Quantum Mechanics
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Lecture - 12
Linear Vector Space (LVS) - I

Okay, so I am going to get on to the linear vector space today and that is the main theme in quantum mechanics. We are going to mechanically follow a notation or a language and you need to get used to this language. So I will go slowly on this language by showing analogy with your conventional three dimensions, some problems were already given in conventional three-dimension right. Why is the three-dimensional space called as a vector space?

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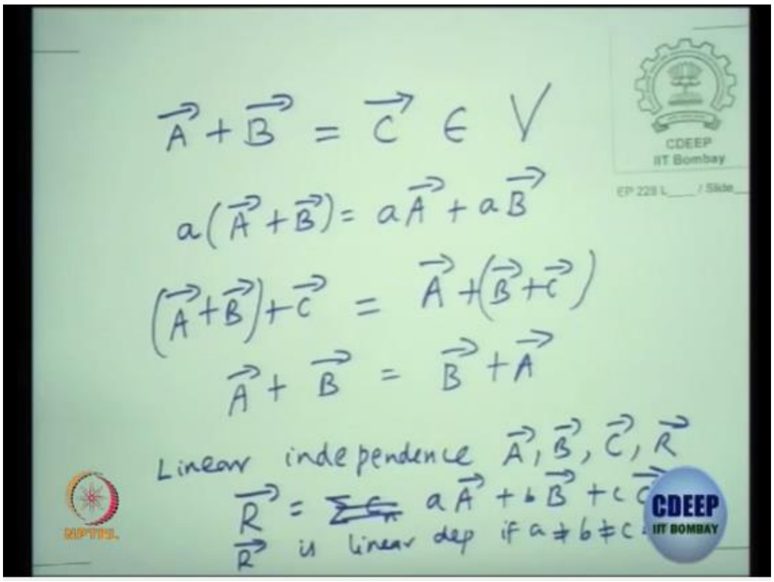

So you have vectors linear combination of two vectors which is going to give you another vector which is an element of the linear vector space, let me call it as capital V. So set of vectors belong to a space which I will call it as the vector space V and $\vec{A}$+$\vec{B}$ should be $\vec{C}$. Some scalar multiplying $\vec{A}$+$\vec{B}$, then will satisfy is that right or else can also have $\vec{A}$+$\vec{B}$+$\vec{C}$ satisfying associativity property right and vector addition.

They are commutative in the sense order does not matter okay. So then you call this these properties which are satisfied you call this phase to be a linear vector space right and this is satisfied in your three dimensional world okay. So there the condition is that if you give an arbitrary vector $\vec{R}$, if I write this as so suppose each of those are given and if this arbitrary vector $\vec{R}$ is a linear combinations of those vectors.

Or if you are given a set of vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{R}$ and if $\vec{R}$ is some linear combination of let say $\overrightarrow{C_n}$ or let me also write it as a $\vec{A}$+b $\vec{B}$+ c $\vec{C}$ and if these coefficients a, b, c are 0 then you would say that the set is a linear independent. If a, b, and c are nonzero, then this vector is dependent on the other vectors. You all learnt this in your math course right. So this R is linearly dependent if a is ≠b≠c≠0, could be, a, b, and c could be equal.

But remaining the scalar coefficients can be nonzero. Then, you will say it is linearly dependent okay.

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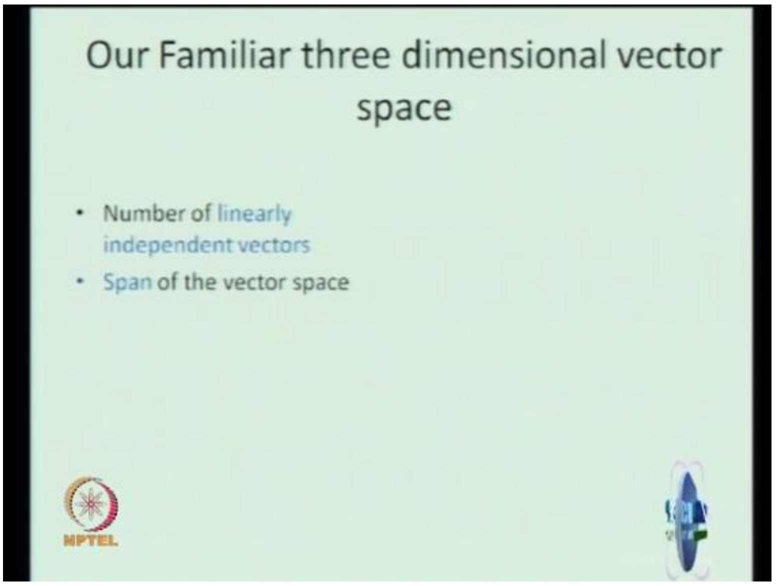

So just to put it in a form of a flowchart or a set of points, so you can look at least in three-dimensional space in this physical space how many linearly independent vectors you will have right. I have sent here three-dimensional vector space which is what we mechanically talk as x axis, y axis and z axis. Many times you would not even know what is the dimension, you have to figure it out in general okay.

