Quantum Mechanics
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Lecture - 27
Compatible vs Incompatible Observable - I

So today let us first kind of recap what all we had done in the last few lectures and then my theme today will be even in the observables you can separate them into a compatible observable and incompatible observables okay. So this is the theme for today's lecture. Before I get on to this, let us recap some of the things which we know. Operators acting on the Hilbert space which corresponds to observables, what are the properties?

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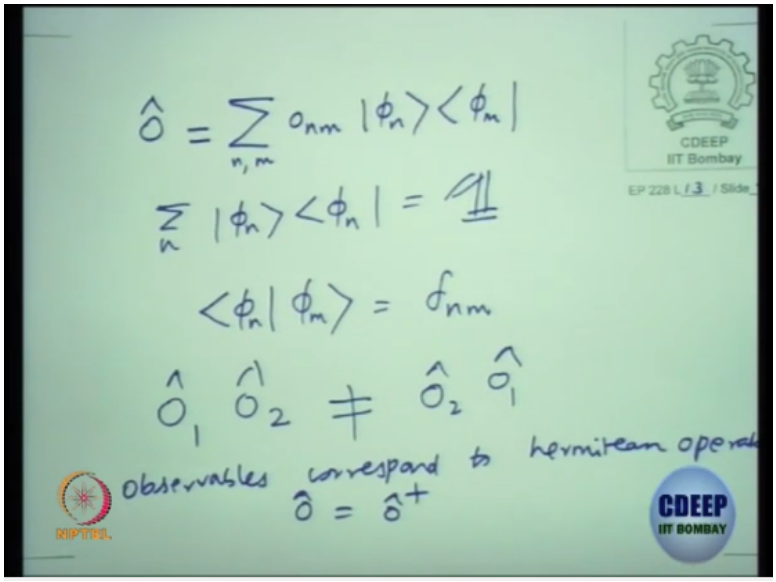

So we have said that the linear operators O hat is a linear operator and you can give in a finite dimensional space some kind of a representation which will involve Onm phi n phi m right. So these are the complete set of basis. What do we mean by complete set of basis? It is summation over n, this is n, n summation over n phi n phi n is an identity operator and for convenience we use them to be orthonormal also, to be delta nm right.

If you have two operators O1 and O2 then this in general is not equal to O2 O1 okay. So this is in general not equal to O2 O1 and whenever you have products of each of the observables in your theory will be given by a Hermitian operator. Observables correspond to Hermitian operators. So they have to be O=O dagger, complex conjugate and transpose if you give a matrix representation.

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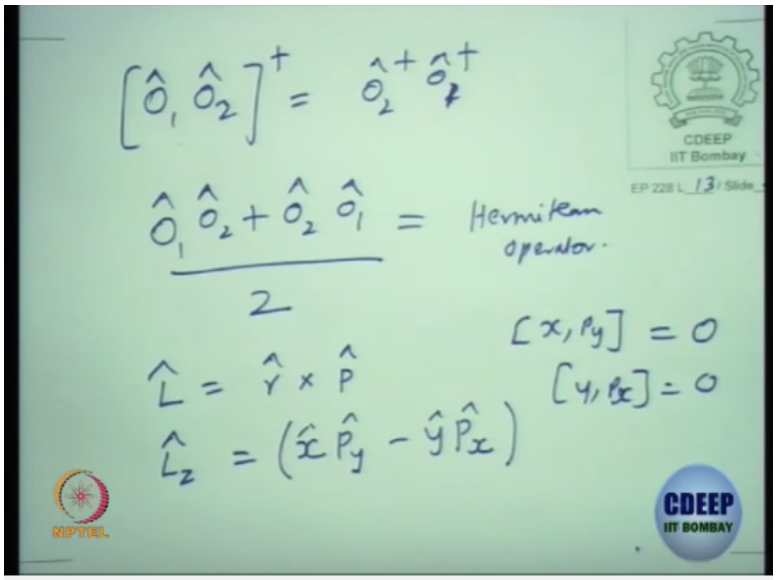

If you have product of two Hermitian operators and if I try to find the dagger of this that is O2 dagger O1 dagger sorry. So this product of two Hermitian operators is not Hermitian because this O2 O1 is not same as O1 O2. So what do we do? We need to write a Hermitian operator which means we will define O1 O2+O2 O1 to be a you can normalize it by 2 because we are repeating it twice in comparison to classical mechanics.

