Quantum Mechanics
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Lecture - 13
Linear Vector Space (LVS) – II

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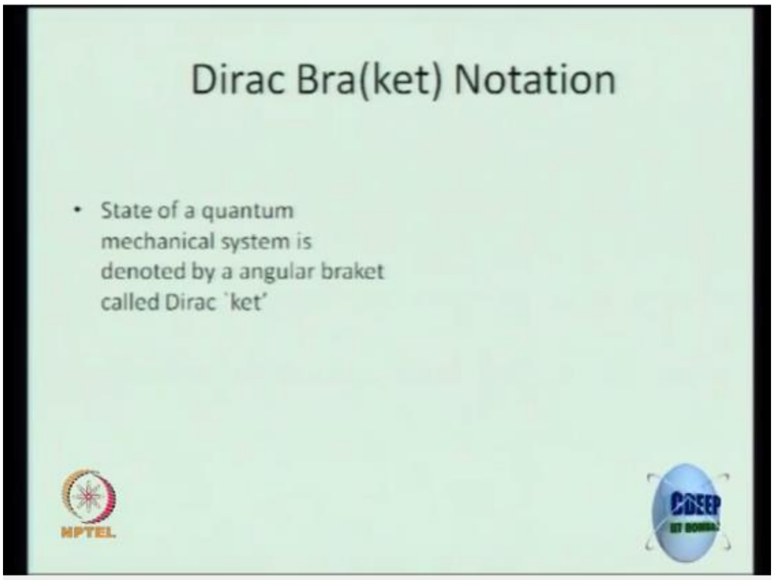

Okay so I am going to bring you to this notation of Dirac introduced it in 1920s, a notation which is some kind of an angular braket, it is called bra ket notation which we are going to follow okay. State of a quantum mechanical system is denoted by this kind of an angular braket which is called the Dirac ket okay. So let me try to introduce this for you.

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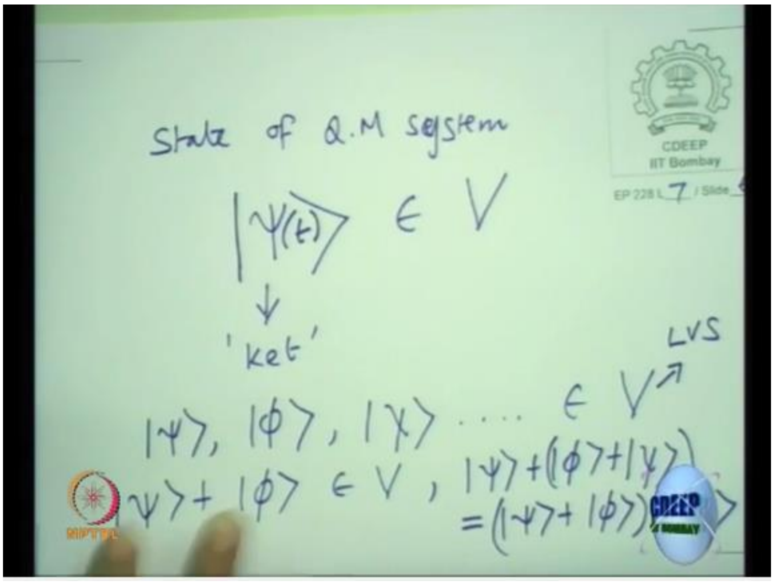

So state of quantum mechanical system, we denote this by an angular braket with a Greek letter $|\psi\rangle$, can be $|\psi\rangle$, $|\phi\rangle$ is a kind of notation which we are going to use and in principle the state of the system contains all the information about the system and it can be evolving as a function of time okay. The state is denoted. This $|\psi\rangle$ is an element of some vector space V okay. So this is the notations we are going to use, not the vector $\vec{A}$, vector $\vec{B}$ and so on.

We are going to use an abstract notation which is an angular braket and this angular braket is known as ket. So you can write again in this notation. Suppose the vector space if you have $|\psi\rangle$, $|\phi\rangle$ and suppress the time right now, $|\chi\rangle$ dot which are elements of a vector space and what do you have to do? You have to satisfy those familiar properties which you are studying. What are those properties?