So here the linear independence is the definition I have said and you have to also know if you are given a space, how do you span the space okay, how do you span the space? Span means you should be able to write any arbitrary vector in using those spanning vectors okay. Suppose let us take our same familiar R_3, so this three-dimensional space is sometimes called as R_3 okay.

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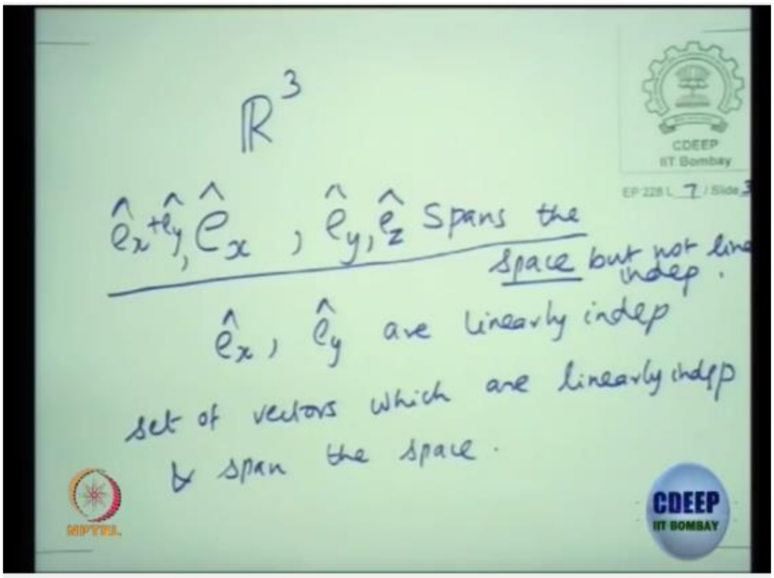

R_3 is a notation where this 3 denotes x, y and z axis okay. So here you could try to write that your familiar unit vector ex, ey spans the space it is alright? No, so you need to add ez which spans the space and you cannot completely write $\hat{e}_x$ and $\hat{e}_y$ alone of spanning the space. You could have written $\hat{e}_x$ and $\hat{e}_y$ are linearly independent correct, $\hat{e}_x$ and $\hat{e}_y$ are linearly independent and to span the space which is R_3 you need three of them.

So you should have a set of, so set of vectors which are linearly independent and what and span the space. This is very important. You could not just look at a set of vectors which spans. This is not the only one which spans the space, you could have defined something else which spans the space, order something else which need not be linearly independent. It could add things like ex+ey, this also spans the space but are they linearly independent?

This set spans the space definitely but not independent okay. So you need a set which spans the space and linearly independent. So there are two possibilities, actually there are many possibilities, I can.

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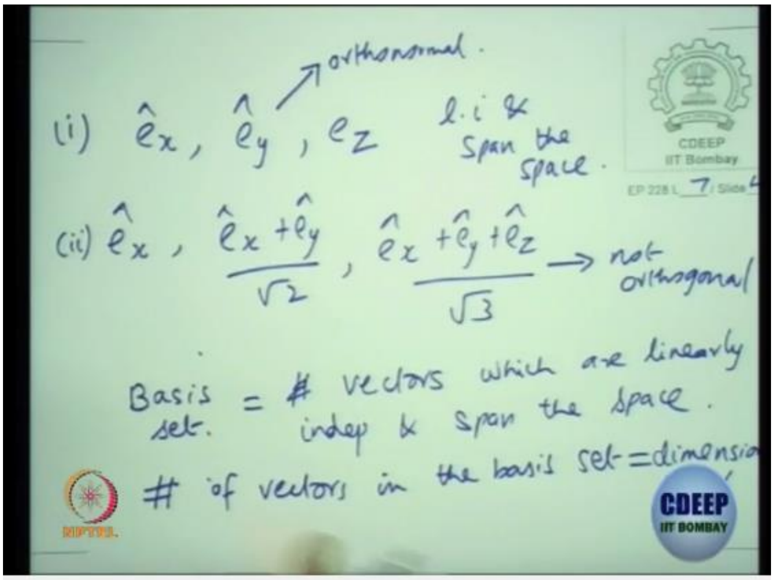

You could have $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$, I will put l dot i as linearly independent and span space. It could also have something else like $\hat{e}_x$, $\hat{e}_x$+$\hat{e}_y$/$\sqrt{2}$, $\hat{e}_x$+$\hat{e}_y$+$\hat{e}_z$/$\sqrt{3}$. Why am I putting the denominator, so that they can be like a unit vector. Are these also spanning the space? They are also spanning the space. Are they also linearly independent? Any arbitrary vector you can write it in terms of these three.

