Quantum Mechanics

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Lecture - 28
Compatible vs Incompatible Observable - II

So let me just get on to this compatible and incompatible observables okay.

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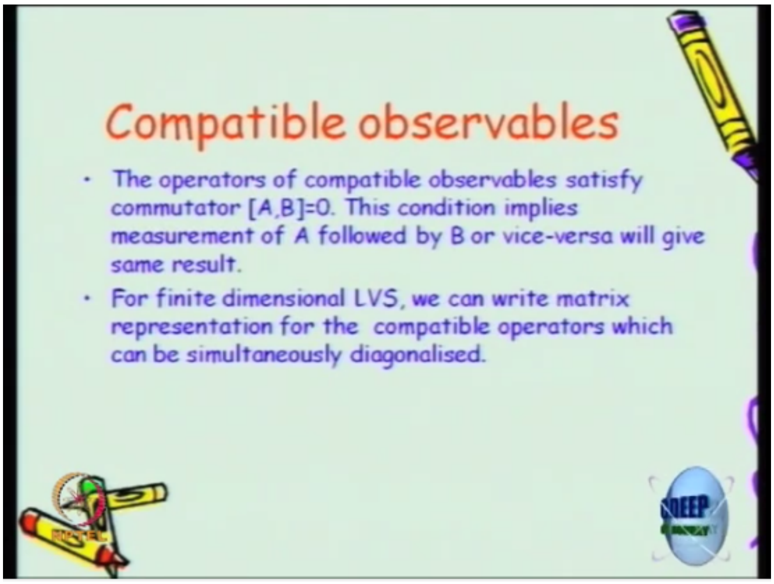

So as I have already said the operators of compatible observables satisfy commutator A, B=0. There could be a set of compatible observables with A, B to be 0 okay. This condition what does it mean in the context of measurement? I had already told you when you do the measurement on an arbitrary state psi of t, it will collapse to one of the eigenstate of that operator correct.

So this condition will imply that if you first measure A operator on an arbitrary state, it will collapse to some eigenstate of the A operator. If you try to measure the B operator on that, this what will it give, So the claim is it will give the same the order if you do it in this order or the other order, the result which you will get will be whatever measurement you get for A and B will be the same.

So for finite dimensional linear vector space, I have always stress the fact that you can write for these operators a matrix representation and if you write a matrix representation then two matrices commutator being 0 means that you can simultaneously diagonalize the matrix.

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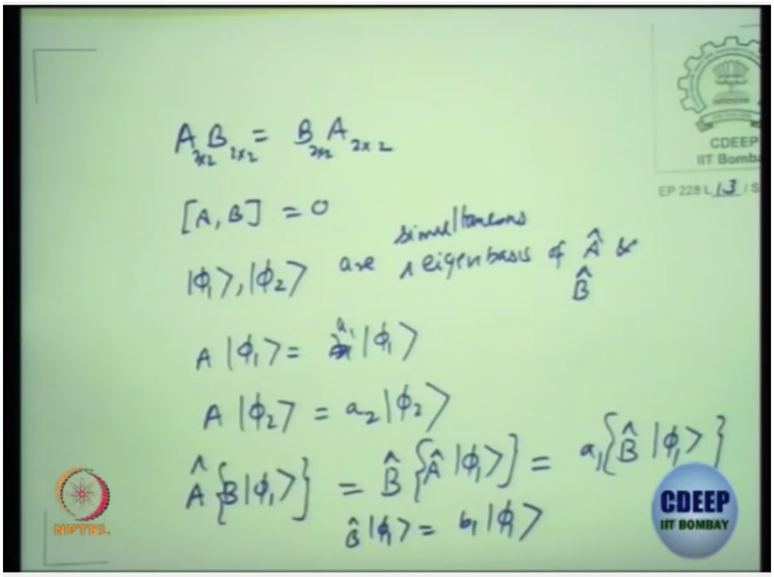

So if you have AB=BA, suppose let us take 2 x 2 matrices, so this is the meaning of A commutator B=0. If phi 1 and phi 2 are the basis, eigenbasis of A operator, what is that mean? Phi 1 is lambda 1 phi 1 or let me call it as a1 phi 1, A on phi 2 is a2 on phi 2. Let us use this on the state, let us do that on that state. So A on B on phi 1=B A on phi 1 but A on phi 1 is a1, a1 is a number eigenvalue, B operator on what have I got.

