempt some of these problems without trying to rigorously try and solve the differential equation, that is the main aim.

So, because I have given this form and if you plug it in you will see that the operator differential operator which you have here or the Hamiltonian operator these two sides are Eigen functions exactly like your time independent theories, but you can read off from here that there is a frequency Omega squared is eB/m, eB/m whole squared is Omega square and it is a shifted oscillator.

With this shift Y0 dependent on X component quantum number kx and the magnetic field was proportional to the magnetic. So what will be the solution now? We can try to write that is Fkx of Y. Now you also see that the kx dependence is showing up where the peak or the wave function shift will happen. If kx is 0 it will look like your usual harmonic oscillator in the Y coordinate; if kx is nonzero if it is positive, it will be shifted in the positive Y-axis; this negative will shifted in the negative Y-axis.

So picture clear? This step please work it out just substitute this one into this equation and verify that this is there, okay if you have not already done please fill it up.

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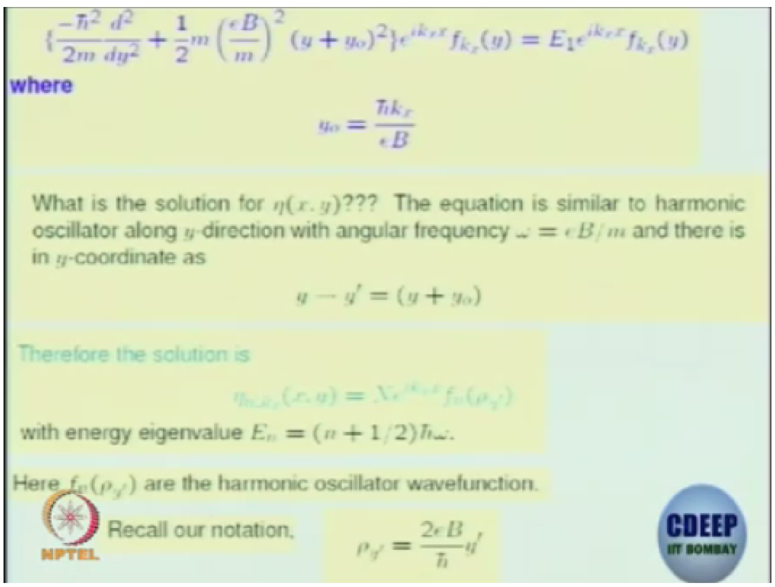

So again it is a shifted oscillator. And the frequency of the oscillator is also dictated by the given equation which is eB/m. Clear? So what will be the energy? 1/2 h cross Omega will be eB/m for the ground state. Another states is n+1/2 cross Omega, Omega is eB/m. So finally we see that we can actually find the solution like this e to the Ikx and then Fn with shifted Y2 Y prime and appropriately write it in terms of the in terms of the new coordinate by putting the m Omega /h cross.

Eigenvalue will be this will have in n+1/2 h cross Omega where Omega is you eB/m. Is that the energy or something more is there? The Z coordinate energy should be added to this, yes.

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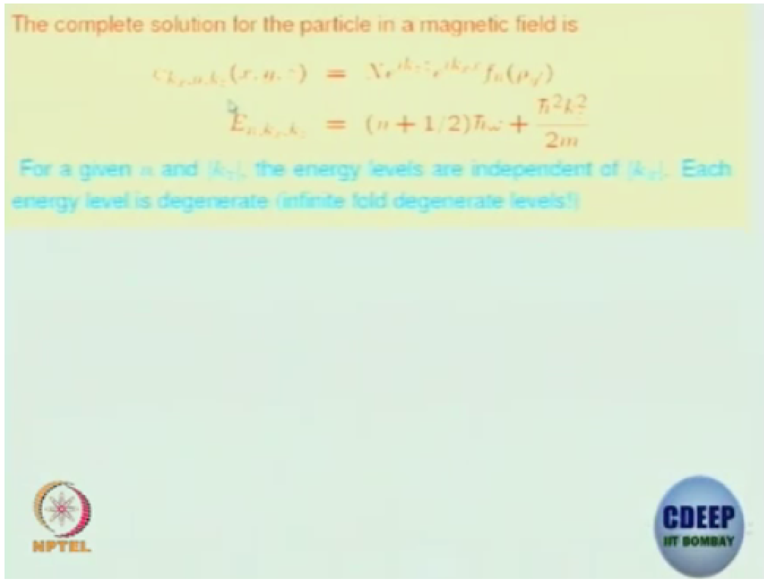

