Quantum Mechanics
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Lecture – 10
Conditions and Solutions for One Dimensional Bound States - II

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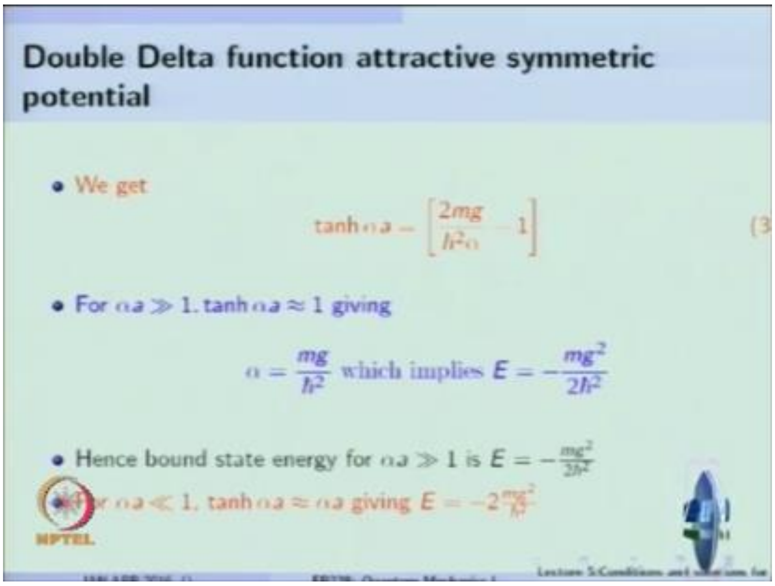

So, looking at this equation 3, there exists an energy, if it satisfies this equation that is all I can say, okay, so let us look for the trend, what are the trend; α a is very much > 1 is the first one, so which means you all know what to stand hyperbolic in that limit, it can be approximated to 1, is that right, so from there you can determine in such element that the energy has to be - mg squared divided by this g, do not confuse it with; this g is the coupling which was in the front of your V of x, okay.

So, this g squared actually tells you that the energy is proportional to g square, sometimes you know, whether g is positive or negative, the expression is the same but if you take instead of - g as + g in your potential, the concept is gone because the global potential, global minimum potential will be 0, so you have to have energy positive.

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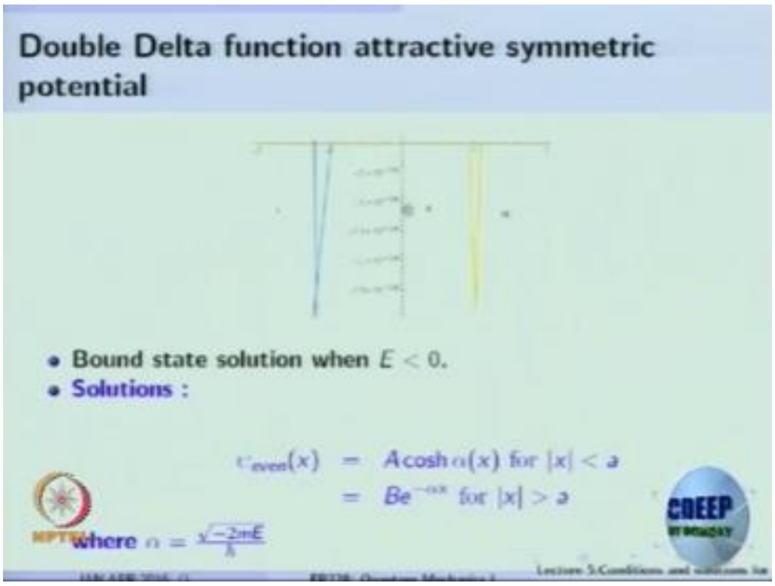

So, you cannot have the ground state, bound state, if I take the; if I take this and flip it okay, if I make this instead of attractive, if I make it repulsive.

