Quantum Mechanics

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Lecture - 14
Linear Vector Space (LVS) – III

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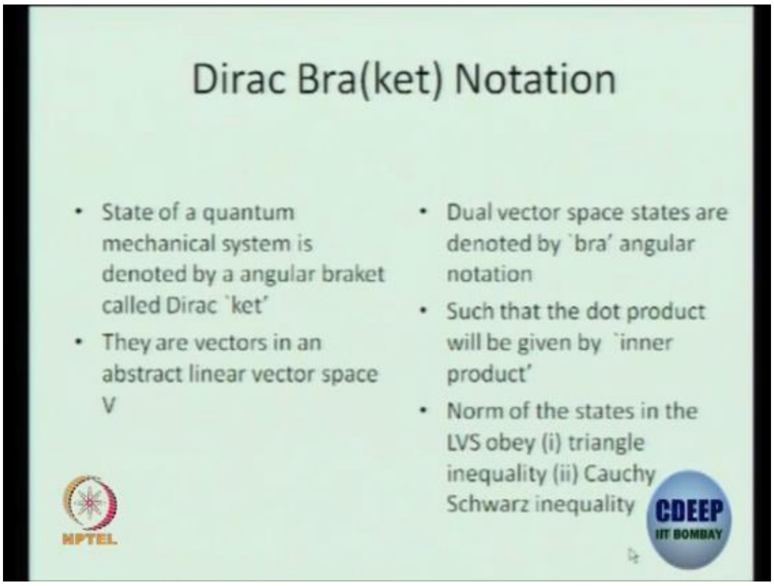

Today now what I will confine is to prove this triangle inequality which you are familiar in linear vector space language and Cauchy-Schwarz inequality. So what is Cauchy-Schwarz inequality and the triangle inequality in this language?

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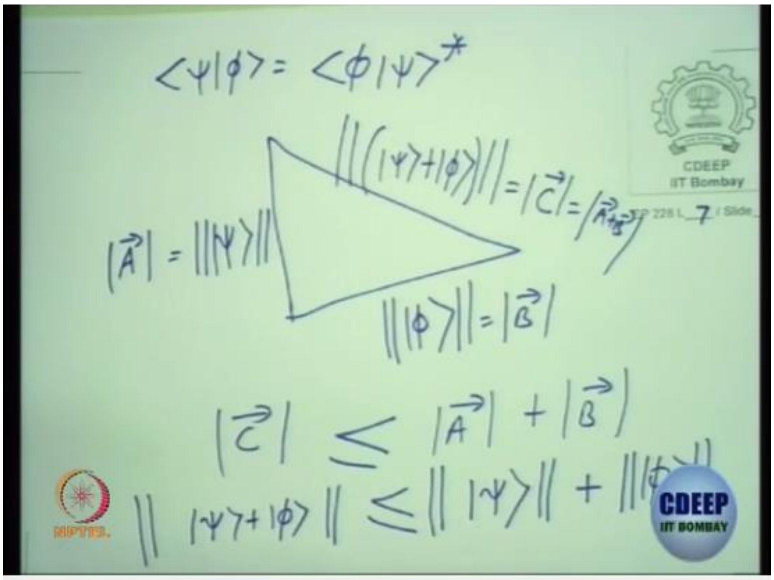

So if you take a norm of $|||\psi\rangle+|\phi\rangle||$, you can write a triangle, you can call one vector as norm of $|||\psi\rangle||$ and norm of $|||\phi||$, it is a length of this side of a triangle. Then what will this be? If this is a ψ vector and this is a ϕ vector, you need to take $\| |\psi\rangle + |\phi\rangle \|$ and then take the norm so correct. Yes, everybody is getting used to the notation. If I write $\vec{A}$ vector, $\vec{B}$ vector all of you will be familiar right.

If suppose I call this as $|\vec{A}|$ vector, this as $|\vec{B}|$ vector, so I call this as $|\vec{C}|$ vector which is nothing but $|\vec{A}+\vec{B}|$ vector. What is the triangle inequality? $|\vec{C}| \leq |\vec{A}| + |\vec{B}|$. I am just mapping it, whatever I have written here is also true for an abstract notation in any dimensional vector space, could be complex or real vector space, it does not really matter.

