Quantum Mechanics
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Lecture - 17
Tutorial 3 - Part I

We have come to tutorial 3 and we will be discussing few problems on linear vector space and in previous 2 tutorials, tutorial 1 & 2, we have already seen few problems on particle in a box and there were several problems wherein you find out the energy, eigenvalue, wave function, etc. I hope you have already tried by yourself those problems, but if you have not tried please go back and try it at least once and it will be very helpful to you.

So now we will start with third tutorial, in which you have few problems based on linear vector space and let us start doing the first problem. So in the first problem and here there are.

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1. $|V_1\rangle$ and $|V_2\rangle$ are two normalized kets such that $\langle V_1|V_2\rangle = i/2$. Define

$$|X\rangle = 2|V_1\rangle + (2-3i)|V_2\rangle$$
$$|W\rangle = (3-2i)|V_1\rangle + 2|V_2\rangle$$

Find the norms of $|X\rangle$ and $|W\rangle$ and determine $\langle X|W\rangle$ and $\langle W|X\rangle$.

2. Consider three kets $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$ such that

$$\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 2;\ \langle e_3|e_3\rangle = 4;\ \langle e_1|e_2\rangle = i\sqrt{2};\ \langle e_1|e_3\rangle = 1+i;\ \langle e_2|e_3\rangle = 2$$

Use Gram-Schmidt procedure to construct an orthonormal set of three kets.

These are all more math oriented problem where you have to solve few steps and we can start with the first problem, which says that there are 2 normalized ket given to you and some definitions of these ket's in terms of a new ket is given. So we have to find out the norm of the 2 kets. So first is |X> and the other is |W>. So you have to determine the norm.

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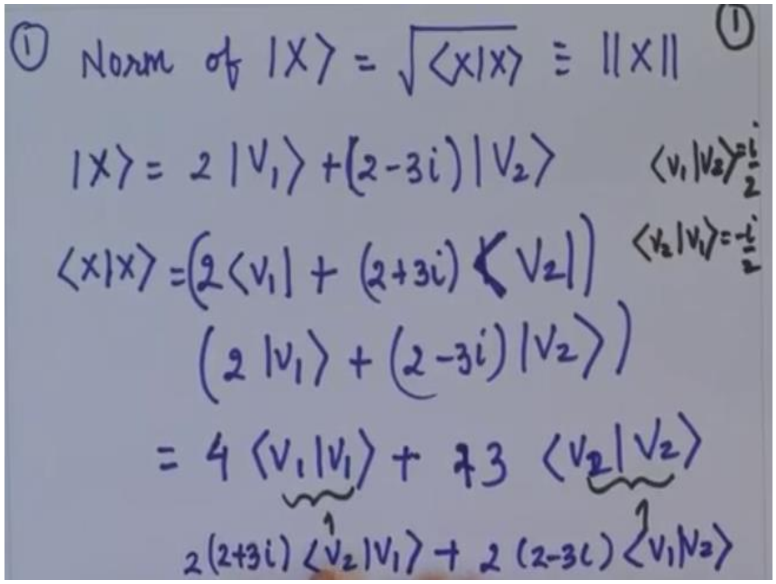

So we know that the norm, just a recap norm of a vector say |X> is given by $\sqrt{< X|X >}$, which is also represented as ||X|| okay. So we have to evaluate the norm of these vector |X> and |W> and we know that |X> in terms of the normalized eigenkets $|V_1 >$ and $|V_2 >$ is 2$|V_1 >$ + (2 - 3i) $|V_2 >$. So this is the first one in terms of $|V_1 >$ and $|V_2 >$. Now in order to evaluate this, we will have to start by doing <X|X>. So this would be just simple just one example to illustrate.

I will solve his completely the others I will not give in between steps. So all this is a |X>. Now you have to calculate the <X|. This is the <X|. So in bra ket notation, the norm is represented in this manner. So now ket will become bra. This normalization or this number will remain as it is and the complex number will be a complex conjugate, you have to write. So 2-3i will become 2+3i times this bra.