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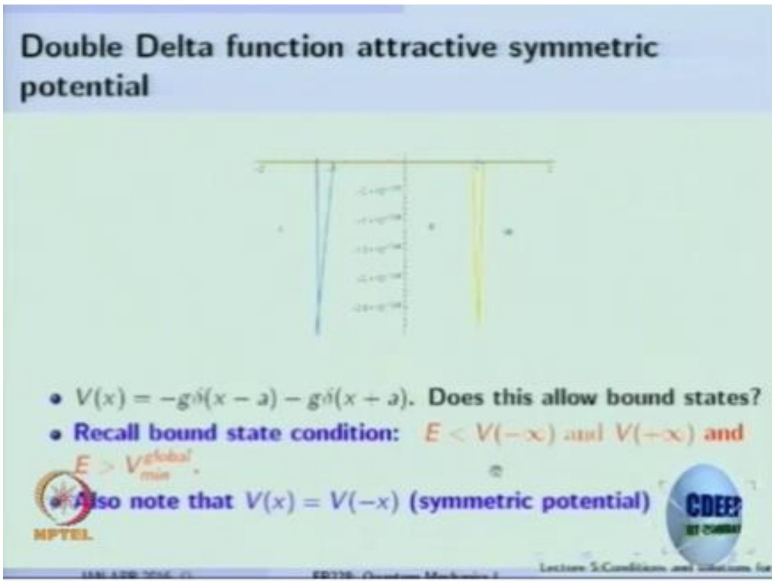

Even though, the calculations can go through mechanically, the theorem tells you that these 2 theorems will prevent you from saying that you cannot have a bound state problem, okay, bound state has to satisfy this as well as this and this will not be possible, if I put a plus sign okay. So, at least for α a, a is the point where you have this delta potential, delta function potential, dirac delta.

If α is very much > 1, you do have an energy solution and you can get that energy value also (()) (02:52) but that is only when α is very much > 1, what about; so we say that there is the ground state energy for α a very much > 1 and the ground state energy is - $\frac{ma^2}{2h^2}$. What about α a; much <

1, what about that condition? Look at the equation 3 again, so you can substitute a hyperbolic α a as α a.

And then simplify this above equation and you can show that the energy will be minus twice in this way, so in fact it is doubling something wrong, 4 times, make a mistake, so this will be α a, so then you will have an α a, α square and we check it, if it is 4, I will check or you want me to do it on the board, let us try it here okay.

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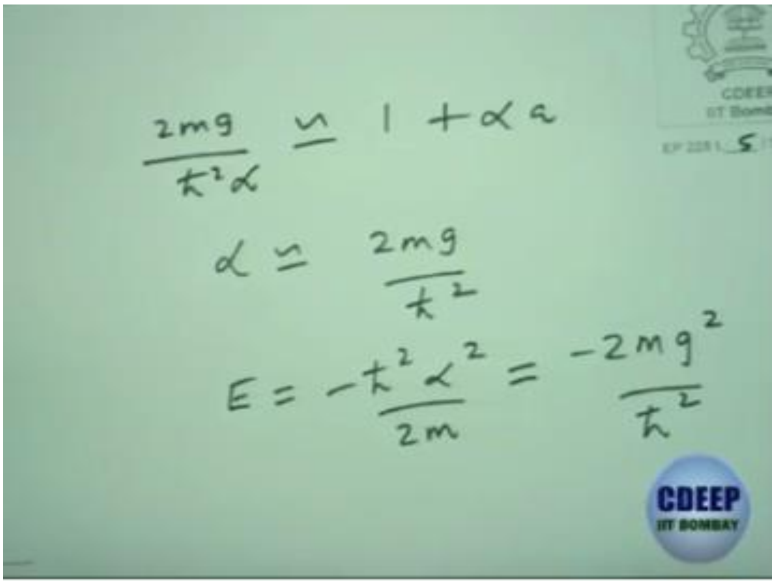

So, I have $\frac{2mg}{\hbar^2\alpha}$ to be approximately 1 + α a, I have just taken this 1 to the other side okay, and here you will get α to be $\frac{2mg}{\hbar^2}$, E will be -$\frac{\hbar^2\alpha^2}{2m}$ which is -$\frac{2mg^2}{h^2}$, am I made a mistake, someone was saying there is a 4 factor, okay, so there is a factor of 4 between the α a much > 1 and α a much < 1.