But this property will be satisfied, what is the property? If I take $\| |\psi > + |\Phi > \| \leq \| |\psi > \| + \| |\phi > \|$ right. This is what we call it as a triangle inequality. So I want you to try and prove this where you know how to write this norm as an inner product and you have to use the property that $\langle \psi | \phi >$ is $\langle \phi | \psi \rangle^*$ right. We have these properties that $\langle \psi | \phi >$ is $\langle \phi | \psi \rangle^*$.

Then, you need to prove this. This was the proof at some point either in the tutorial or in the session but before we do it get used to this Dirac braket notation and try and prove the triangle inequality.

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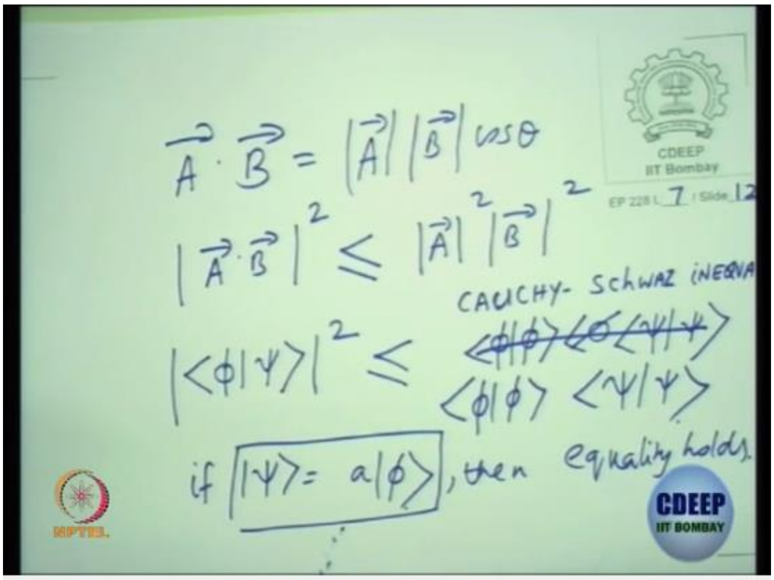

So the next one is $\vec{A} \cdot \vec{B}$ is formerly you would have written it as $\|\vec{A}\| \|\vec{B}\| \cos\theta$. So $\vec{A} \cdot \vec{B}$ is generally less than the magnitude will be less than norm A and B because cos theta. What is the values of cos theta? The range is from -1 to +1. So if you take the positive magnitude, then the left hand side should be less than or equal to.

The same thing for the inner product, we need to replace the dot product by an inner product for an abstract vector which finds a quantum mechanical system and prove this property okay. How do I write this? I could write this $\vec{A} \cdot \vec{B}$ as I said, you would have written it as $\langle\psi|\phi >$ but then I will take mod squared, that is invariant under ψ ϕ interchange right. $\langle\phi|\psi>$ is $\langle\psi|\phi\rangle^*$ but mod squared of $\langle\psi|\phi\rangle$ is the same and this will be $\leq \|\langle \phi |\phi\rangle\|^2 \, \|\langle \psi |\psi\rangle\|^2$.

So I am just squaring both sides okay what I am doing what else is not right $\langle \psi |\psi\rangle$. Is that right? Maybe let me write it again. It is $\langle \phi |\phi\rangle\langle \psi |\psi\rangle$. So this is the Cauchy-Schwarz inequality. When will it become equal? if $|\psi >$ is a times $|\phi>$, then equality holds right. What is this condition? If $|\psi>$ is some linear multiple of $|\phi>$, then $|\psi>$ and $|\phi>$ are linearly dependent or in the language of your vectors, they will all be collinear, they will all be in the same line or if you do not need a different, it is a same, it is linearly dependent.

For such linearly dependent situation, the equality will hold and in general the inequality will be satisfied, so this is what is called as the Cauchy-Schwarz. So in principle you know how to prove this. I want you to redo the same thing in the context of this language notation of the ket and bra with set of properties which an inner product satisfy and you should be able to get some results on this.

I will be following this notation from the next lectures and so on, so if you do not know this notation, you will be completely lost, so please play your hands on writing down what these notations are.