$|\psi\rangle+|\phi\rangle$ should be an element of V what else $(|\psi\rangle+|\phi\rangle)+|\chi\rangle$, you can also satisfy this associativity property same as $|\psi\rangle+(|\phi\rangle+|\chi\rangle)$ exactly like what we did for the vectors in R_3 that there could be a long list here. So long list which I have for an abstract vector space, I do not know what is the dimensionality of this vector space, I do not know the linearly dependent, the set which spans the space.

Now I am putting it in this abstract notation. Is that clear? I really do not know what they are and our aim is to fix all these things given a quantum mechanical system, what is the basis set, how to write the states in terms of the basis set and also we need to give a prescription or an equation to determine just like a Newton's law. All this Newton's law what does it tell us? It tells you the trajectory of the particle X as a function of T right.

You get position of a particle in Newton's law as a function of T solving those equations. X as a function of T you will not be able to get unless you give a set of equations. Similarly, I need to give an equation, an empirical equation to solve this to find the time evolution of the state and before even we go into it is this vector space a linear vector space, if this is a linear vector space then these properties have to be satisfied.

If these properties are not satisfied, then it is not even a linear vector space. So these are things which we need to know and whatever a $|\psi\rangle$, $|\psi\rangle$ has all the information about the system okay. So this is an abstract notation slightly different from your wave functions. We will come back to the wave function later. Right now let us take the state of a system is encoded in this angular braket state which is called as a ket vector which can evolve in time.

There is a prescription equation which is the Schrodinger equation actually that how it is written in this Dirac notation we need to see and we need to satisfy all these properties. So this also could be commutative right, the combination of adding two states in a vector space, you can interchange the order, it will be again an element of this vector space to satisfy this associativity property.

So in fact the set if it satisfies this property, you call it as a linear vector space okay. So then we say that this is a linear vector space if it satisfies these properties in the language of this angular notation. So how do we define so that is also the scalar multiplying which I did not write but you can put it in?

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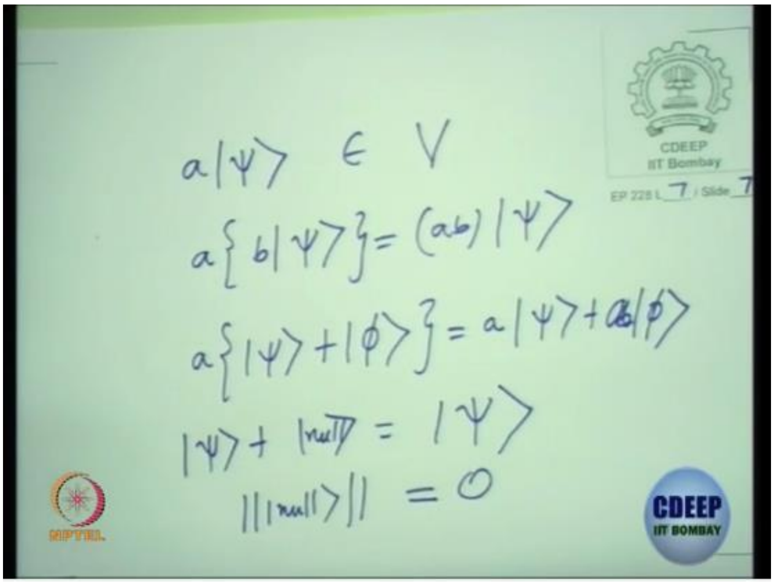

So if you have a scalar on |ψ⟩, this is also an element of the vector space and you will have a and then you have a (b|ψ⟩), this will be ab multiplying on |ψ⟩, this is also true. This also I wrote last time on the R_3 vector space. If you have something like this, you can do the distributive right something like this. Sorry a on, a on |ψ⟩+a on |ϕ⟩. These are things which you do it on a vector space; you can do this here also.

If ψ adds to something and gives you |ϕ⟩, what do we call the state? Does it do anything to the state? In your R_3, it is like a vector with components 0 0 0, it is called as a null vector. What is the null vector? Null vector is one whose components are all 0 or the length of the null vector will be norm of the null vector will be 0 right. So if you add a null vector to |ψ⟩, you get back the same state.

A null vector will have norm to be 0 but how do you find the norm, that brings us to the concept of you need to find a dot product.