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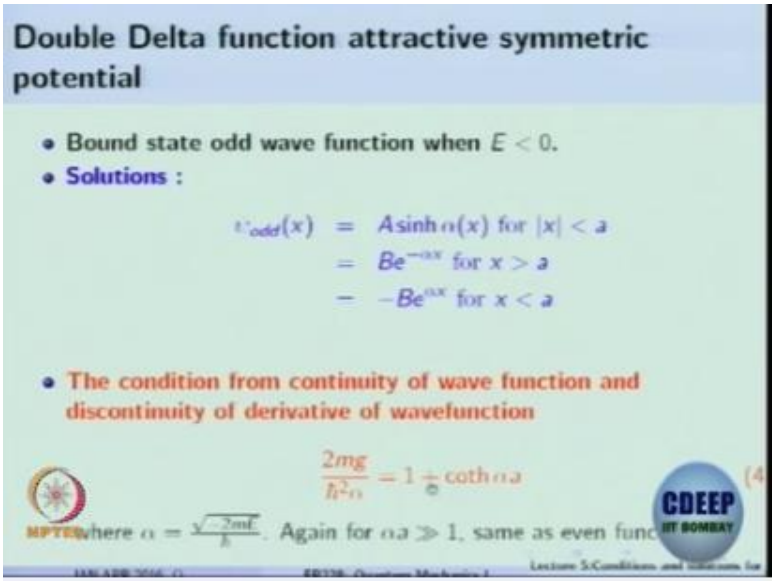

So, we can also do an equivalent solution as I said since it is a symmetric potential instead of even function, we could have also worked with odd functions, so in a context of odd functions, sin hyperbolic is an odd function and you will also have $B\ e^{-\alpha x}$ and - B$e^{\alpha x}$ for the 2 regions, so that this region before a and after a are all odd, exactly like what we did for the even function please redo it again, wave function continuity derivative of a function continuity.

And what is expected is that you will get a instead of that tan function, you would have a cot function, again you can do α a much < 1, sorry much > 1, in that case cot hyperbolic is again 1 for large α a, there is no distinction, the leading term is e to the α a, so you will have the same as what you got for the even function. So, very high if you fix an a and if you want α to be very large that means, when – E, energy will be very large, magnitude of the energy.

Then both, the even function or odd function, do not distinguish, they give the almost the same energy, this is what is the trend is.

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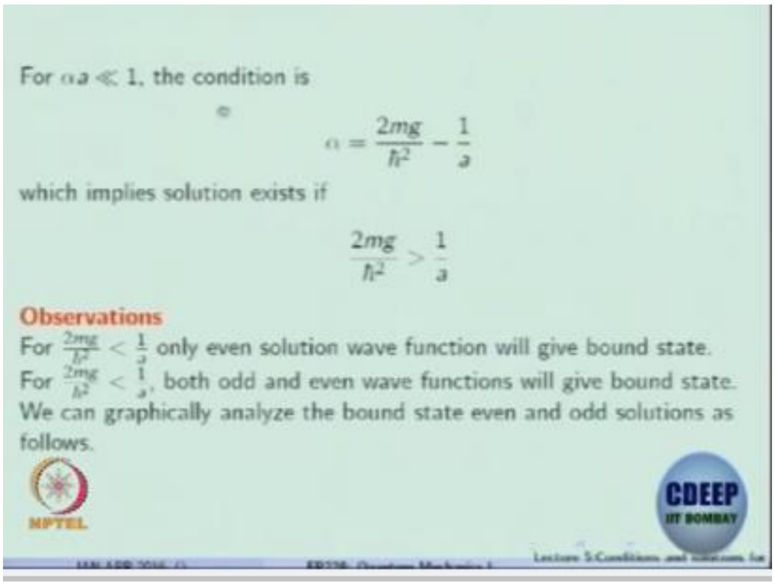

For α a very much < 1, take the above equation; the cot hyperbolic function and find the solution, so you get α to satisfy some condition like this and I have already said α is positive which means, there will be a solution if and only if your, a value satisfies this condition. So, for low alphas for a fixed a, if this condition is not satisfied then, you may not get the odd solution but you will definitely get the even solutions.