So the total solution in three dimension for a particle subjected to a magnetic field. We can try to use an appropriate choice for the vector potential, magnetic field is along the Z direction, you can make an appropriate choice for a vector potential and I try to give an (()) (24:30) for the separable function e to the ik z, z is separated but then Z and Y are connected, okay so this is Y prime is actually a function of xk, it is a function of kx. So this is the total solution the quantum numbers are orderly N, you also have a kx and a kz.

And the corresponding energy N sorry the energy for a N excited state in the Y coordinate but it can have a kx which is contribute into the energy in the X coordinate in general and kx. But we have seen that the kx and n are coupled in such a way that the energy depends only on the harmonic oscillator kind of shifted harmonic oscillator. So there is no kx dependent energy. It's clear? kx will only show up in the shifting of the harmonic oscillator.

The harmonic oscillator get shifted by coordinate Y to Y prime, the Y prime is dependent on kx but the energy does not get contribution because there are only two in equations the E1 and the E-E1. E1 was n+1/2 h cross Omega, E-E1 was h cross squared kz/2m. That is no kx. So what is this tell you? It is a degeneracy. kx can be anything for a given n and a given kz which means there is a infinite for degeneracy in the wave functions.

You can have many possible wave functions for different kx but a specific in the kz, highly degenerate, infinitely degenerate. Okay, pictorially I can show.

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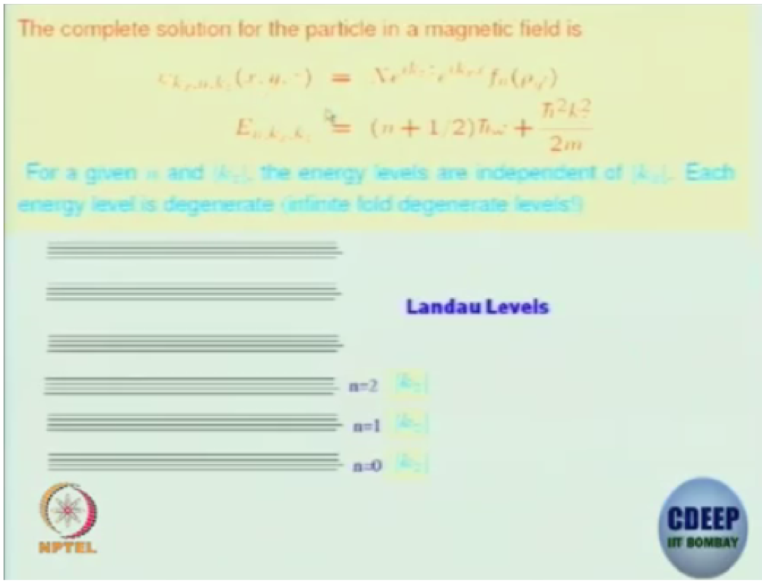

If I look at the ground state n=0 with a specific kz I have put it, no I am not very good at putting Infinite, nobody is good at it. So I just put in many lines to say that, each level you can have many wave functions depending on choice of kx to be different. But all of them will share the same, okay. So this is what is the called as sometimes as Landau Levels. So I have taken you through a small tour where I took the coupled harmonic oscillator, I said you can do, undo it by appropriately rotating your axis then I added or the only oscillator in electric field.

And we had a trick of completing the squares where it becomes like a shifted harmonic oscillator. Then I went to a little more complex problem where I added a particle in the magnetic field which should satisfy Lorentz force, which will give you a Hamiltonian and with a suitable choice of the vector potential you can end up getting the wave functions to be highly degenerate for a specific n and specifically kz, kx can be individual.