What is this equation? See the extreme left and the extreme right, B on phi 1 is also an eigenvector of A operator with the same eigenvalue a1. So B on phi 1 should be proportional to they are not degenerate eigenvalues, eigenstates, you cannot have two different what is that LV that B on phi 1 should be proportional to phi 1 that is it or B will also share the same eigenbasis of A operator.

So what have I tried to prove here? I have tried to use the fact that it is compatible observables or the corresponding operators are commuting. Once I take the corresponding operators as commuting, then I get an equation which is like an eigenvalue equation. For simplicity, we have taken a two-dimensional linear vector space and a matrix representation which is 2 x 2 matrices.

Take the basis which is an eigenbasis of A which is non-degenerate, a1 and a2 are not equal and in the process you find another eigenstate B phi 1 and that means for a completeness of these two-dimensional linear vector space B phi 1 should be proportional to phi 1 or equivalently we have proved that B operator on phi 1 is some b1 times phi 1. So this means both A and B can have the same, this can be eigenbasis of A and B which is what we call it as a simultaneous eigenbasis of A and B.

You cannot do this if they do not commute, you can do this only if they commute and we are also assuming the fact that both the eigenvalues of A operator and B operator are non-degenerate. You could have had the same eigenvalue with the different linearly independent state, that case we are not considering, we will come to that.

So I have proved for you that for finite dimensional linear vector space, we can have matrix representation for the compatible observables which can be simultaneously diagonalized. Now you are clear? When I say it is a simultaneous eigenbasis, it means that given an A operator and B operator which are 2 x 2 matrices, you can find so you can find you have an operator. How do you diagonalize the matrix?

You try to find some S matrix, similarity transformation which will give you a diagonal matrix right. Matrix mechanics everybody has done. Now the claim is that this S matrix is the one which gives you the eigenvectors. So when I say they are simultaneous eigenvectors, it means it is a same S matrix which should also operate on this which will give you the B diagonal, it cannot be a different S matrix.

Both A and B are simultaneously diagonalized by the same diagonalizing matrix which gives you the two eigenvectors which are the same eigenvectors for both A and B. The earlier statement was if you do two consecutive measurements A followed by B, A measurement and then followed by B, the final result which you get will be the same as doing B followed by A, the order does not matter, that is what is the first statement.

The second statement is the commuting operators in a finite dimensional vector space need not be two-dimensional, even in n dimensional n x n matrices, you can simultaneously diagonalize by the same similarity matrix which means the eigenbasis of A and eigenbasis of B will be 1 at the same.

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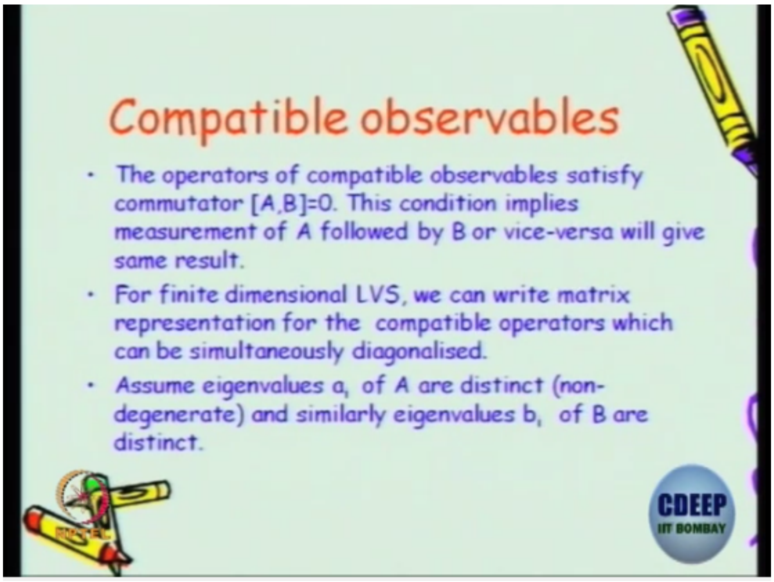

So this is the first case which I said for the 2 x 2 matrices. Assume eigenvalues ai of A are distinct, distinct means they are non-degenerate and similarly eigenvalues bi of B are distinct okay, so both are non-degenerate. This is a very special situation. You could have situations where some operators may have two eigenvalues being same which will give you the degenerate situation right.