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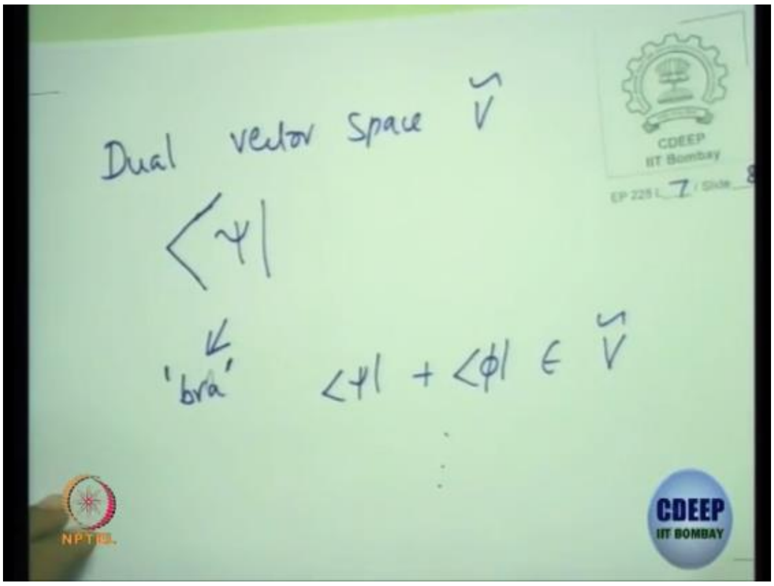

But in any abstract linear vector space, you have a dual vector space V tilde which could be different from V, most of the situations $\tilde{V}$ will look similar to V, in fact in R_3 you take vectors and take a dot product because $\tilde{V}$ is exactly similar to V okay but in general a dual vector space will be different from your original vector space where we denote states by a different angular braket.

This is called bra okay, Dirac introduced this notation, we all universally follow it. So this is the one which is the angular braket which is written in the opposite fashion okay. So this bra notation which we introduced, this will also satisfy the same set of properties which you saw for the vector space V but in this notation so you will have ⟨ψ|+⟨ϕ| should be an element of $\tilde{V}$ and so on. So why have we introduced this notation?

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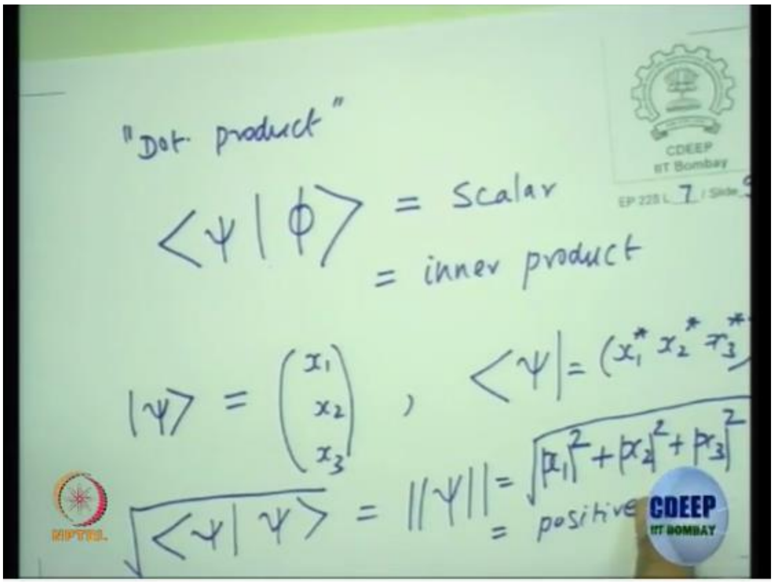

So with this notation, we can define an inner product "dot product" which we studied in R3. We could write some ⟨ψ|ϕ⟩, a ket contracted with the bra. This is supposed to be like a dot product which will give you a scalar and this is sometimes called as a inner product okay. So to make contact with your familiar R_3, let us do this; let us write the ket vector as a column is suppose I want to write a ket vector |ψ⟩ as $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Then, to get a number you are all familiar with matrix mechanics. The corresponding dual vector will be denoted by $\langle\psi| = (x_{1^*}, x_2^*, x_3^*)$. These are elements of R_3 that means real entries; the corresponding dual vector is a row. You have all done this many times. So this is a row or it is the transpose of this in real vector space. R_3 is a real vector space because the entries are all real and you know that what is the length of the vector when I ask which is square root of this which we denote it by ||ψ|| that is just your $\sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$.