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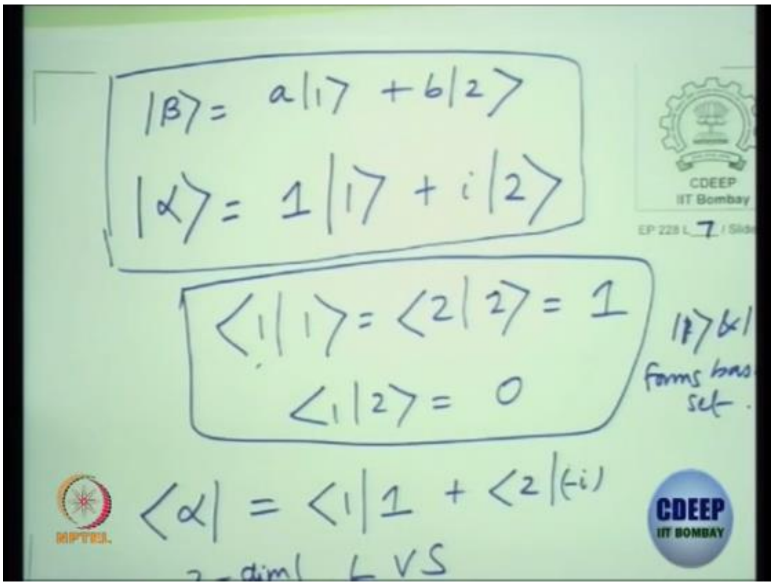

For example, if suppose I give you a vector, suppose I give you a state let say $|\alpha >= 1|1 > +i|2 >$ and I also say that $\langle 1|1\rangle\ and\ \langle 2|2\rangle$, this is just a notation, you can use Greek letters or any arbitrary, so this is 1, I give $\langle 1|2\rangle$ is 0. What is the dual vector to the $|\alpha >$ vector? So $\langle\alpha|$ what is it? So you can write this as $< 1| + (-i) < 2|$ with the coefficient 1 and then 2 with a coefficient i star which is –i okay.

So when we want to take the inner product of $|\alpha >$ with $\langle\alpha|$, you first have to convert the coefficients to complex coefficients and then we can proceed with this to find the duel vectors okay. So this also brings in me a condition of what is linear independence and span of the vector space even though I have given it here for some arbitrary vectors in terms of this. What is the span looks like from this vector?

There are two vectors which is required to define this α. So the span set should definitely include the $|1 >$ and $|2 >$ but an arbitrary β if I want to write, if it still involves $|1 >$ and $|2 >$ for any arbitrary vectors would be some a $|1 >$+b $|2 >$. This kind of tells me so |α> and |β>, |α> is some specific state, |β> is some arbitrary state, any arbitrary state I am able to write it in terms of $|1 >$ and $|2 >$ linear independence on spanning the space since any arbitrary |β> is in terms of $|1 >$ and $|2 >$, it spans the space.

And we have also written a condition that $\langle 1|1\rangle$, $\langle 2|2\rangle$ is identity and this $\langle 1|2\rangle$ this is 0 not identity. So what is this? What is this condition? It is a basis set is |1> and |2> forms basis set, you all agree? |1> and |2> forms basis set and they are linearly independent. They are in fact orthogonal also, is orthonormal. These dot product is these basis dot product is 1.

The dot product of these two bases is 0 which is your orthonormal basis and since it spans, any arbitrary state can be spanned by using these two states and it is satisfying this property linear independence and spanning the space, so this |1> and |2> forms the basis set. So what is the dimensionality of the vector space? 2, dimensionality of the vector space is the number of states in the basis set.

So this will define for you a two-dimensional linear vector space, it could be complex because a and b in principle could be complex coefficients. Is this clear from this example? That whenever you have to write the dual vector for any ket, you need to make sure that the coefficients, any scalar multiplying a ket if I ask you what is the dual vector.

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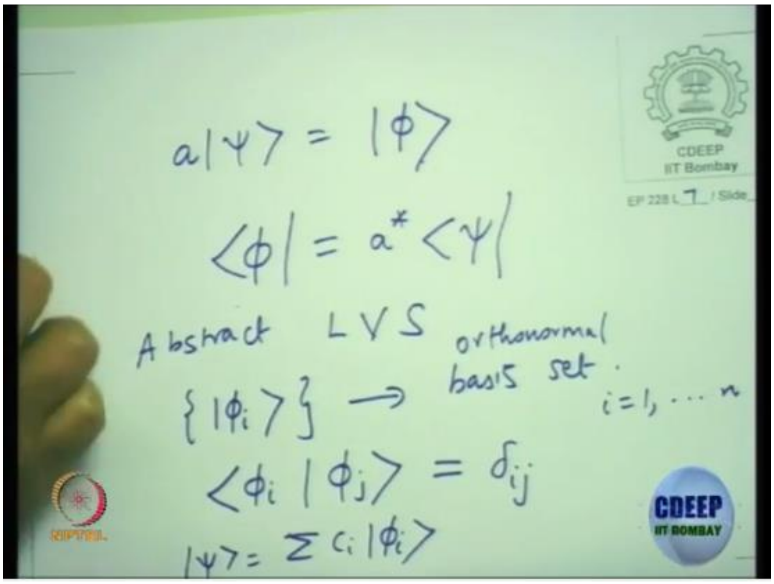