Energy eigenvalues for example when you look at a particle in a two-dimensional box, if you look at the first excited state, there are two linearly independent states right with nx=1, ny=2 or nx=2 or ny=1 in a two-dimensional particle in a box where you will have the same energy eigenvalues, nx, ny formally.

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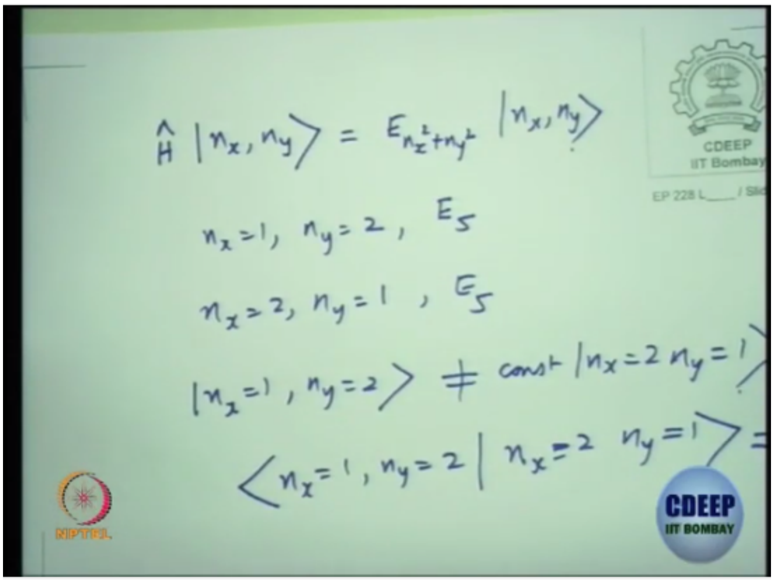

These are various notations where you can have the Hamiltonian operator for a particle in a two-dimensional box which will give you En x squared+ny squared. If nx=1 and ny=2, this energy will be E5. If nx=2, ny=1, this is also E5 but that states which I write nx=1, ny=2 is not proportional to some constant times nx=2 ny=1. They are linearly independent, in fact you can explicitly show that nx=1, ny=2 and nx=2, ny=1 is 0 or nonzero.

If you say it is linearly independent and you are looking at a basis which are orthonormal basis, then this is. So this is a degenerate situation. One observable A if I take it to be an energy operator for a two-dimensional particle in a box that will be a degenerate situation. I am not looking at the situation, I am looking at one-dimensional particle in a box, let say that there are two operators each of the eigenvalues, the position operator for example in a one-dimensional box, it is going to be distinct energy sorry position eigenvalues.

But there could be situations like this, some functional operator; you can have a functional operator as x squared+y squared. Then all the points on the circle will have the same eigenvalues but the states will be different. So that is where you will start seeing degenerate and non-degenerate. Just for simplicity, let us look at most of the 2 x 2 matrices or 3 x 3 matrices.

Let us take all the eigenvalues of A operator are distinct, no two eigenvalues are same and similarly eigenvalues bi of B are distinct. What will happen when you do the measurement now?