So this is what we will call it as a Hermitian operator. Is that right? So this is what you have to keep in mind whenever you are given find the expectation value of a product of two operators, you need to make sure that that product either those two operators are commuting or those two operators are not commuting okay. This is what you have to remember like for example angular momentum is r cross p.

So if I write this Lz operator, how will I write this? As xPy yPx but why do not we do the ordering here. Classical mechanics no this is quantum mechanics, we still write quantum angular momentum operator without adding a Py xy, so your x and py they are commuting operator. This argument is only when O1 and O2 are not commute. O1 O2 is different from O2 O1 but if O1 and O2 are commuting O1 O2 is same as O2 O1.

That is the definition of a commutator, so whenever we know that x with py component is 0 we do not need to worry about the order that is why we did add a pyx/2 same with other. So xy the angular momentum operator even in any quantum mechanics book this itself is

Hermitian, a trick question like this can confuse you and you can write the Hermitian combination like that which is not right okay.

So I am slowly getting onto you that there are two kinds of operators. There could be operators which are commuting; there could be operators which are not commuting. So the commuting operators are what we call it as a compatible operators and non-commuting operators are non-compatible operators or incompatible operators and then what did we do further in the last lecture.

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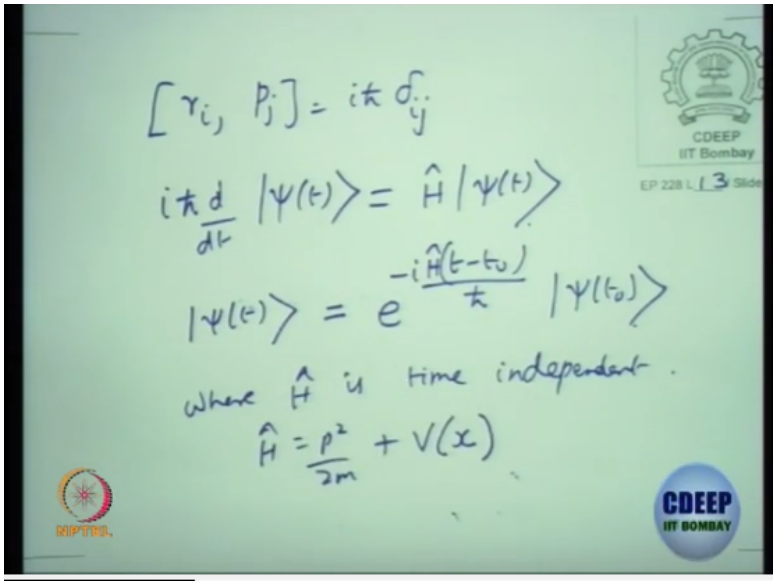

We went on to writing, we had this postulate ri, pj is ih cross delta ij in one dimension I wrote this identity but you can rewrite it this way. We also had an evolution equation on psi of t to be H of psi of t. This is the equation, empirical equation which will allow you to get the state of a system at any time t which is evolving due to the Hamiltonian and then we try to find psi of t for time independent potential energy.

If this becomes time dependent Hamiltonian, then you cannot do this because this H is independent of time, you can rewrite this as –iH operator t-t0 over h cross on psi of t0. This is the solution for this equation where H is time independent, this is very important which is equivalent to H is made of kinetic energy+potential energy, we are doing nonrelativistic quantum mechanics.

So H is P squared/2m and V in principle it can be a time dependent electric field and so on. Then, it will be a problem, so here we are going to consider to be only dependent on x, maybe

you can also consider other situations where we can put a P dependent but for simple systems like hydrogen atom or harmonic oscillator, they are all time independent. So these are only position dependent potentials I have taken.

But you can take potentials which are functions of x and p even there this time evolution of a state, the state at t is related to t0 by this operator which is called as the time evolution operator.