This is the first term and the second term would be 2$|V_1 >$ + (2 - 3i) $|V_2 >$, okay. Now it is very simple to evaluate this. This will give me 4$|V_1 >< V_1|$. Let me calculate this terms then we write the cross terms square term will give me. This will give me 2+3i * 2-3i will just give me 4 minus and minus plus, so 4+9 that is 13 $|V_2 >< V_2|$ okay and we know that these are normalized eigenkets. So the norm of this would be 1 okay and then the cross terms would be 2*2 + 3i this, this with this will give me $|V_2 >< V_1|$ + this on this will give me $|V_1 >< V_2|$ okay.

We are also given in the data that $|V_1 >< V_2|$ is nothing, but i/2 and so the complex conjugate of $|V_1 >$ and $|V_2 >$ will be this will be - i/2. So I will use this.

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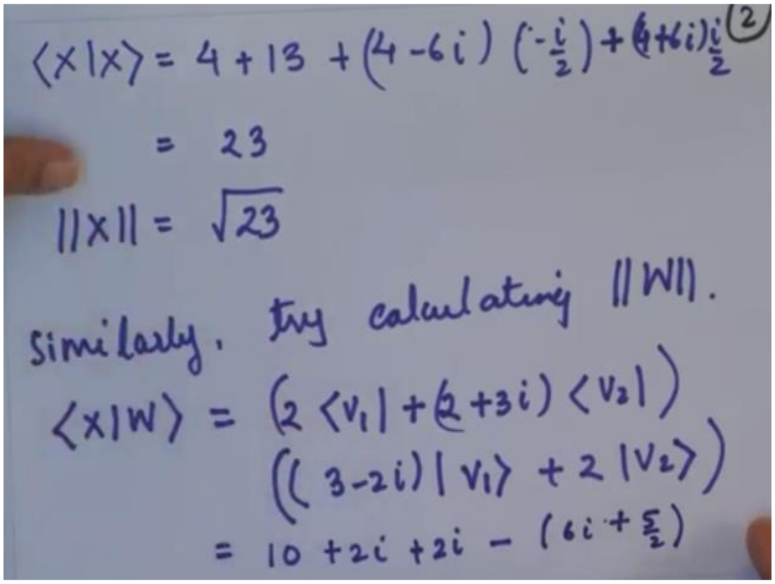

You have to calculate for <X|X> will be simply 4+13. So you have here, you have seen here, you had 4 from here, 13 from here and then you have the other terms that is 4-6i is one term from here* -i/2+, we have 4+6i*i/2. So this would simply give me 23. You add up all this and you obtain the ||X|| is square of 23 okay. So this will give me this result okay. So we have calculated the norm of ||X||, which will be now square root of 23.

This is what we obtained by calculating ||X||. Similarly try calculating ||W||. In a similar manner, you can just calculate a ||W||. The next part of the question is evaluate norm of <X|W>. So <X|W> same way we will take the X term which is $2|V_1 >$ + (2 - 3i) $|V_2 >$ on, we have W as $2|V_2 >$ + (2 - 3i) $|V_1 >$ okay. So this can also be easily evaluated. You have $< V_1|V_1 >$ will give me 1. This will be the term. We have to multiply this coefficient 2*3 - 2i.

Here I have 2+3i*2 and $< V_2\ |V_2 >$ So this will give me, this will give me what? This will give me 6 and 4, 10 and this will give -4 and +6, so +2i and the cross terms would give me, I just have to evaluate these. So this will give me 4i/2 that is -2i okay. $< V_1|V_2 >$ is +2i and remaining term same way I will have 6- I will have to evaluate 6- this will be +6 okay. So this will give me -6i+5/2 on simplification I obtain this.

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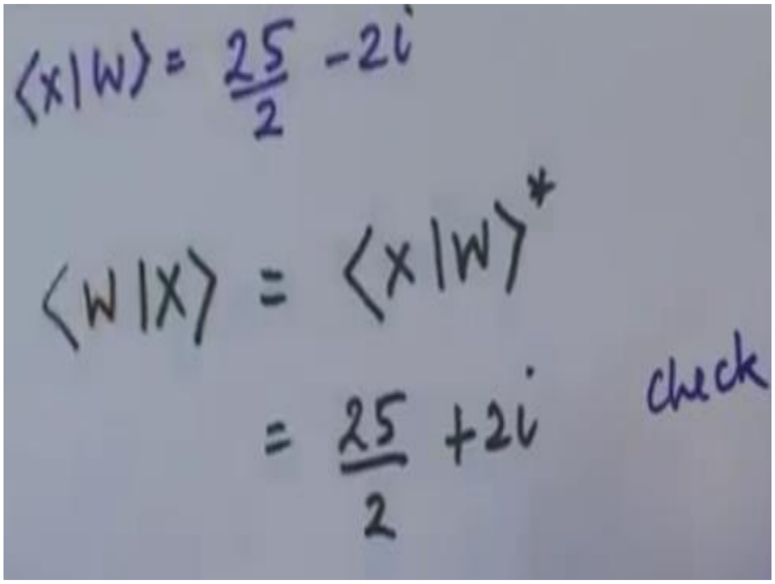

