Quantum Mechanics

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Lecture - 29
Tutorial - 5 (Part I)

So in tutorial 5 again we have the same pattern of 5 problems and you have learned in your lectures how one obtains or solve commutator operator.

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3 Consider a three dimensional ket space. If a certain set of orthonormal kets, say $|1\rangle$, $2\rangle$ and $|3\rangle$, are used as base kets, the operators $\hat{A}$ and $\hat{B}$ are represented by

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}; \quad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix}$$

where both a and b are real.

(a) Find the eigenvalues and eigenvectors of $\hat{B}$.

(b) Check that $\hat{A}$ and $\hat{B}$ commute.

(c) Find a new set of orthonormal kets which are simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$. Specify the eigenvalues of the operators corresponding to each of the new kets. Does the specification of eigenvalues completely specify each eigenket?

So there will be a problem based on commutation to obtain a commutation relation. Then you have to calculate the commutation between the operators or in terms of matrices. So that will cover the part of the tutorial.

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4 Assume that the states $|u_i\rangle$ are eigenstates of the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

with energies E_i. Show that for an arbitrary linear operator $\hat{A}$

$$\langle u_i|[\hat{A}, \hat{H}]|u_i\rangle = 0 \ .$$

Using the above, compute $\langle u_i|\hat{p}|u_i\rangle$, the mean momentum of the state $|u_i\rangle$.

And in the latter part you have to obtain some expectation value of momentum operator or the energy eigen state of Hamiltonian.

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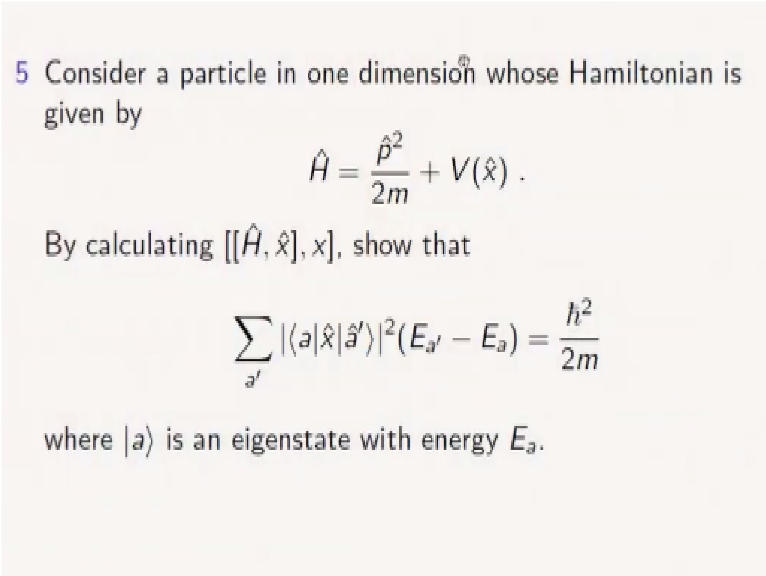

And there is one more exercise where in you form the commutator expression, you have to evaluate some relation. So coming to tutorial 5, the first problem is interesting because here you are given 2 states of a quantum system. It has basis state psi + and psi -.

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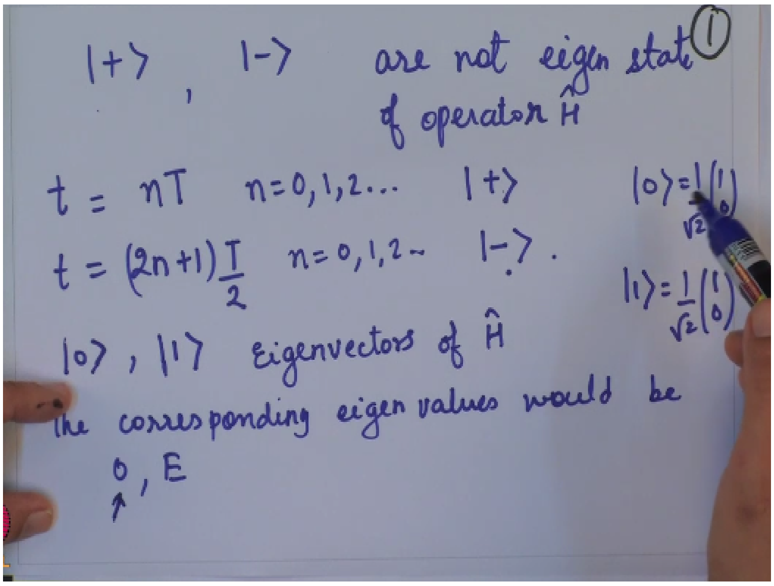