Their sum of squares should be possible, is norm of the vector ψ should always be positive and it should also be finite, should be finite right. It should not be infinity, norm of the length of the vector cannot be infinity, should be positive and finite. These restrictions we can see in R_3whenever we take vectors we make sure that you do not take vectors with length infinity, you work with finite vectors and the zero vector is with zero components x_1, x_2, x_3 are all 0.

And you get the length of this vector right. Suppose instead of R_3, I want to go to a complex vector space like we have done complex plane, complex plane has one z coordinate but I could look at two z coordinate in an abstract complex vector space. If you do such things suppose $x_1$is complex, x_2 is complex, x_3 is complex suppose. If I take the inner product, what will happen to this?

It will become complex right. If x_1 x_2 x_3 are complex, $x_1$squared+x_2 squared+x_3 squared, x_1 squared will be complex, x_2 squared will be complex, x_3 but then this meaning of calling it as a length of a vector loses its meaning. You do not want a complex quantity for a length of a vector. What is the better option, which you do mechanically in matrix mechanics? Matrix is what you do, you do not only transpose, general matrices have entries which are complex conjugate and transpose which is what is called as an adjoint operation.

So this is also like a matrix, it is not a square matrix, so when I go to this dual space in principle I should take the complex coordinate correct. If I do that then this one will be $|x_1|^2$+$|x_2|^2$+$|x_3|^2$. Then, this will be positive, so in an abstract vector space if I take a vector defined by at least for the finite dimensional.

So this will give you the number of components here tells you that any arbitrary state what is the dimension of the space, so suppose I take it to be 3 dimension, let it take the abstract state as written in terms of these components in a column vector. The dual vector which is denoted by this angular braket, this one is called ket, this one is called bra. The corresponding dual vector should be written interchanging the column to a row.

And also complex conjugating the coefficients so that the inner product which you are all defined so far for the specific vector and its dual vector this one and this one that will be positive definite if we define the ket as a column and the bra as a row with complex conjugation. Now my next question is this ψ with ϕ what is the property ψ with ϕ satisfies? ψ with ϕ is like an inner product of two different vectors okay.

You have a vector ψ, you have another vector ϕ and it is a dot product of two different vectors. What happens? Does that remain same if you interchange, if you put ϕ to be in the dual vector space and ψ to be in the vector space? So what is the property it satisfies?

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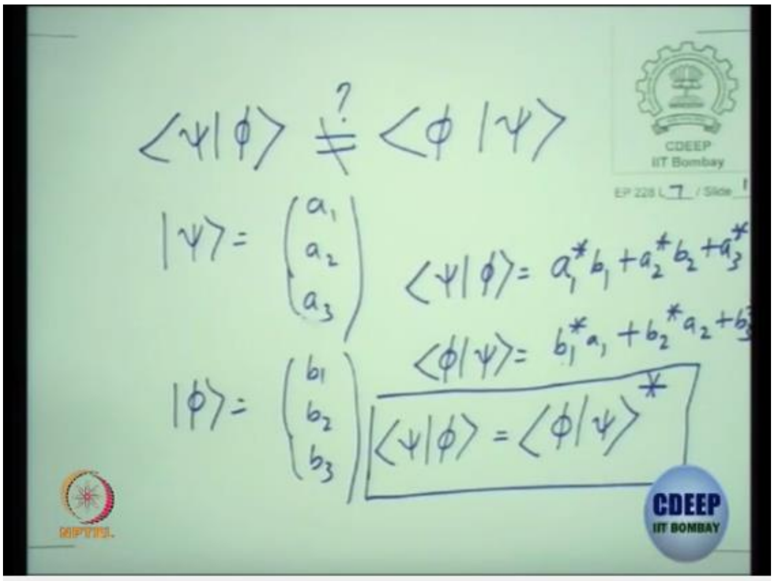

$\langle\psi|\Phi\rangle \neq \langle\Phi|\psi\rangle$? Phi ψ no so suppose I take $|\psi> = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ which are complex and we take $|\phi> = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ which are complex. What is <ψ| ϕ⟩, which is $a_1^* b_1 + a_2^* b_2 + a_3^* b_3$. What is <ϕ|ψ>? $b_1^* a_1 + b_2^* a_2 + b_3^* a_3$, so what do we see from this? ⟨ψ|ϕ⟩ is $\langle\phi|\psi\rangle^*$, you take the inner product and complex conjugate it.