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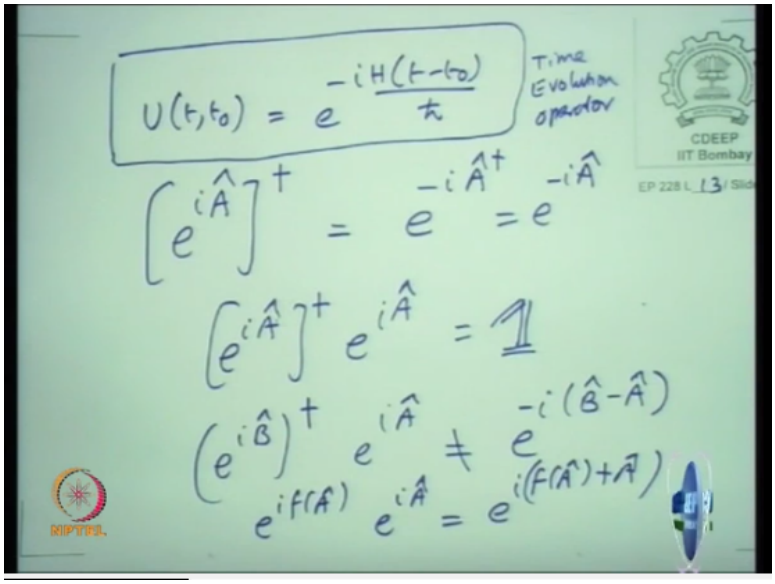

So the time evolution operator U of t, t0 is e to the –iH t-t0/h cross has this property that by the way any e to the power of iA hat where A hat is an observable, A hat corresponds to an observable. Once I say A hat corresponds to an observable, A hat is A hat dagger okay. If I take the dagger of this, it will give you e to the power of –iA dagger but A dagger is same as A right, so this is not Hermitian.

So exponential of i times Hermitian operator is not a Hermitian operator but it has this additional property that e to the iA dagger with e to the iA that is going to be identity but this you could do because the exponential power operator is the same as this operator. This also I said in the last class. Suppose I give you e to the iB hat dagger e to the iA hat, this is not equal to, you all agree, this is not equal to this in general.

It will be true if it is commuting operators okay. If B and A are commuting, then you can write it this way. So trivially I have taken here the same operators, so that is why this becomes identity. So I am slowly trying to stress the fact, you can have classes of operators

where some of the operators are commuting. There can be some of other operators which are not commuting.

If they are not commuting, you have to be careful when you take products of exponentials of operators but if they are commuting then you can directly add exponentials of the powers. In fact, you can write e to the if of A operator e to the iA operator, you can rewrite it as e to the if of A operator+A operator, this is allowed, so some playing around with these operators with the notion that the order matters is very important okay.

So the time evolution operator which I have written here, this is what is called as a time evolution operator for time independent Hamiltonians. This helps us in finding the state at any arbitrary time given the state at an initial time t0 and this operator has various properties.

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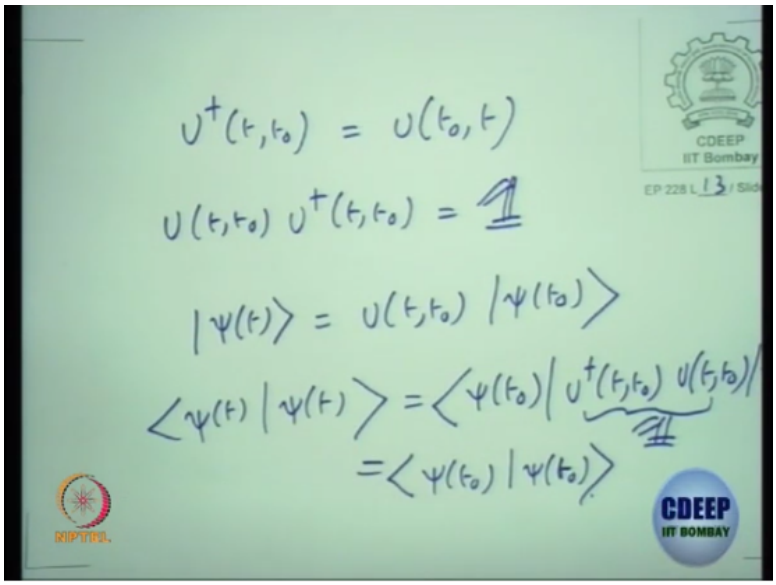

They are this operator is supposed to be unitary same as t0, t, you all agree. This is one property and you can also show U t, t0 U dagger t, t0 is identity. So you can write psi of t as U t, t0 psi of t0, you can write psi of t with psi of t to be psi of t0 U dagger t, t0 U of t, t0 psi of t and use the property that U dagger U is same as UU dagger which is identity okay. Once you use that then it becomes what happens this one is identity, so the norm under unitary evolution of any state is preserved right.