When you simplify this further, what you obtain is <X|W> comes out to be 25/2-2i okay. I was just simply adding the real and the imaginary part. Now in the second part you are asked to find out okay this would be nothing but the complex conjugate of the one which you have calculated at <X|W>. So this can be simply written this way. So it will be a good exercise to check this okay. So just check this and find out whether you obtain same answer.

Now in the next problem what we have here is that you are given 3 eigenkets okay, you are given 3 eigenkets. They are e_1, e_2, and e_3 okay, 3 eigenkets are given to you, such that you are given the relations on of each eigenket with each other with respectively and using the Gram-Schmidt procedure, you have to construct an orthonormal basis of these eigenket. So in order to calculate the orthonormal sets of these 3 kets, what one does is used Gram-Schmidt, which will project out each component in terms of the orthonormal vectors.

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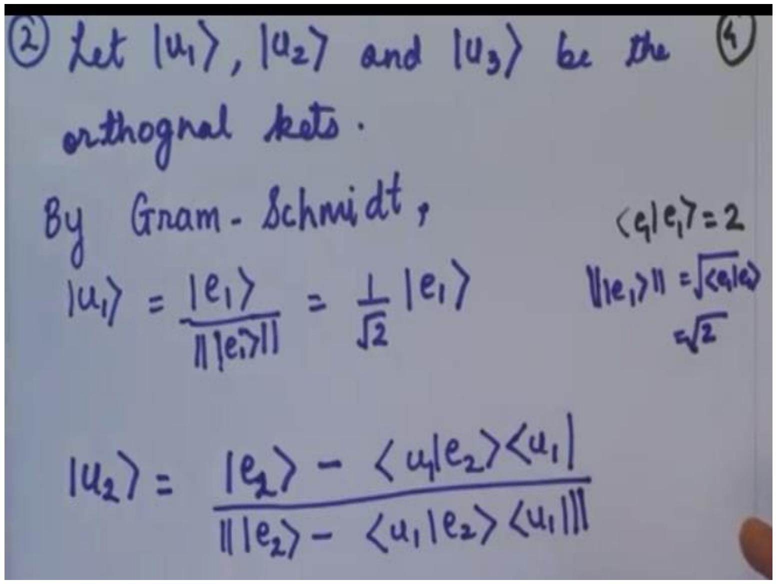

So now to start with we have the first we start writing let, so this is problem number 2. Let $|u_1>$, $|u_2>$, and $|u_3>$ be the orthogonal kets. You have to obtain u1 in terms of these kets $|e_1>$, $|e_2>$, $|e_3>$ you do in terms of these kets $|e_1>$, $|e_2>$, $|e_3>$ and similarly for $|u_3>$. So we start doing by Gram-Schmidt. So by Gram-Schmidt procedure, what one does is now you start with u1 and again just a remark that you can start with any of these I mean $|u_1>$, $|u_2>$, $|u_3>$ is just a representation of 3 orthogonal set of eigenket.

So you can start by representing $|u_1>$ in terms of $|e_1>$ or $|e_2>$ or $|e_3>$ first, it does not matter. It will give you a different sets of eigenket. So I start $|u_1>$ by writing $|e_1>$ as norm of $|||e_1>||$, okay. So we have calculated the norm, we have see, we have seen already what is the norm of a vector okay. So now I will, I have started with u1 in terms of the eigenket $|e_1>$. So this will be simply. Now we have in, we have given that $< e_1|e_1 >$ is nothing but 2 okay.