The complex conjugate of this will give me this right. This ordering is unimportant, calling it as $b_1^* a_1$is same as $a_1 b_1^*$. The star of that is same as this four. So this is the property in a general complex vector space. If it is real, left hand side is real, right hand side is real, complex conjugation is not required but in general the vector space could have complex entries and you need to make sure that the inner product of any two vectors will be if you interchange the dual vector and the vector, it will be complex one.

So please get used to this notation so that is why I am spending some time so that you are familiar with this notation and then you will see how powerful this notation is okay.

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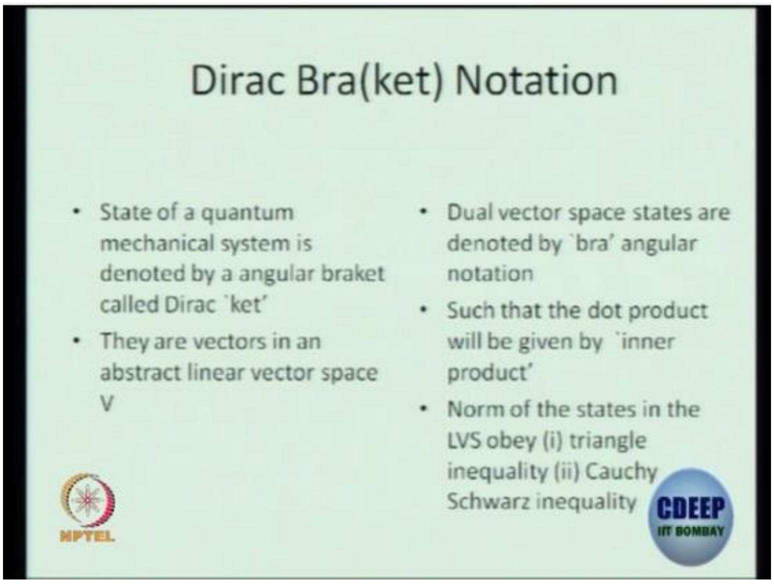

Here this kind of a block summary of what I said right now which is the Dirac bra ket notation. State of a quantum mechanical system is denoted by an angular braket which we will call which was called by Dirac as ket and they are vectors in an abstract linear vector space even though I am trying to map it to a known three-dimensional vector space, you can do it for in general and abstract vector space whatever I said as long as you follow the axioms.

Is that clear? I did this specifically with R_3 in mind so that you understand but you can do it for any n-dimensional vector space that n could be very, very larger. A dual vector space are denoted by another angular braket which is called as bra and then the dot product you need the dual vector space otherwise you could not define this inner product. So the inner product is the one which will give you the length and other scalars.

Immediate question which you all know when you sum up two vectors in the triangle if a, b and c are the norms of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ which denotes the sides of a triangle. What are the properties it should satisfy? Sum of two sides should be greater than or less than, sum of two sides should be greater than or equal to the third side, so these are things which you all know and we need to prove this in the Dirac braket notation okay.

So let us try and prove this. Straightly a couple of questions for you, if you have a set of matrices given, does that form a vector space? Let us take set of 2 x 2 matrices. Do they form a vector space? Yes, you can add two more to matrices to get another matrix which is also a 2 x 2 matrix so it forms a vector space, so there are various examples. You can take a set of real numbers, they also form a vector space; set of complex numbers, they also form a vector space.

You can also have column vectors with a column with n entries; they form an n-dimensional vector space and so on. So you can have examples. You can also have examples where you can take a differential equation and find a solution, there can be many solutions to the differential equation and linear combinations of the solution will also be a solution of this differential equation actually the equation which you have seen.

Stationary states are solutions; linear combinations of the stationary states are also solutions know, you can start doing this. So they also form a vector space. All the stationary states set also forms a vector space so that various examples we can do.