In the product of the state with itself is called square of the norm and it is same whether you work it out t or work it out a t0. So using this equation we also went into the position representations right, so how to make contact with your Schrodinger equation time dependent

Schrodinger equation which you have learnt in the wave function formalism, we went into that.

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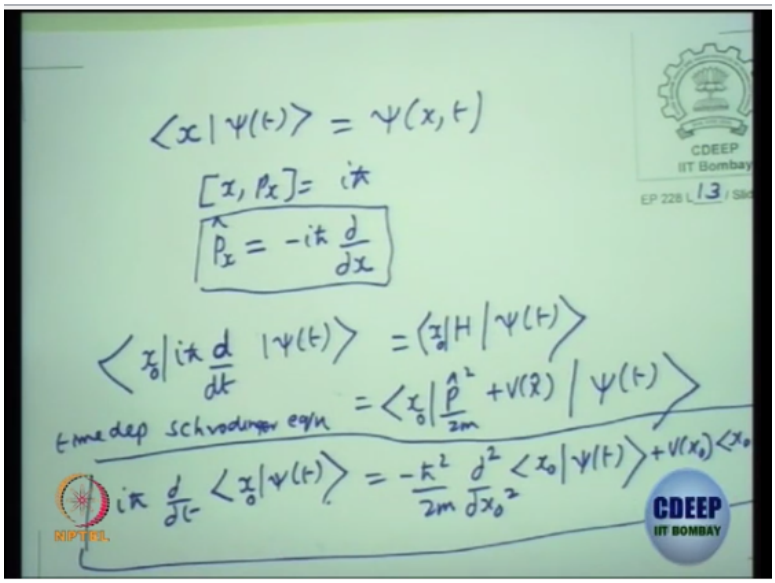

And we had psi of t with projection onto the position basis. This is what we called it as a psi of x, t and we also tried to use the commutator of x with px=ih cross to determine the representation for the Px operator in the position space as ih cross del/del x and after we did this, now we could write ih cross d/dt of psi of t you can take a position state here will be H x on psi of t.

H is P squared/2m+V of x you are taking time independent potentials, so let us write that as x with P squared/2m+V of x operator on psi of t okay. So you can do this V of x on psi of t or you can also operate V of x on x which will you prefer? Since x operator eigenstate is x, it is preferable to operate here and we also know what is the position representation for the momentum operator, so we can operate it on this okay.

So we can rewrite, so even though this d/dt is on psi of t and this is t independent we can write this piece as ih cross del/del t of x with psi of t, this is allowed and here it is - h cross squared/2m. If you are getting confused with this x at x0 maybe you can put an x0 here okay. That will probably clarify so you can put an x0 here some specific position, pick a point x0, so this will be del squared/del x0 squared on x0 with psi of t+V of x0.

Now it is all numbers so an eigenvalue with x0 with psi of t agreed and we have defined x0 with psi of t or x with psi of t a psi of xt, so what do we reproduce? So this one is your wave

function at a position x0 and this is the familiar differential equation which we studied in the wave function formalism which we have reproduced starting from the equation on the ket vector psi of t. Is that right?

So this is nothing but your time dependent Schrodinger equation, Schrodinger equation for the position space wave function. We do not say this as position space wave function because you will mechanically do wave function which is in the position space but this particular exercise you can say that this is a time dependent Schrodinger equation for position space.

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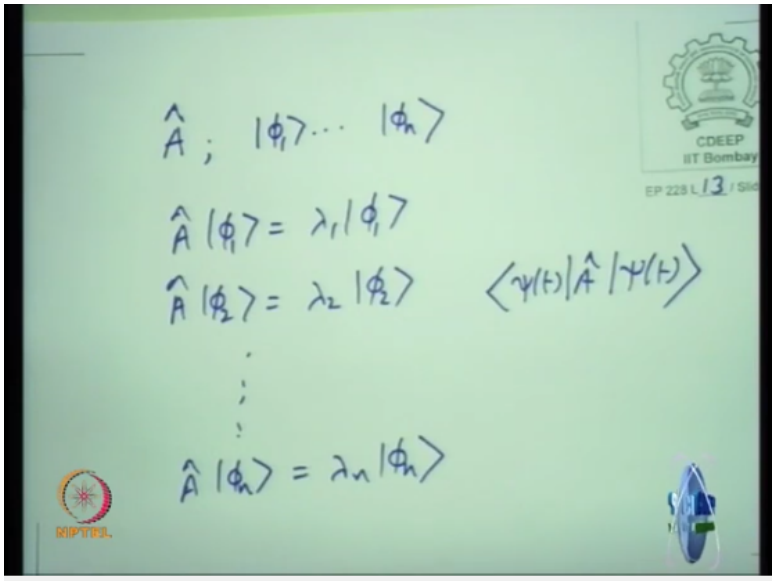