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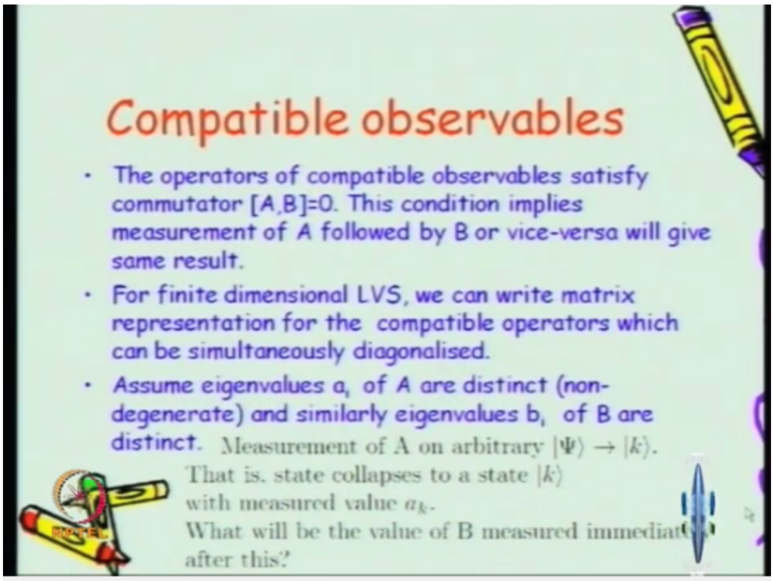

Suppose you do a measurement of the operator A or observable corresponding to the operator A on arbitrary psi. So I have just replaced your phi n by this n or phi k by k okay. So this is just to further simplify notation, do not worry about why we are not writing, so we are going to always write instead of phi n, I am going to just use the shorthand notation here. Remember that phi n eigenvalues lambda n and that n is what I am keeping track of and I am not really going to keep track of the phi.

This is just the matter of notation. It is just a variable, dummy variable okay. So measurement on an so you initially have a system given to you prepared in a state psi, that is what you are given, you are trying to do a measurement of an observable A on that arbitrary state in which it is prepared. So we already argued that it will collapse to one of the eigenstates of the A operator but which eigenstate nobody knows before measurement.

After measurement, the person says I got ak as eigenvalue. Once he says I have got ak as eigenvalue then I know that it has collapsed to the state k in phi k as k now. Then, if you do B measurement immediately after this, what will be the B value? It will be bk right. So this is what is the system collapse is here. What will be the value B measured immediately after this? We can definitely say even before measuring, it has to be bk.

And you can measure and say it is in bk because once it has collapsed to the eigenstate, it is going to remain in that eigenstate right. It has collapsed to one of the eigenbasis of A operator but once it has collapsed into the eigenbasis of the A operator, it will remain in that eigenbasis which is the simultaneous eigenbasis for the B operator as well because it is compatible observables, both A and B are commuting.

That is why even before measurement of B, after you do the measurement of A, you can precisely say what should be the value of the B measurement and you can verify okay.

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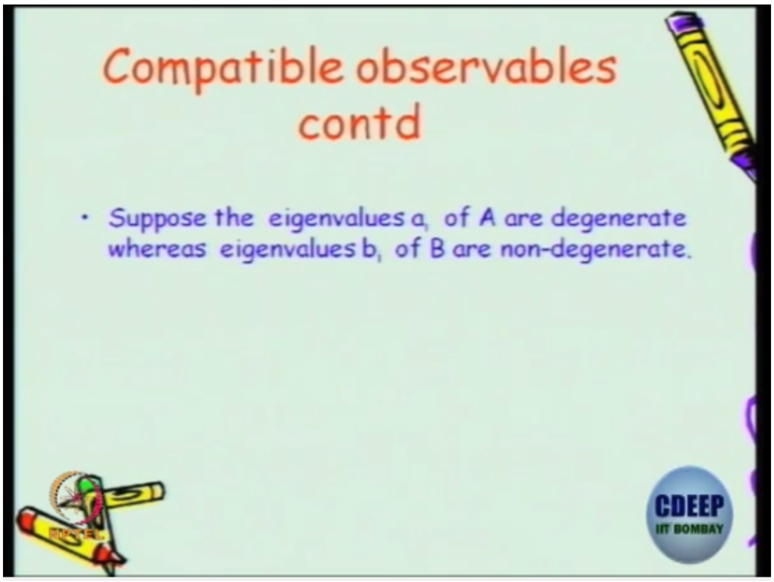

What is the next complication we can do? This is what I was trying to say. Suppose say eigenvalues ai of A are degenerate. Let us take a1 to repeat twice, the remaining of them are distinct okay. That is what is the degenerate situation and whereas you take eigenvalues bi to be non-degenerate, b1 to bn are all different, so let us take that simpler situation.