So psi + and psi - are the 2 basis state and these basis states are not eigen states of operator given to you some operator or Hamiltonian operator H, let me denote it by H. So these 2 are not the eigen state or eigen basis of the Hamiltonian operator and what one does is when you make a measurement of the energy of the operator so when you operate this operator on some basis you obtain the corresponding energies.

So if I operate this operator on eigen vectors, I get the eigenvalues. Now here you are given these 2 eigen kets and these eigen kets are such that when so the measurement is made at different times. So time t = 0, t/2, t, 3t/2, 2t et cetera. So when time is some n times T where n = 0 1 2 et cetera. so at these integral times the system is in state +, while at time when it is a half or you can call it as half integral.

When the time is half integral that is T/2 3T/2 et cetera your n can take value 1 2 3, the system is in state ket -, so again we are sticking to Bra and ket notation. This is a ket + and this is ket -. So state + when t is integral multiple of time. Time is integral multiple of some nT and the eigen vector or the ket is - when time is half integral multiple of capital T. So you have to construct a 1 parameter of parameter Hamiltonian which reduces the above physics and give a relation between the parameter and time T.

So now let us get started, we assume that the Hamiltonian operator is an eigen vector as an eigen vector 0 and 1 okay and in the previous set of examples you have seen that the eigen vector is written as the notation we used was 1 and 2 here we have 0 and 1. So 0 will be nothing but and 1 will be, let us assume that Hamiltonian operator has eigen ket 0 and 1, you can see they are orthonormal and normalized.

So these are eigen ket or eigen vectors of the Hamiltonian operator. Now when I have these operators the corresponding eigen values, eigen values would be 0 and E okay. We can choose one of the value as 0 because the global minima and it will not affect the value of the measurement. So now these kets can be represented in terms of the basis state + and -.

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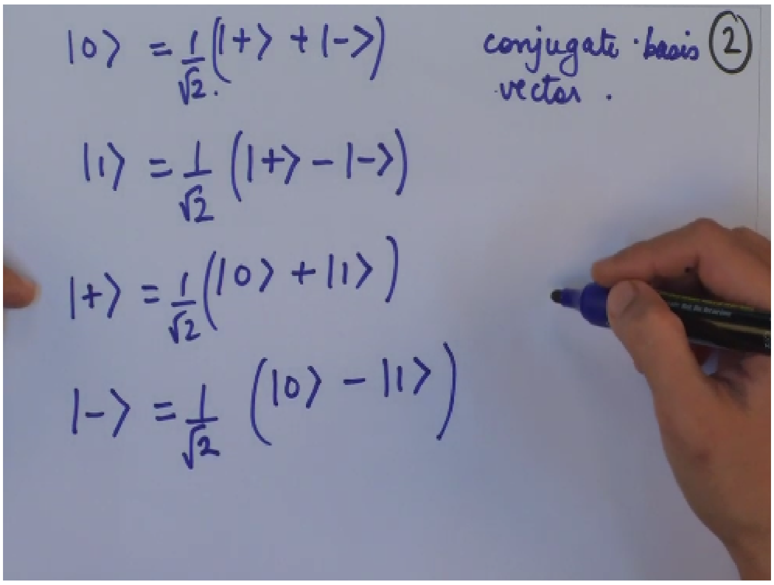

So these kets are 1/root 2 okay. So these are called conjugate basis vector. So these are called conjugate basis vector and state or ket 0 is represented as 1/root 2 ket + ket - sum of these 2 and the corresponding ket 1 will be, so in the conjugate bases you have ket + + ket - as the eigen ket 0 and the ket 1 is represented as + - - ket okay. So this you can invert this in terms of ket + and ket -.