So we have to calculate the norm of e1 okay. So this will be nothing but square root of $< e_1|e_1 >$, which is nothing but √2, okay. So we will have this as 1/√2 $|e_1>$ okay. So we have obtained $|u_1>$ in terms of $|e_1>$. This is one set, one of the elements of the set of the 3 eigenket. Similarly, u2 in order to obtain $|u_2>$ you will write u_2 in terms of $|e_1>$, $|e_2>$ - you project out in $|e_1>$ in terms of u1, so you will have $< u_1|e_2 ><u_1|$ you use a projection operator of $|u_1>$ to project out the $|e_2>$ component and you will have to divide this by the norm of $|e_2> -< u_1|e_2 ><u_1|$ okay.

Now it is very easy to calculate this. Now once you know $|u_1>$ in terms of or rather you can write $|u_1>$ here in terms of $|e_1>$ and then you are given the relation between the 3 eigenket, when you operate one on the other. So what you can do is first let us see what will be.

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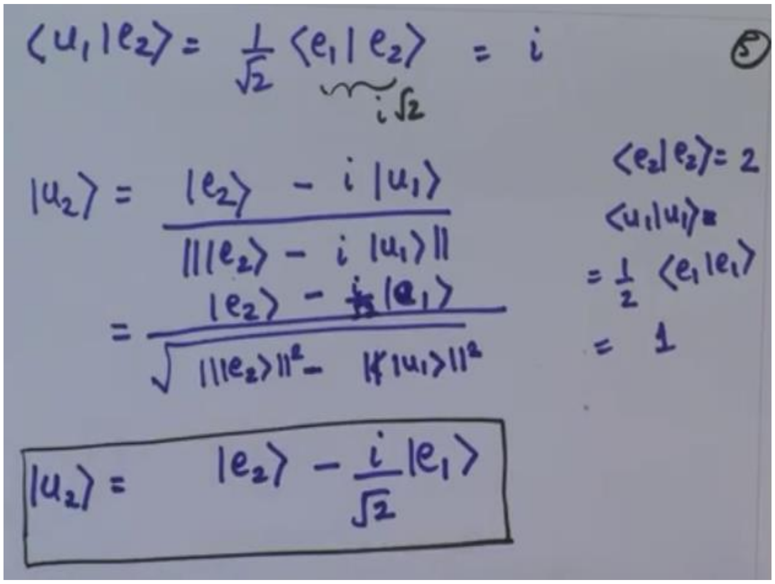

What will be $< u_1|e_2 >$. So u1 as we have just now seen is 1/√2 $< e_1|e_2 >$. Since we have real component, we can write this directly. The complex conjugate of <e|e> will be <e| with this normalization factor and you have you are given that this $< e_1|e_2 >$ is i/√2. So this will be nothing but i okay. So again this u, I can write it as $|u_2 >$, I can write it as $|e_2 >$- i $|u_1 >$. So I will just substitute for $|u_1 >= |e_1 >$. I am sorry this has to be a ket.

So I will express this as because I am writing this is a projection operator. So $< u_1|e_2 >$ is we have calculated it to be i. So it will be i*$|u_1 >$ okay divided by I will calculate term, norm of this quantity which I wrote just now as $|e_2 >$-i $|u_1 >$, okay. Now it is very simple to evaluate u1 we know. So now $< e_2|e_2 >$ we know, the bra and ket of $|e_2 >$ is 2 and $< u_1|u_1 >$ is simply is simply 1/2. What we have here in terms of $|u_1 >$ when we write, it is $< e_1|e_1 >$, which is nothing but 1, okay.

So this will be 2-1 because when I take the norm, it will be$\sqrt{\langle e_1|e_1\rangle}$, okay. Let me write here this will be nothing but $\sqrt{\||e_1 >\|^2 - \||u_1 >\|^2}$. So how will I write $\||u_1 >\|^2$ and in the numerator I have $|e_2>$-$|u_1>$ will be i/√2 $|e_1>$. So the final answer for u2, I can simplify it to $|e_2>$-i/√2 $|e_1>$, because this is 2-1, so you have 1. So this is what I get for, I can get $|u_2>$.