Even solution energy was – 2mg/ $\hbar\omega$, so this kind of indicating that at low energies, even solution; the ground state solution will be even and it will be present and low energies; the odd solution may not be present, if a does not satisfy this condition, this is all from math, we not; we try to put a continuity, we have tried to put a discontinuity in the derivative function we know what the discontinuity factor is, right.

We have subsisted and we ended up trying to get an equation, which is the cot hyperbolic function but then we try to find the trend for α a to be very much < 1 and the trend seems to be done, if the a value satisfies this criterion, you will find the solution for α a < 1, so what are the observations; 2mg/$\hbar^2$, suppose is < 1/a, even solutions will anyway exist but odd solutions will not be there, so this should be greater.

Sorry, the second line should be > 1/a which is this condition, then both odd and even functions will give bound state, please correct this, I will also corrected when I put the file, so this second one should be greater, then both odd and even functions will give bound state, so this is from algebraic way of doing it and getting the trend but it would be nicer to actually plot your solutions and see whether you get a non-trivial value for α; α and hence the energy.

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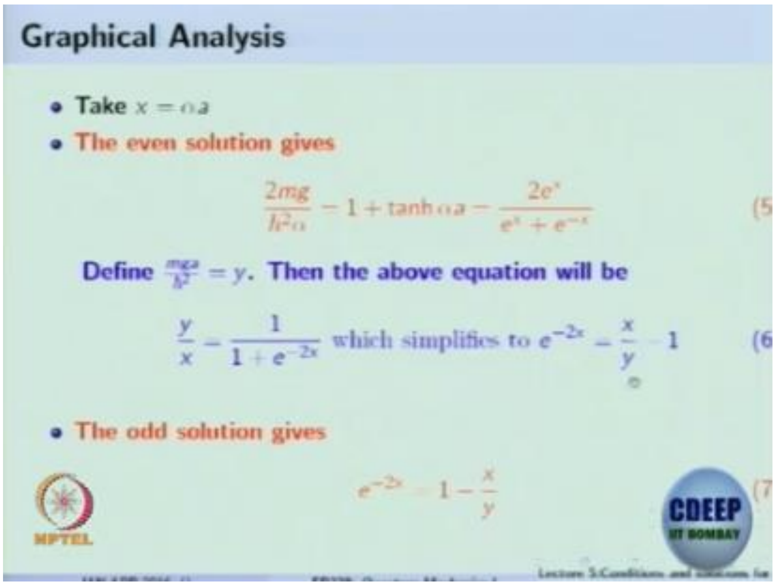

So, how do we go about it; the graphical analysis, so it is a similar way in which I am going to do it for this case, so now I am going to take a defined; do not confuse this x with some position or anything, I am calling some variable x as = α a, just for convenience, so the even solution which I have, the solution for the condition satisfied for the energy when you take even wave function solution that can be rewritten in terms of the x variables.

You can also define another variable y as mg a / $\hbar^2$, so then the above equation will be this equation can be rewritten by putting an a and then a below, if you do that then, you will have a y, actually has a 2y. I think, the 2 will cancel on both sides, so its simplifies; this equation can be further simplified, equation 5 can be further simplified subsisting y as y over x proportional to this.

And you can write an equation which is $e^{-2x} = \frac{x}{y}$-1, what typically, how do you find a solution to this? I am incidentally interested in x and for a specific y, you can take y to be some value because is dependent on the coupling constant, the position of your Dirac delta function, the mass and h bar squared, it is a constant, y has nothing to do in the plot, you can fix y to be some value; 0.1, 0.2 okay.

And you have the right hand side as x/0.2 -1, and left hand side is e to the - 2x which is a function of x, right hand side is also a function of x because y is a constant, so you can plot e to the - 2x as a function of x, you can plot x by a specific value of y, you can take y to be > 1, <1, > 1/2, <

1/2 whatever and you can plot, this is what we do. If both left hand side and right hand side get some intersection point for a specific value of y that intersection point will give you what is that x and looking at that x, you know what is α.