So this can be simply inverted in this manner okay and using this you can obtain some relation between the eigen values for different time.

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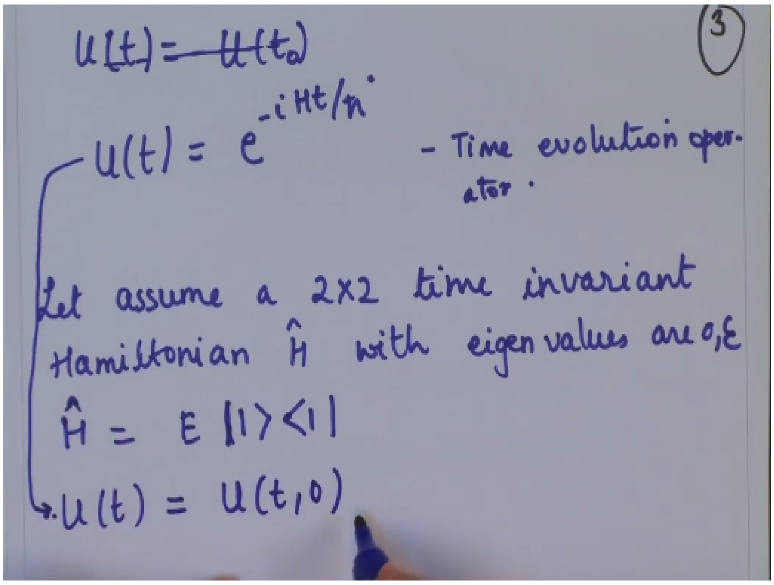

The evolution operator for or the unitary operator which evolves with this will be U of t will be e raise to –iH, this is the unitary operator or time evolution operator okay. So this is the time evolution operator in terms of U0 I will have to write it later. So let us assume a 2 * 2 time invariant Hamiltonian H with we have assumed that the eigen values are 0 and E. So 0 and E correspond to the energy eigenvalue.

Now the Hamiltonian operator with 0 eigen value can we can take or assume presumably because the global phase is invariant under such measurement. So it will be immaterial. So the Hamiltonian operator now I can write as 1 1 okay. So again coming back to this notation, so Hamilton operator will be written in this form okay. You have to recollect the problems we have done in tutorial 3 and tutorial 4, we have done such problems.

So now when you write the unitary operator coming back here, this unitary operator it is evolving from time T = 0 to time T = some T. So time evolution operator is given by this. We have to operate this. Suppose we operate this operator on some basis vector or basis state.

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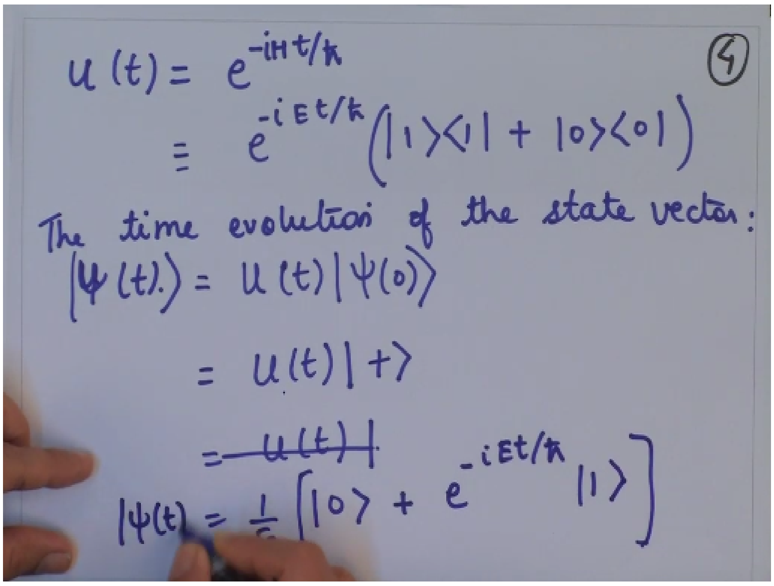