If you have a vector |ψ> and I multiply a scalar and call this vector as some |ϕ>, what is this one? This one will be a^* <ψ|, this a star you can put it this side or that side, it is just a number okay. Many of them make this mistake, mechanically when they write the dual vector they do not complex conjugate the coefficients. Please remember this and it plays a very crucial role.

Sometimes observable quantity which you will evaluate will become complex, you do not want that to happen. So when you go to a dual vector space in real spaces it does not make difference, so that is why when you did your three-dimensional R_3, there was no problem but if you are doing complex vector spaces which is what happens in all your quantum mechanics problems so I have taken a simple example of a two-dimensional vector space.

And two-dimensional vector space also naturally appear in quantum mechanical systems okay, so you have a two-dimensional vector space and for such cases when I have to find the dual vector, I have to make sure that the coefficients which multiply it has to be complex conjugated and whichever is this ket I have write it as a bra.

So formally we will write in an abstract vector space, abstract linear vector space we will say that we have a set which is a basis set okay with the property that $\langle \Phi_i | \Phi_j \rangle$will be δ_{ij} to make it orthonormal basis set. So by basis set now you should know it spans the whole space and they are linearly independent and once I say it is orthonormal then it satisfies this condition.

I have not said what is the i running from, so i could take values from 1 till n where n could be arbitrary, need not be 3 or 2 as I was discussing and n can also run to infinity. You have infinite dimensional vector space. If n is finite, then it is an n-dimensional vector space, can have infinite dimensional vector space. Here we have taken it to be a discretized index that could also be a continuum. Here so many possibilities are there.

So you can have all types of vector spaces but this property has to be satisfied and this basis set means any arbitrary state, you will be able to write it in terms of summation over $C_i|\phi_i >$, that is the meaning of saying that it spans the space, you can write any arbitrary state in terms of that and these two conditions will be sufficient to look at any abstract space okay.

So then this brings in for me one small thing which I heard from one of you in the beginning of the lecture that you have done Gram-Schmidt orthogonalization right. So let us just go over the Gram-Schmidt orthogonalization in this notation. So this basis states which we have is orthonormal set but generally you can be given as I gave you in the beginning. I gave you a simple example which was a non-orthonormal but still a basis set right.

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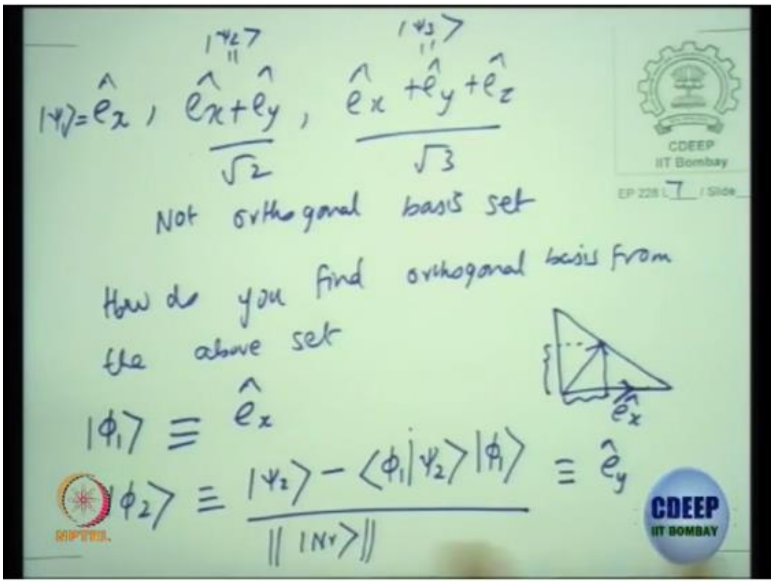

I had an $\hat{e}_x$, $\hat{e}_x + \hat{e}_y$ and then $\hat{e}_x$+$\hat{e}_y$+$\hat{e}_z$, you could normalize them. This is a basis set in three-dimensional real space which spans the space but it is non-orthogonal and how do you go about finding the orthogonal set is by the so this is not orthogonal basis. How do you find orthogonal basis from the above set? This is you are familiar, you first define your first state $|\phi_1>$ to be equivalent to $\hat{e}_x$ and then how do you find $|\phi_2 >$?