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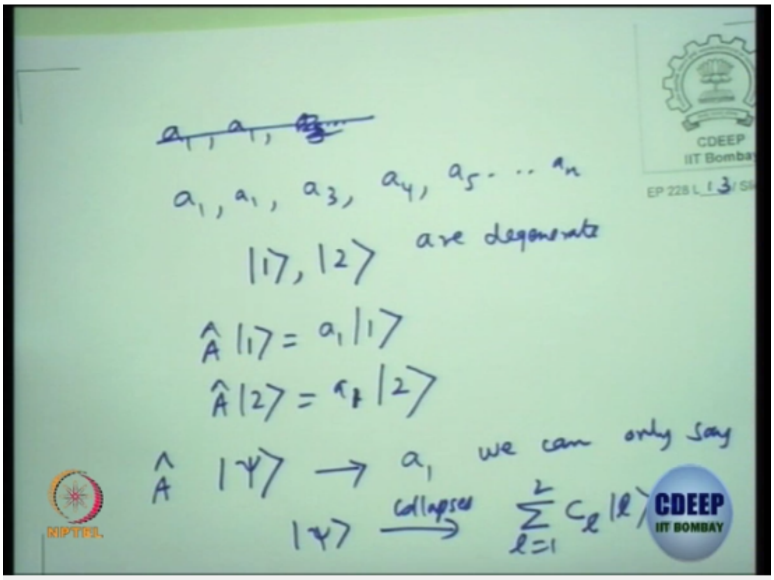

So let us take a1 and we tied a1, a2, a3 so let us continue here a1, a1, a3, a4, a5 dot as distinct eigenvalues. There is a degeneracy, two-fold degeneracy for the eigenvalues the first and the second eigenvalues. What does it mean? The state 1 and state 2 are degenerate. So A operator on state 1 will be a1 times 1, A operator on state 2 is a2 I should have written but a2 same as a1 on 2 because of the degeneracy.

So now when you do a measurement on an arbitrary state, suppose you take an arbitrary state psi and you try to do a measurement of the A operator, we said it should collapse, what is an experimentalist get? He says I get a1 eigenvalue suppose. Can you predict the state now? What can you utmost say? It can be either one of them or it could also be a linear superposition of them.

When he says he has found eigenvalue as a1, the system state has collapsed to we can only say psi collapses to some linear combination of 1 and 2. So summation over let us write Cl l where l is 1 to 2, I do not even know what they are. The Cl can have a system, which of the set of Cl's I am measuring I cannot say without touching the system. You do not know beforehand, you only know that before measurement, it will go into one of the eigenstates for the non-degenerate case.

For the degenerate case, you just get an eigenvalue in your hand that it has collapsed specifically to 1 or specifically to 2 that equation you do not have before, whatever you miss, think about it okay. This is one small scenario which we are taken.

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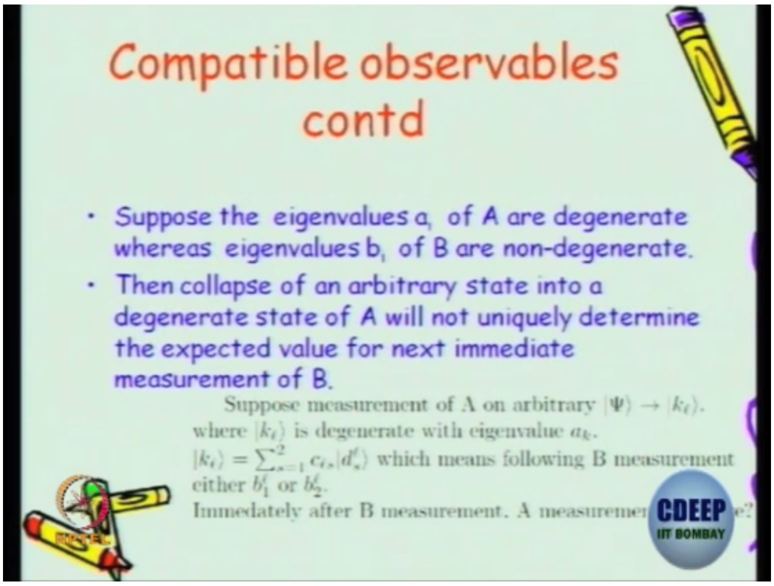