You will have U of t is nothing but iHt upon h cross which will be equivalent to writing this e raised to –iE t upon h cross in terms of the basis 1 and 0 we have this. So at time T = 0, we can operate. So the wave function, so let me write the time, evolution of the state vector. So this is the state vector. So now we want to evolve the state vector such that you have U of t psi of 0.

So from time T = 0, U will, U of t is the unitary operator which will evolve the wave function or the state vector with time so that I can write it as U of t. So when time T =0, remember when time T = 0, the state is + and we have seen that this in the terms of conjugate bases we have + ket is written in terms of 0 and 1. So this is U of t. I will have in terms of 0 and 1. So I can again substitute for these and directly write 1/root 2 0.

The evolution operator I have used to get exponential - i Et upon h cross state 1. So this is the time evolution operator.

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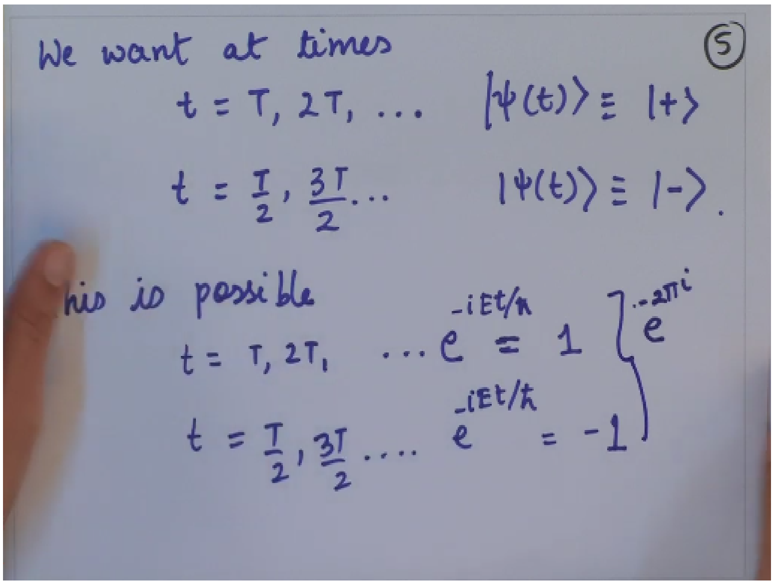

What we want is that at times t = T, 2T et cetera, psi of T that is this ket should be + and at times t is T/2. So 3T/2 et cetera psi of t should be in state -. So how will we rewrite this psi of t as + and -, this is possible when our t goes from one state that is t is in terms of the Hamiltonian operator. So in terms of the time evolution operator we have written this with the exponent.

So time t will be equal or periodic only when this expression T, 2T comes back to some value, that is to say that exponential of iEt upon H cross comes back to some value say 1. So this periodicity you get when this exponent is 1 and for time T/2, 3T/2 et cetera this same exponent would give you a value -1. So at these 2 times this is possible only when your exponent is 2pi i okay, for both cases.