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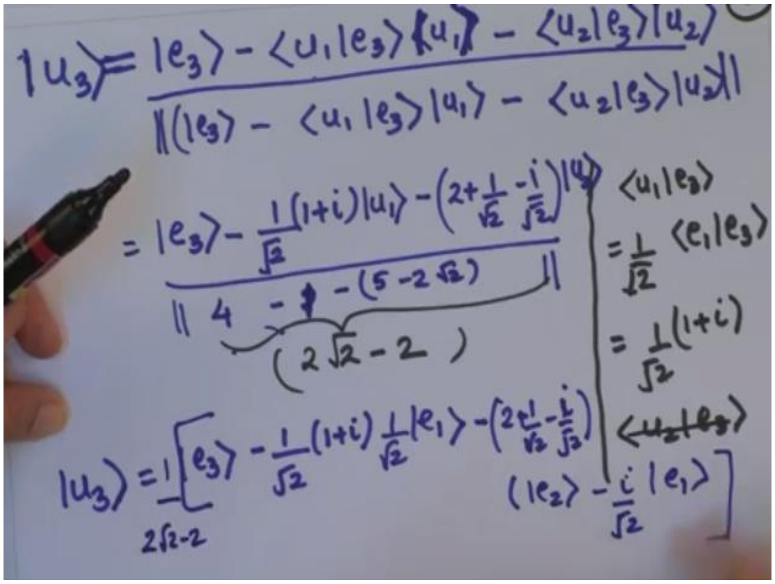

And ket 1 was this $|u_3>$ is nothing but $|e_3>$ then the projection of $<u_1|e_3>|u_1>$ and again I wrote it wrong okay minus you have projection of $<u_2|e_3>|u_2>$, okay and the norm of this entire thing that is $|e_3> -<u_1|e_3>|u_1> -<u_2|e_3>|u_2>$, okay. Now again we have to, what we have to do here is first we will calculate what is $<u_1|e_3>$, okay and $<u_2|e_3>$. So just recall we have this $|u_2>$. So first we will do for u1. So $<u_1|e_3>$ will be nothing but $|u_1>$ is 1/√2 $<e_1|e_3>$.

And what we have $<e_1|e_3>$ as1/√2 (1+i) okay and now what we do here is we substitute for $<u_1|e_3>$, $<u_2|e_3>$ will be nothing but what was my $|u_2>$, $|u_2>$ was this is my $|u_2>$, sorry times $|e_3>$. So this $<e_2|e_3>$, we have it as 2. So 2-i/ √2 and $<e_1|e_3>$, is 1+i okay. So that is nothing but I can write this as this will be –i, so this will be +i, so if 2+1/√2-i/√2. So this is my expression for $<u_2|e_3>$ okay. So this I will substitute in this expression.

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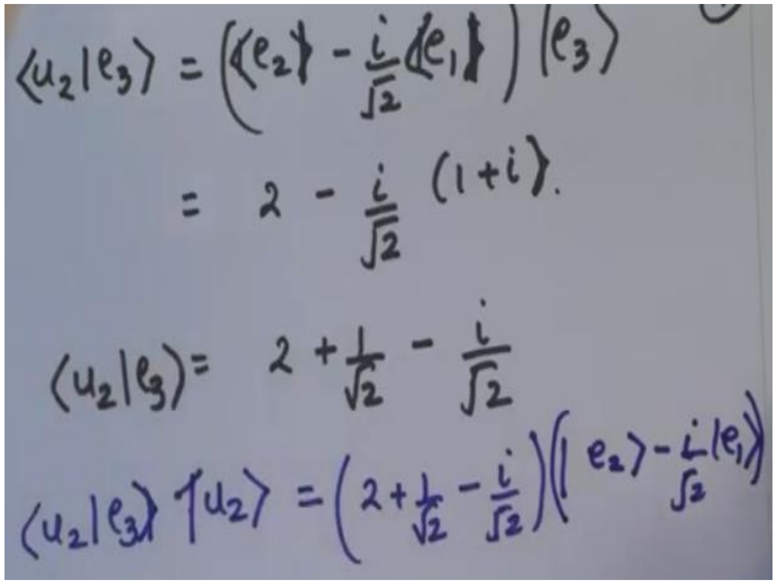

So let us write this. So this will be, I have now and again $< u_2|e_3 >$, we have calculated and $< u_2|e_3 >$ with let me highlight this again $< u_2|e_3 >$ and $|u_2 >$, let me explicitly write this here as this will be the expression 2+1/√2-i/ √2. This will the expression on $|u_2 >$, $|u_2 >$ was nothing but $|e_2 >$-i/√2 $|e_1 >$ okay. So we can keep it as it is here or we can just substitute it in the end okay. Mostly it will be good to substitute in the end, because it is a lengthy expression. So now let me come back here.