So whenever I give you an operator A, you can try and say that I would look at complete set of basis phi 1 till phi n and A on phi 1 if it is lambda 1 phi 1, A on phi 2 is lambda 2 phi 2 and so on. Then, what is this basis called? This is a complete set of basis. If it so happens that the A operator on this basis gives you eigenvalue equation, then we call this to be the eigenbasis of the A operator and the set of eigenvalues of this.

Typically, when I asked to find expectation value of A operator, suppose I ask you what is the expectation value of A operator in a state at time t, it is in some arbitrary state psi of t. Ideally, you should insert the complete set of basis identity operator which is an eigenbasis of the A operator, so that you can proceed. This is what we did in the earlier when we try to get the Schrodinger equation we went into the position basis.

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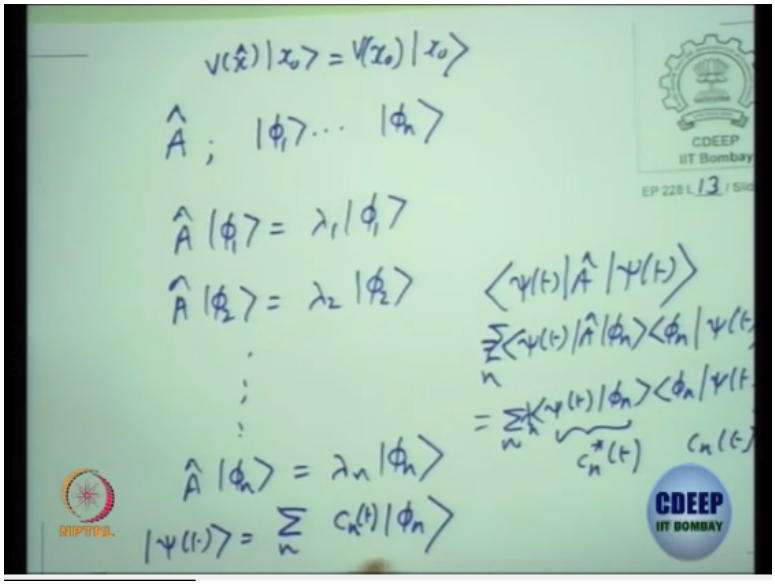

So V of x operator on a position state x0 is V of x0 on x0 so we went into a convenient position state basis to do our calculations. Suppose you have an abstract operator A, you try to go to the eigenbasis of that operator and things will simple. You could also work in some other basis where it is for a finite dimensional vector space where it is an off diagonal matrix, need not be a diagonal matrix.

But if you go to a diagonal matrix, things will get simplify, you can still find the eigenbasis of any operator and you can start inserting. So what I am trying to say is if I want to find this, I can write psi of t A operator phi n phi n psi of t and summation over n. So this is a complete set of states which I introduced okay. So then it simplifies to summation over n psi of t phi n and you can put a lambda n out.

A on phi n is lambda n and then you will have a phi n psi of t. Now can we remove this by summation over n phi n phi n as identity, not possible why? Because there is a lambda n eigenvalue but then this you know what it is. This one is the probability of the state psi to have a specific you know energy or A operator eigenvalue as lambda n that is what you can say, that is the coefficient.

And this will be the star of that coefficient. If suppose you write psi of t as summation over n Cn phi n where will you put the t dependence now? For every operator will you try to find the basis states, you do not put the t dependence, which one will carry the t dependence? Will Cn be independent of time or it will be dependent on time. In general, you want to make a time

dependent state and you write the basis as time independent basis and the coefficient will carry the time dependence.