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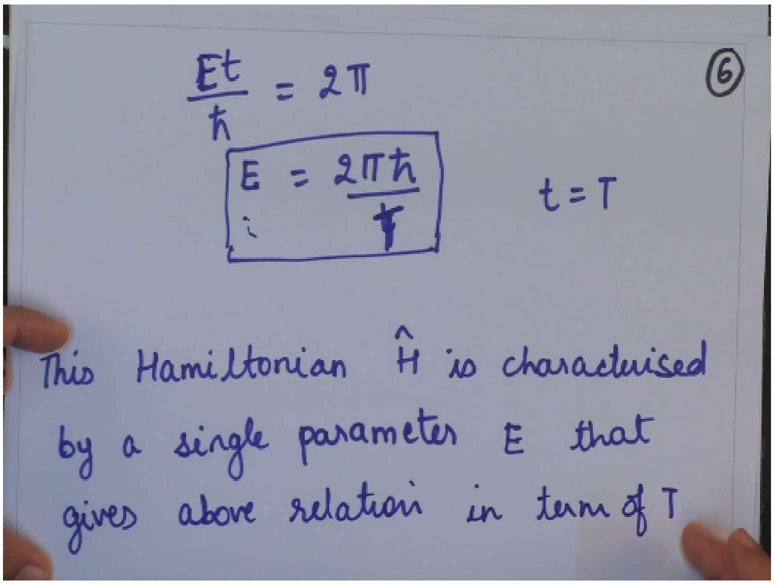

So Et upon h cross should be 2 pi and so the relation between E and the time t. So now at time some time T. So this will become a capital T we are saying that t is capital T. So we have here the single parameter which relates the energy to T and energy is related to the Hamiltonian. So this Hamiltonian is characterized by a single parameter E which gives the above relation okay.

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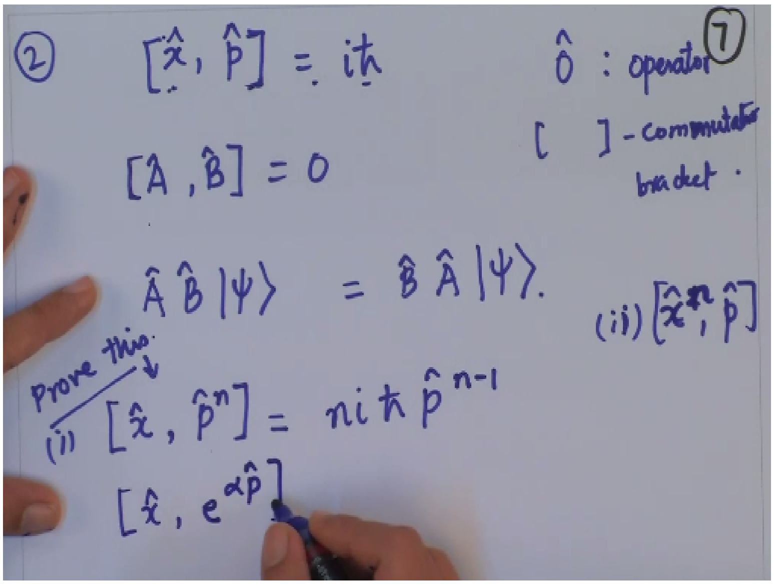

You have I think already done some relations like x. So there are commutators like x, p. So famous commutators like x, p and then you have you can check whether a Hamiltonian operator commutes with the momentum operator, this position vector. So let me put a cap here. So x cap, p cap where x and p are the operators and operators are represented by a hat or a cap.

So this is an operator representation okay and this is called as a commutator brackets okay. Two operators commute, operator A and B commute that means that A and B when you put in the commuted bracket it is 0. You will see in coming lectures that why do we check whether 2 operators commute or not. So when 2 operators commute they form a simultaneous eigen ket.

So the measurement of A do not affect the measurement of P that is what it could mean. So when you operate an operator A B or eigen vector, this will be equal to, so interesting part of these 2 operator is that the measurement of A and B are independent and hence you can form a simultaneous eigen ket for this vectors and this will be useful for coming lectures. This is a slight background and as we know that operator X and P gives ih cross.

This is the relation we obtain and I think you might have proved some time or if you have not proved this please try your hands on prove this. I will not discuss the proof in this tutorial but you must try out x, pn. You can also try, this is 1, you can also try what do you obtain n operator p. So x raise to n on P. So x on p n this I will give you this result which you can use in the problem.