Once you know α, you can determine what is the energy because different days will give you different alphas and you can plot graphically and get a solution to the curve, is that right, so will you try to plot this yourself when you get back and see what is happening, when does it change for what values of y, whether this y plays a role and so on okay, it will play a role in the sense that we saw that odd and even functions solutions had a requirement that 2g; 2mg/ $\hbar^2$ has to be > 1/a.

There is some constraint which you will kind of play a role that there will be a turning point, why, below which only even solutions or above which I do not know one of them, it will not be there in the other regime both the solutions will be there, okay so please play around with this and get a feel of the solutions okay. Similarly, with the cot hyperbolic α in the above equation which is for the odd solution, we got this condition.

I am rewriting this condition in terms of the same x and y variable, okay please verify this also, you can see here it is x/y – 1, this 1 – x/y. so the curve itself as you can see will have an intercept of – 1 here whereas, this one will have a slope which is negative okay, you can see always, kinds of trend and you can plot them. So, what is the ultimate thing I am trying to say, the intersection point of the left hand side curve and the right hand side curve for a specific y will give you the energy for an even solution.

Similarly, the intersection point of the right hand side and the left hand side for the odd solution, give you another energy, you can also check whether this energy and this energy are one and the same or different, which one has lower energy, which one has higher energy, all these checks you can do, okay.

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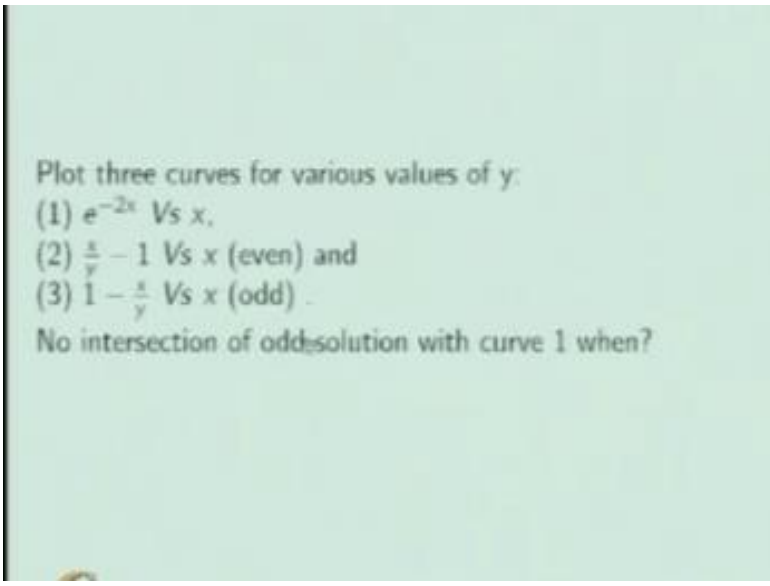

So, plot 3 curves for value of y, e^{-2x}versus x, x/y - 1 versus x, is that the even function; yes, that is an even function and 1 – x/ y versus x for the odd function, so you have to check no intersection of odd solution with curve 1, when this y that condition we already have, if you remember we had this condition; twice mg a/$\hbar^2$ is > 1/a, you can take this a this side and then that becomes y > 1/2.

So, solution; odd solution will exist for y > 1/2 an even solution will exist for y < or = 1/2, y > 1/2 both odd and even will exist because for any y, you will have an even solution, it is not find any condition like this, for the odd solution if y is > 1/2 you will have a solution but you will not have a solution when y is < or = 1/2 also, is that right because α has to be possible, so no intersection of odd solution with curve 1 when y is < or = 1/2.

This is what I have tried to show you from the earlier slide, both odd and even intersect curve 1, this is this curve 1 for y > 1/2 should be plot and check.