So you can actually list, you can keep going but you notice something in this keep going. What is something which you notice? The number of states which is linearly independent and spanning the space is always same right. So basis is number is basis set is actually vectors which are linearly independent and span the space, basis set. So this is one basis set, this is a second basis set and what is the difference between the basis set and basis set 1 and basis set 2.

So this one is orthonormal, orthogonal, the vectors are orthogonal this basis vectors in the set whereas this one is still a valid basis set but not orthogonal. Is that correct? Still a valid set but it is not orthogonal okay. So you can work with any basis, not a problem but conventionally whenever we do all our vector calculus and so on, we prefer to work with orthogonal basis set.

Basis set is a set of independent vectors which are linearly independent and could span the space. The number of such vectors in the basis set is the dimension of V. Here I have taken this vector space to be a three-dimensional familiar R_3but whatever I am giving you here can be extended to an arbitrary vector space where this number would change, it could not be 3, it could be many but it has to satisfy these properties that they should be linearly independent.

And any arbitrary vector if I give you I should be able to write it as a linear combinations of the set which spans the space that is why it is called it spans the space. To have both the condition then that minimal set is what is called as a basis set and the number of vectors in that basis set defines the dimension of the vector space okay. So given a vector space for example R3, you can find linearly independent vectors.

You could have two linearly independent, you could have three linearly independent but you should also make sure if it is two are linearly independent, can it span the whole space or any arbitrary vector can you write it in terms of two linearly independent vectors that is not possible. So you need to take count of both spanning the space as well as the number of linearly independent vectors.

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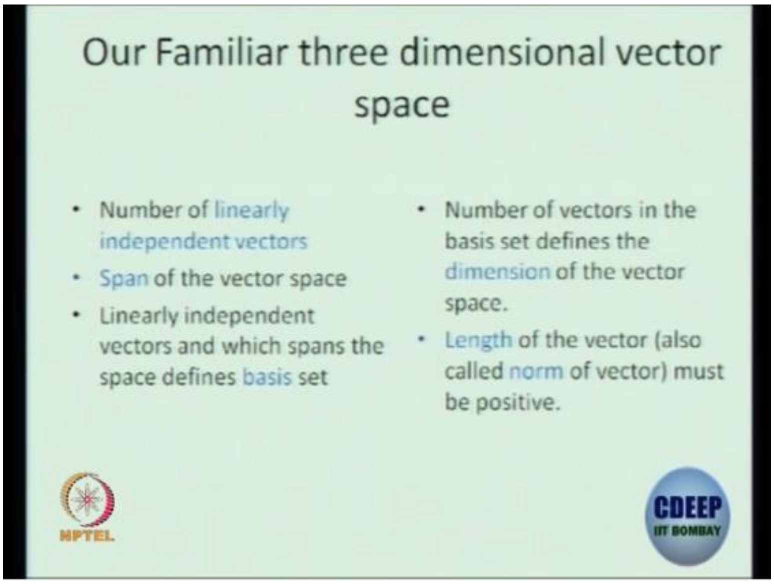

And then this collection of linearly independent and which spans the space, so the linearly independent vectors and which spans the space defines the basis set. The number of vectors in the basis set defines the dimension of the vector space. Then, the next question is you can ask what is the length of a vector, how do you do this in your vector calculus.

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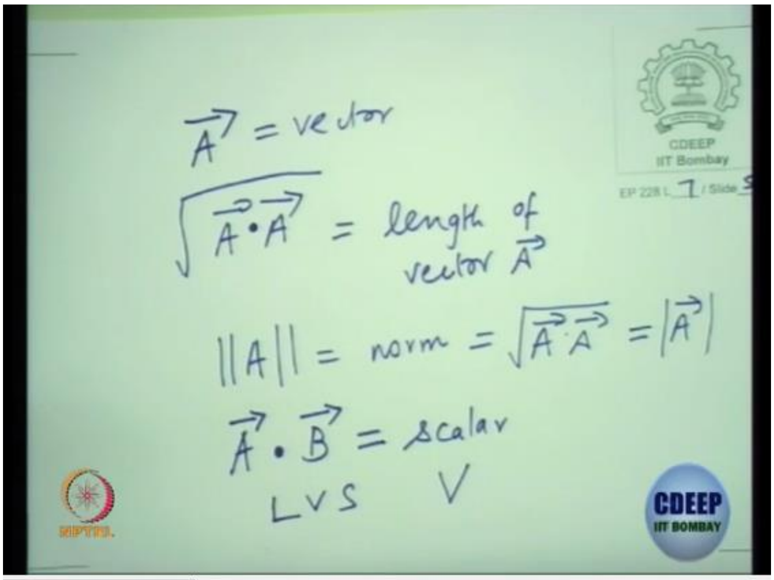