So here what we did was $< e_3|u_1 >$ so this gave us 1/√2 1+i okay and on $|u_1 >$- $< u_2|e_3 >$ this is the expression I have 2+1/√2-i/√2*$|u_2 >$ okay. So this is the numerator. Denominator will be norm of this quantity okay and now other, on further simplification when you simplify this further and find out the norm, what you obtain is this norm comes out to be, this comes out to be let me write you have to explicitly solve this. This will come out to be 2 √2-2.

So this is this norm okay. You have to calculate $< e_3|e_3 >$, which is nothing but 4 correct. This will simply give me -2 or 2, 1, okay. This will give me 1 okay, because I have calculated u1 on this. This is 1+i/√2 square this and you operate again. So what you obtain is just a factor of 1 and there is a minus sign. So you will have a factor of 1, then the square of this term, which will result in -5-2 √2 okay. Then again you further simplify and obtain the denominator.

So the final expression for $|u_3>$ will be $|e_3>$-1/√2 1+i, okay. I can at this point I can substitute for u1 and what was my u1, this is my $|u_2>$, use my $|u_1>$, $|u_1>$ was 1/√2 $|e_1>$. So I have 1/√2 e_1-, I have this factor 2+1/√2-i/√2 times this factor $|u_2>$ is this big expression, which I showed you just a minute ago and this expression okay. That is $|e_2>$-i $|e_1>$/√2. This entire thing multiplied by this normalization factor, which is 1/2 √2-2.

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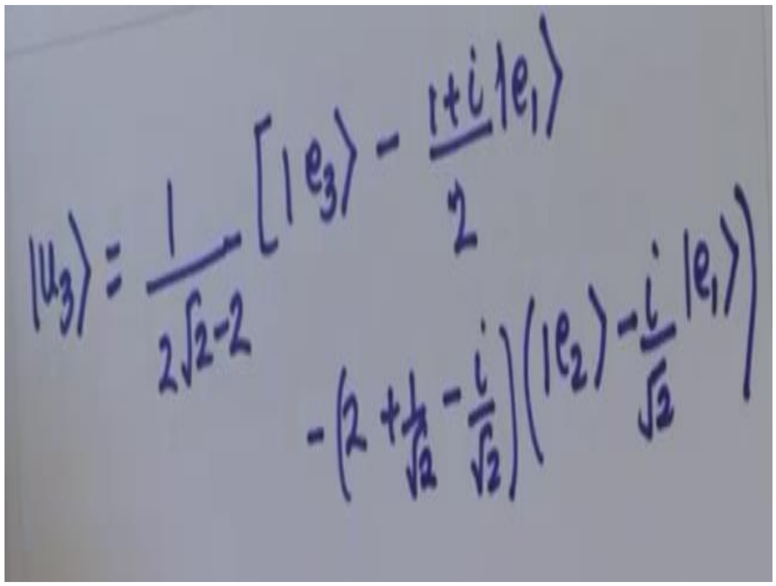

So let me rewrite this expression completely $|u_3>$ is, $|u_3>$ is 1/2 √2-2 times, you will have a factor of $|e_3>$-(1+i)/2$|e_1>$. So we are doing the simplification of this previous step. Here I will get 2. So I have 2, okay on e1. So these are the first 2 terms and the second term and we can keep it as it is, no need to simplify further. So the last term will be, I will write here – 2+1/√2 –i/√2, you can see this okay times $|e_2>$-(i)/√2$|e_1>$.

So these are the 3 orthonormal sets expressed in terms of the eigenket $|e_1>$, $|e_2>$ and $|e_3>$. So let me put everything together. So this is my $|e_1>$, this is $|e_2>$ and this is $|e_3>$, okay. So when you, so you obtain $|u_1>$ in terms of $|e_1>$, $|u_2>$ in terms of $|e_2>$ and $|e_1>$ and $|u_3>$ in terms of this. So you can check whether these are orthonormal sets of 3 eigen, of these 3 eigenkets. Similar problems we will do in the next part of this tutorial, a similar exercise.

So it will be good if you try these problems by yourself, you get hands-on on these problems. They are very simple, they can be solved, they you try to do them in proper steps without skipping any steps initially. Later when you have hands-on on these problems, you can just skip steps wherever you find you can do things mentally. We will see in the next class the remaining tutorial.