So the collapse of an arbitrary state to a degenerate state of A, suppose I know that the collapse gives me the degenerate eigenvalues will not uniquely determine the expectation value for the next immediate measurement of B. This is all I am trying to say. It could be either b1 or b2; we do not know which one, so this formally I had tried to help you in one particular thing.

Suppose measurement of A on arbitrary state psi goes to kl where kl is degenerate with eigenvalue ak okay. So all the l's, so you can take l this to be two-fold degenerate where l is 1 and 2 this is the example I took. So ak is the same eigenvalue, you could try to rewrite the state in terms of the superposition of the eigenstates of the B operator. No harm doing this and you will find that the measurement which you do on B operator on an arbitrary kl, it will collapse either to l=1 or you know the s=1 or s=2 for a specific one.

But which one it will collapse to you do not know till you measure. If you get the a1 eigenvalue, I know the B eigenvalue has to be b1 or b2, that much I can tell but when you try to do the measurement it will be either b1 or b2, which one you will get beforehand you would not do amongst it. So immediately after B measurement what will A give? So it will give a1 only.

If suppose a1 was a measurement before and then the B measurement suppose it gave b2, again you do an A measurement, it will be a1 only okay. The logic is from non-degenerate to degenerate nothing will happen if collapsed to a specific ideal fashion but if you are going from a degenerate to non-degenerate, there is some ambiguity that it can go into a superposed state and then you need to worry about what is the B measurement.

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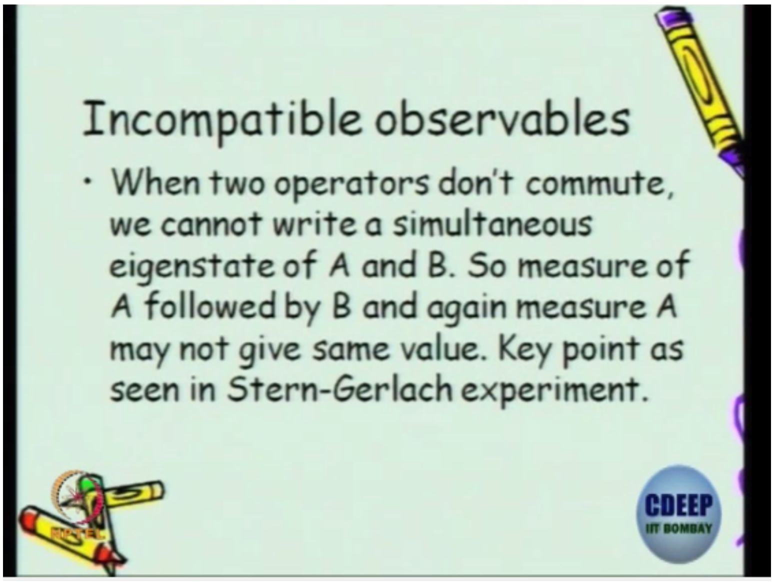

So what is incompatible observables? So far, we have spent a lot of time on compatible observables and even in compatible observables there are two kinds where one observable can have degenerate eigenvalues or non-degenerate eigenvalues. In non-degenerate, there is very simple, they are the simultaneous eigenbasis of both of them. In the degenerate case, there is some certainty that you can rotate in that degenerate subspace to write the eigenbasis for the B operator. This is what I have tried to tell you.

So incompatible observables means they do not commute and we cannot write a simultaneous eigenstate of A and B operator. So measurement of A followed by measurement of B and again measure A may not give C as a value, cannot give you the same eigenvalue, same values, you agree know. This is the key point which plays a very crucial role in the Stern-Gerlach experiment.