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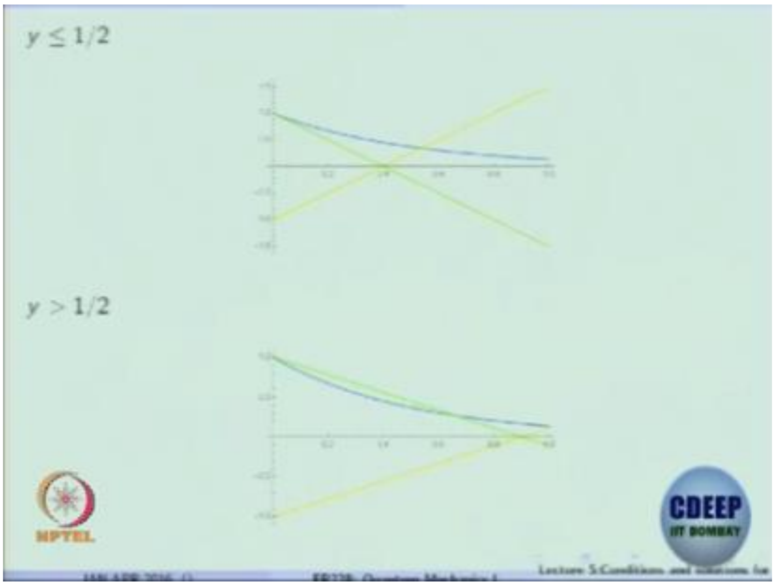

I have tried to plot it in mathematical for you, which curve is what; the blue colour line is what; e^{-2x}, what about this orange colour line; odd solution or even solution; even solution, it intersects at this point this is visible, so there is a solution but odd will start is the green line and the green line does not intersect the C to the – 2x when y is < or = 1/2 so, this I have tried to plot with some y value as 0.4; 0.4, if I take then I find this plot, you can try and check.

Why are there only 2 solutions, what else can you have; barrier is (()) (16:37) yeah and it is a delta function no, but this is also for a specific, you know it is the Dirac delta function is a spike at x = + a and –a, so that kind of triggers a discontinuity and this is the only possible solutions you get, you cannot; when you do a potential well and barriers you do not get many solutions.

No but you do have a for a specific energy of the particle which you send in, you find one solution for different energies you can; yeah but depends on the a, it is constrained by the a also, a is given in the problem, so you will have only one specific energy which satisfies the condition that is what you see, you do not get more than; yeah, see the thing is sometimes when you have tan functions, there will be a series of curves.

Then, there is a possibility of it hits the higher curves but here it is the exponentially damping curve and it is impossible for a straight line, I do not see a physical way of arguing that there will be only bound state but I can think about it but I know that for other cases, when you have oscillatory solutions in certain regions, which is what happens in the case of your particle in a box and so on.

You could have a repeated periodic ones which can kind of give the constraints, so you can have different energies giving you but in this case, it is always damping exponential solution in all the 3 regions say, in such cases there is no periodicity, so there is no way in which you can find more than one solution, this is what I see but I do not, it is not a proof but this what I see, okay.

Okay, so similar thing we can plot for y, so this is not a very clear plot but you can still see that the blue curve is the exponential curve and for y > 1/2 the slope of the odd function changes and it hits this curve, right and similarly, the even function is also hitting this curve. So, what is happening, how do we see which energy is smaller here of course, this energy in y is < 1/2 I see the energy to be here.

But here the energy is increasing and this is; ah, lower negative that is true, right okay, excellent, so this is what we see that the negative energy ground states is going to be lower for the even wave function than the odd wave function okay, it is kind of neat and nice, so I thought I should take this to show it you on a slides, so that its clearly in your mind and you know when to do a difficult problem, how you should analyse them graphically.

And also try to see whether you can analytically go to certain regions like α a very great; very much > 1, α a very much < 1 and then you get a trend, once you get the trend you go back and look at whether the trend is seen for an arbitrary α, α a, so here I am not put in any condition that α a has to be, 1 or α a to be > 1, graph gives you what is the α or what is energy and that I have find out by looking at this intersection.

And as he was pointing it out that for bound states generally, you will have more than one solution but if you had in some regions oscillatory solutions and there could be a periodic kind of curves, instead of this exponentially damping blue curve and then a straight line could cut those periodic curves at many points and then you can say that you will have bound states, when your energy is between the (()) (21:25) be 2 bound state, there will be 3 bound states and so on.

But in this case in the delta function potential, you do see all the 3 regions you have only the exponentially damping or growing functions, so you will have a single curve not; if you had a tan function, you would have a series of curve between $\pi/2$ to $3\pi/2$ and so on, so you will start seeing some periodicity and if you put this other curve of the right hand side, it could intersect those periodic curves at different points.