What do you do? You take the second state and then subtract out. What do you do in let us take it in this okay? You have this state, the first basis I have taken it to be $\hat{e}_x$, the second one is making an angle 45 degrees with the first one but I want to make it orthogonal. How do I do it orthogonal? Take the projections of that vector in this direction, subtract it out and what you get will be this piece.

So let us write this in this notation $|\phi_2 >$. So suppose I call this to be $|\psi_1 >$, call this to be $|\psi_2 >$, this to be $|\psi_3 >$. $|\phi_2 >$ will be ψ_2- projection you have to do. So you will do with $|\psi_2 >$ and take it along $|\phi_1 >$. Will this be normalized? May not be. Whatever is the numerator I have to take the norm of it to make it to call this like a unit vector right, so this whatever you find here divide it by norm of the numerator state, not writing it.

But you understand what I mean right, everybody understands? Let us do this for this case. $|\phi_2 >$ is $\hat{e}_x$+$\hat{e}_y$/√2. You take a dot product with $\hat{e}_x$ right and then this is $\hat{e}_x$, dot product with $\hat{e}_x$ will give you a 1/√2 and then this is $\hat{e}_x$. ($\hat{e}_x$+$\hat{e}_y$/√2)-$\hat{e}_x$/√2 will become $\hat{e}_x$/√2. So $\hat{e}_y$/√2 the norm is again 1/√2. So what will you get here? $\hat{e}_y$ right. So this is what is the Gram-Schmidt orthogonalization written in the ket Dirac braket notation.

And there is something which you can see here, how do you do $|\phi_3>$? You need to take, subtract the component along $|\phi_1>$, component along $|\phi_2>$ and then normalize that numerator and that will give you the $|\phi_3>$. So let us write it.

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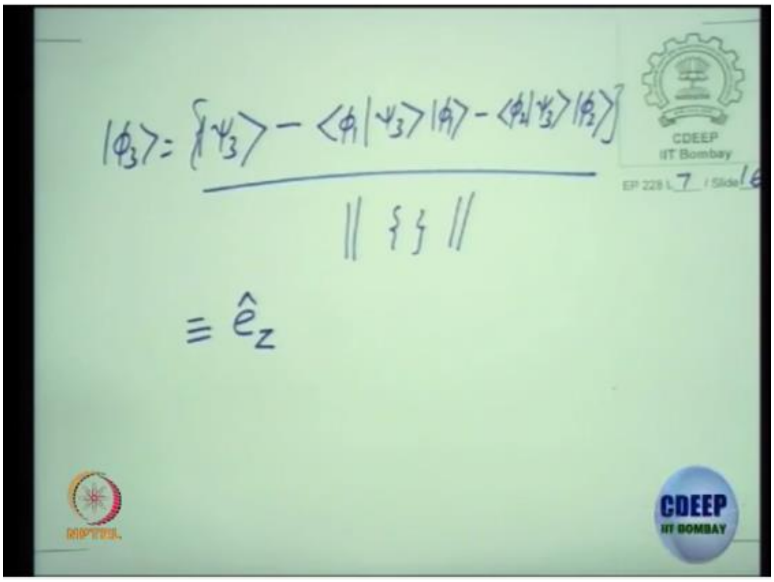

So take $|\phi_3>$ to be $|\psi_3>$-$<\phi_1|\psi_3>$ $|\phi_1>$- $<\phi_2|\psi_3>|\phi_2>$ and then you have to whatever is this numerator that has to be taken the norm so that will give you and this will turn out to be in your specific example as $\hat{e}_z$. So we could do this Gram-Schmidt orthogonalization to find all the 3 basis vectors okay. So we will start continue from here in the next lecture, slowly introduce operators, linear operators and so on and then we will see how to elegantly follow this notation.