How many of you know about Stern-Gerlach experiment? So that they can put it through the spins presence of a magnetic field, this will be one of the themes which will work on the rest some of this lectures following and you will see that the measurement of the x component of the spin followed by y, again come back to the x, you cannot predict, it keeps on oscillating. You cannot say that once I have done the measurement what is the reason?

That is incompatible, some of the incompatible operators I am sure you are all familiar with.

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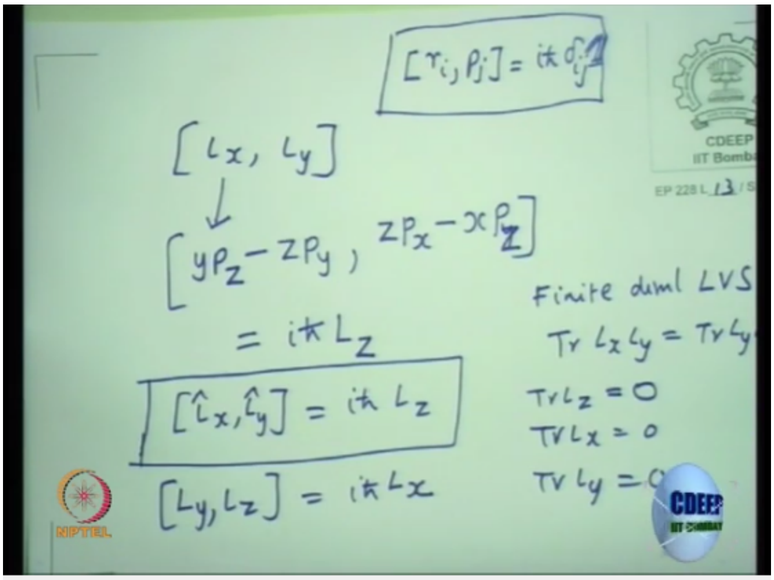

Suppose I ask you Lx, Ly commutative, Lx is yPZ-ZPy and this Ly is ZPx-xPy sorry xPz. Use the properties commutative properties and work this out for me and tell me what you get. Check whether you get this, please check this. You have not done the steps but please use that. What do you have to use? You have to use ri, pj is ih cross delta ij, use that and prove this that Lx, Ly=ih cross Lz, please prove this, exercise for you.

And once you prove this, what can we say about Lx and Ly? Compatible or not compatible? Incompatible observables. So if you try to measure Lx and then do the measurement of y and again Lx, you cannot predict what the Lx measurement is. In the Stern-Gerlach experiment, the intrinsic split was happened of the two beams showed that even though the angular momentum is 0, they have some intrinsic quantum numbers which is the spin quantum numbers.

They also satisfy the same property is what they said okay. So those are the incompatible operators, which makes the systems much more interesting. You do a measurement once; you come back and repeat the measurement you get a new result because of this incompatible set of observables lot of beautiful things happens in the system okay. Classically classical mechanics, we would not think like this but now you know incompatible observables plays a very important role.

Although, some more things if this is going to act on a finite dimensional vector space, the trace of Lx, Ly should be same as trace of Ly, Lx. So which means trace of Lz has to be 0 for finite dimensional linear vector space. So the matrix representation when I write for Lx, Ly, Lz, so this rule is also you can try to find what is the commutator of Ly with Lz, what do you expect?

This is just x, y, z rotation, so this is going to be ih cross Lx. So it is same argument you can show the trace of Lx should be 0, trace of Ly to be 0. Here what happens? By the same token, there is an identity operator trace on the left hand side if you do it is 0, right hand side trace of identity, can it be 0? So then we say that we cannot give a matrix representation for an infinite dimensional Hilbert space.

Matrix representations are possible only for a finite dimensional Hilbert space. That is why this equation is allowed in an infinite dimensional Hilbert space and if claim this is true in finite dimension, then these traces of them have to be 0. So all these properties will be satisfied by the spin operator in Stern-Gerlach experiment that they will be traceless, these matrices which we will find they will all be traceless, so it is very consistent okay. Okay So I stop here.