So that is way you may have more than 1, I think this answers your question, okay so, I want to give you a couple of other things, how many of you know Fourier transforms; everybody knows okay and how do you see Fourier transforms in quantum physics; momentum space okay, so let me try and give you a little bit more flavour here and then we will stop.

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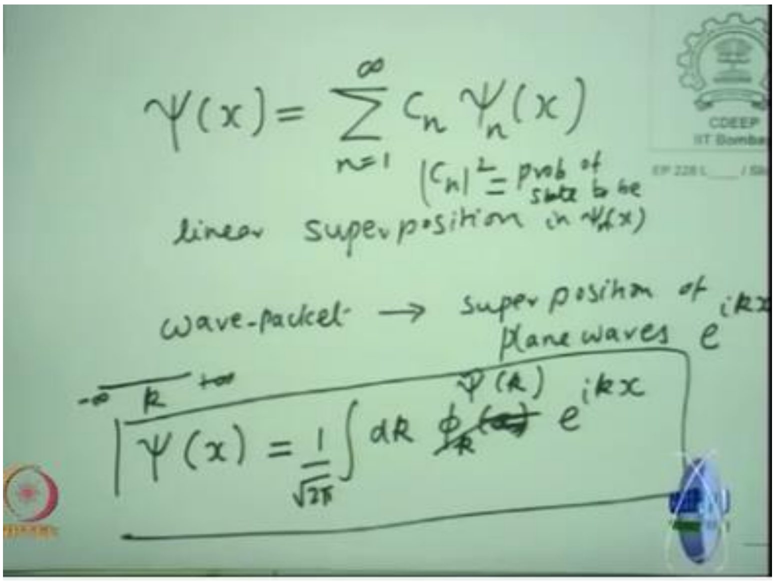

So, what we have seen is at least in a particle in a box, we could write ψ of x as a linear combination of $c_n\psi_n(x)$, where $\psi_n(x)$ are stationary state solutions, yours Schrodinger equation, so this is what we call it as a linear superposition but we could take wave packet as a superposition of; superposition of plane waves, plane waves are typically denoted bye to the e^{ikx}, free particle.

How do I write that? ψ(x), I could write it as; I could take a summation if their case are distinct but if the k is continuous going from - ∞ to + ∞, how do I write this; integral dk, just like you have a probability, amplitude for a state; arbitrary state to have you know what is C_n right, $|C_n|^2$ is the probability of state to be in $\psi_n(x)$ of or have energy, this is what you will see.

So, similarly I can put a probability factor times e^{ikx} because there will be some normalization here, some books follow 2π, I will get back to that so, what have I done is exactly similar to a discrete linear superposition but in the case of a wave packet you, superpose various plane waves, so various wave vector or equivalently wavelengths and if there are continuous distribution for the wave vector K unlike here n is discrete.

It goes in discrete steps that is why we need to put a summation, you could equivalently do a superposition which is an integration here, so when to put an integration, when to put a summation, so Fourier transform; so this expression is nothing but your familiar Fourier transform you have studied, it is nothing but a solution for a wave packet which is the superposition of all possible plane waves with the probability of finding it to have a wave vector k is given by mod of ϕ k x.

ϕ k does need to depend on x, sorry does not need to depend, why did I put it, okay, thank you, yeah π k depends on k, okay we very precise it should be dependent on k, so formally in all your Fourier transform you typically would have replaced this by a $\tilde{\psi}(k)$ and you will call $\tilde{\psi}(k)$ as a Fourier transform in the momentum space and ψ(x) , it is a Fourier transform of ψ.

And there are various other properties of Fourier transforms you know right, so like I will give a couple of problems based on this Fourier transform for a free particle where you can put your k to be $\frac{\hbar^2 k^2}{2m}$ to be the energy and you can rewrite it in terms of that and you can try and do what is the evolution of ψ(x) but this is the concept that whenever you have a continuum, you can replace it by a; you can also do an inverse.

You can try to find a specific wave number from the superposed wave packet, it is an inverse Fourier transform and these things are visible when you do quantum physics, okay, so let me stop here.