So if you take this then you know what is this. What is this? Cn star of t and what about this, Cn of t. Is that right? From here you can write Cn of t as phi n with psi. Is this straightforward that you can see that there are these are the tricks you have to use when you do the computation, go to the eigenbasis of an operator and try to use make use of this eigenbasis. Most of the time we should make the calculation simpler means you can always go to one particular basis, which basis depends on what computation you are going to do.

If suppose I had a Hamiltonian where the potential energy was only dependent on momentum, then what will you do? Which basis is convenient to work? Momentum basis right.

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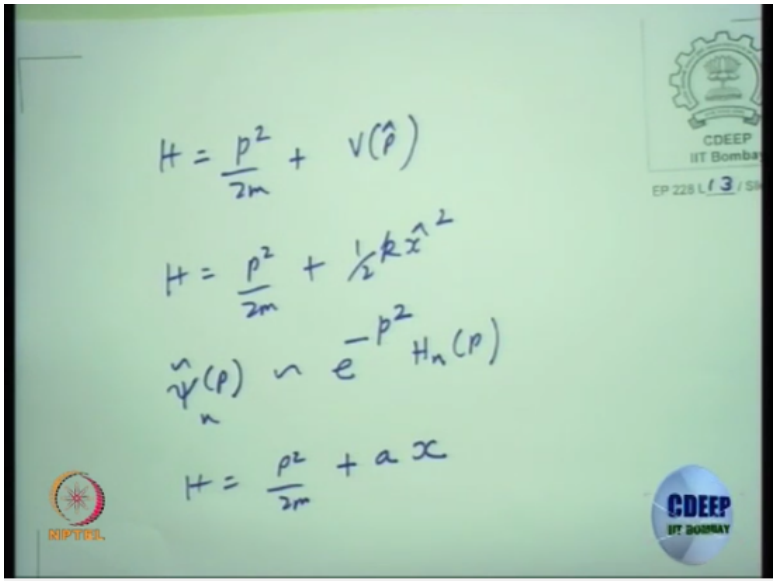

If suppose I had a Hamiltonian which is P squared/2m+V of p suppose, then you can work in the momentum basis. Equivalently, if you take harmonic oscillator, you have P squared/2m+1/2 kx squared. In the earlier case, we try to write the position space representation for the momentum but here both p and x are both quadratic. I could either work with the position space wave function or I could work with a momentum space wave function.

So what will be your expectation? It is symmetric in p and x. So if you try to write what is the psi tilde of p the form apart from these factors of k and m, the form will go like the same

form, which is the linear term or this one will go like this, times you will have Hn of p apart from those factors of m and k, which is required to make it appropriately dimensionless.

You expect the same thing because the Schrodinger equation where you had del squared/del x squared will have a del squared/del p squared if you go to the momentum space. So because of the symmetry between both being quadratic, you can either work in the position space or the momentum space but if I had given V of x which is some arbitrary function then that thing that trick does not work.

So you go to the position space and use this as d squared/dx squared. Here because of the symmetry between p and x, you can play around. Incidentally, this symmetry can also be like the p and x are the in classical mechanics of the phase space variables. So this is like some kind of a transformation of you can also do rotations in the phase space. If you put m=1 and k=1, it is like you want p squared+x squared to be a specific energy in classical mechanics and do rotation in the phase space.

So a lot of inbuilt, intrinsic symmetries like there was symmetries which I was talking about V of x=V of -x but you can start seeing that there is the px interchanging symmetry here. So maybe the functional form I do not even need to work it out in the momentum space. You can do inverse Fourier transform and Fourier transform to work out the momentum space wave function, this also you can do.

But looking at the harmonic oscillator Hamiltonian, maybe the k and m appropriately you have to choose and get the momentum space wave function, think over it. Some of these things you can have a thinking and we can discuss later. So incidentally this is also one of the interesting exercise where if you have this bouncing ball problem, I was giving you which is p squared/2m+you know a linear term in x some constant times x or constant times z and take the pz squared/2m.

This is a one-dimensional problem. Which basis is convenient here? It becomes a first-order differential equation if you go to momentum space and then you will be able to solve it and finding the momentum space here and then do that inverse Fourier transform to get the position space. So some of them you can exploit, which basis to choose depends on the situation and convenience.

And all the basis whichever basis you do and do the calculation, the calculation should be correct because another person who has it on a different basis and your answer you can relate it by a transformation which relates one basis to the other. That also we have seen in a two-dimensional linear vector space okay.