But it would be I would suggest that you must prove this identity. So this we are going to use in this tutorial 2. So in second question you are given 2 operators position operator x hat and the momentum operator p hat and you have to write this operator evaluate the commutator of these 2 operator. So let us try evaluating the commutator bracket or the commutation relation of x and alpha e raise to alpha p.

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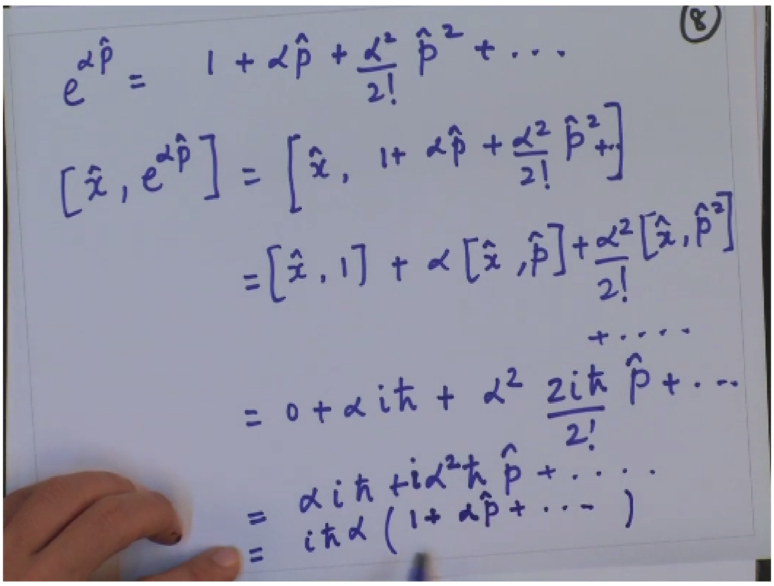

E raise to alpha p cap I can write this as 1 + alpha p hat + alpha square upon 2 factorial p hat square and so on. You can write the exponential series. Now when I do this, this would be commutation of the position operator with these terms of the exponents. This is a simple way to do +…. So the first part this is the first term, second term would be this, third term would be x hat p square + all the terms.

So remember the operator times a number will always give you 0. First term is 0 + alpha ih cross. Now second term you will use this here it will be square, this square n is 2. So you have 2 ih cross p. So this is 2 ih cross p + alpha square 2 ih cross you have a p hat upon 2 factorial. So what will be the result of this alpha i + plus you have the other terms. Alpha ih cross + you have 2, 2 will get cancelled.

You have alpha square h cross p had, I have not missed anything okay. I have a factor of i also + n terms. So in general form can I write it as i so this I can take ih cross alpha and I have 1 + second term would be alpha p hat + so on. So this is nothing but again you get an exponential series.

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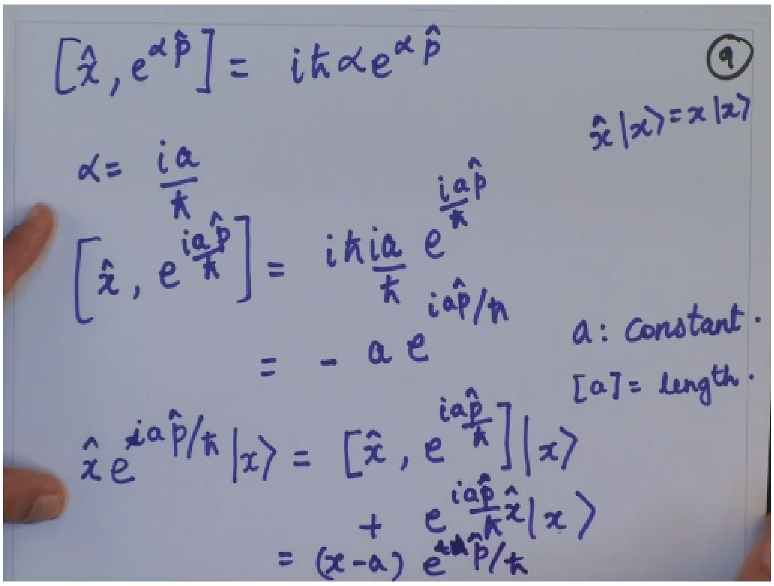