Whenever you are given a vector $\vec{A}$, if you want to find the length of the vector, you define $\vec{A}\cdot\vec{A}$ and take the square root. This is what you will say is length of vector $||A||$ which is formally sometimes written as norm, this is what we call it as norm which is nothing but $\sqrt{\vec{A}\cdot\vec{A}}$, sometimes people also write in this vector calculus as vector $\overrightarrow{|A|}$ magnitude of the vector right.

But in this process I have brought in some concept here. I introduce a dot product right. I have brought in dot product and what is the property of the dot product? If I take one vector $\vec{A}$ and another vector $\vec{B}$, if I put in a dot product, this will give me a scalar. What is the scalar? A scalar is one if you rotate your vector $\vec{A}$ and rotate your vector $\vec{B}$ by the same angle, then the dot product will remain the same.

Because the angle between those two will remain the same okay, so that is why a vector why is it called as a vector? If you do a rotation then you know how it will transform, you know how the vector $\vec{A}$ under rotation, suppose I have point the vector in one direction, you know how it will transform under rotation by some angle about some axis. So it will go to under rotation and become a new vector.

$\vec{A}$ will become a new vector, $\vec{B}$ will become a new vector but $\vec{A}\cdot\vec{B}$ will remain the same. So it is a scalar and when $\vec{B}$ is=$\vec{A}$ you get the definition of a length of a vector or norm of a vector and these are things which we need to have a handle that vectors, length of a vector, dot product of a vectors in three dimension and how do we go to an abstract notation is what we should see now okay.

With this knowledge on three-dimensional R_3, we would like to go to an abstract linear vector space which is a V. How to find the basis set? Looking at the basis set number of bases in that set, number of vectors in that basis set will give the dimension of the vector space. How to define a dot product? What is dot product call in an abstract linear vector space? How to find a length of a vector in a linear vector space?

So these are things which we would like to get a handle off and that is called as a tool in quantum mechanics. If you understand the tool, you will be able to do things much neatly and clearly and that is a universal language which most of us are following in the world and I want you to learn that notation okay. Is this clear? It is all known to you, I am just putting the notation, so that when I go to the abstract space you are clear with the analogy okay.

So the norm of a vector will always be positive is something which you all know, the length of the vector, the mod magnitude it is never a negative quantity.

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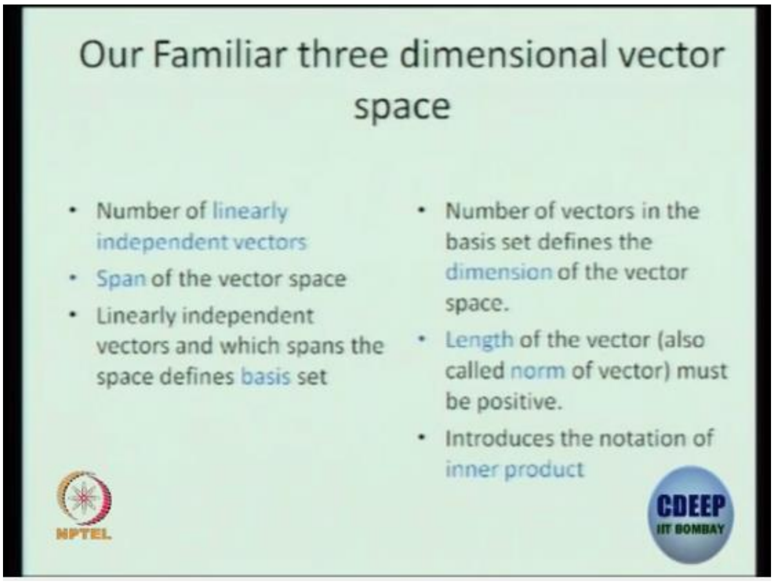

So this dot product is sometimes called as an inner product, most of the times in quantum mechanics we do not call it as dot product, we call it as an inner product. We will come to it. Inner product is one which will give you a scalar which is what we saw the dot product gives the scalar and we call this as an inner product.