So this would be x e raise to alpha p hat is nothing but ih cross e raise to alpha p hat. So this is what I get after performing and I have missed the alpha which I put here. So this is the expression I obtain for commutator of x e raise to alpha p. Now if I replace alpha by my question is ia upon h cross. So let me replace alpha by ia upon h cross, e raise to ia p cap upon h cross will be this would be h cross this will be ai h cross e raise to i a p cap upon h cross.

That is nothing but – a e raise to –ia p cap upon h cross. So this is what I have obtained after finding the commutator of these 2 operators. Remember a is a constant, a is a constant and it is having the dimension of length. Dimensionally it is the dimension of length. You can see from here. It is very simple to see from here. Now the second part of the question is show that x is the eigen state of this operator.

The eigen state of this operator I have an exponent e raise to ia p hat upon h cross I operate on x. So when I operate x hat on x and I obtain x times x okay. So x is some eigen state of a positive operator you have to evaluate that. So now if I operate this on e raise to ih cross. So now we have the commutation relation. So if I write x hat as this operator I define as commutator of x ia operated on x- or +.

Let me see this is x e raised to i then I have a -, so I will have a + e raised to ia p hat upon h cross on x. So this is what I have, this would give me, this relation we have seen gives this commutator gives -ae raised to ih cross and operating x on the vector will give me x. So I have x-a e raise to ia p hat upon h cross.

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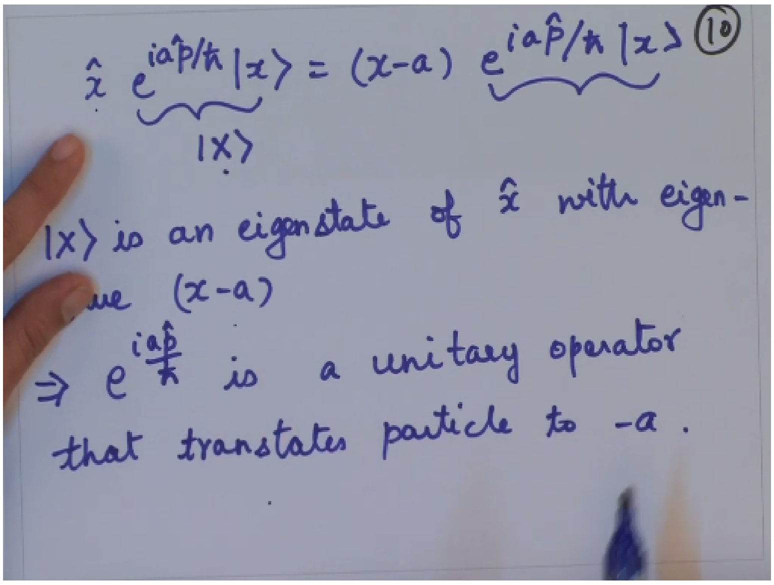

So the final step is e raise to ip hat upon h cross x will be x-a e raise to iap hat upon h cross x. this is what I had correct. So what do we observe? This is some another vector x. So what do we obtain here is that e raise to ip upon h cross x is an eigen state. This operator let me call it as some capital X. So this capital X is an eigen state of operator X with eigenvalue x-a. So this operator is the eigen state of, this ket is an eigen state of the operator small x with eigenvalue x-a that implies that e raise to ip cap upon h cross is a unitary operator.

What does this unitary operator does? It translates x to x-a, translates particle to -a okay. So if this operator is operated on x it should give me same x and ket X, capital X back, but here it does what it does is, it gives me x-a that means it translates the particle to –a. So this is a very good exercise wherein you understand the operator how the unitary operator comes into picture for this particular problem in the previous problem also you have seen a different flavor, so with this we stop here.