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Quantum Infromation and Computing

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Modul No.05

Lecture No.24

Period Finding and OFT

In the last lecture we have introduced the concepts of discrete integral transforms and extended our ideas to define what we called as the quantum Fourier transform, just to recollect what we did last time is to say that.

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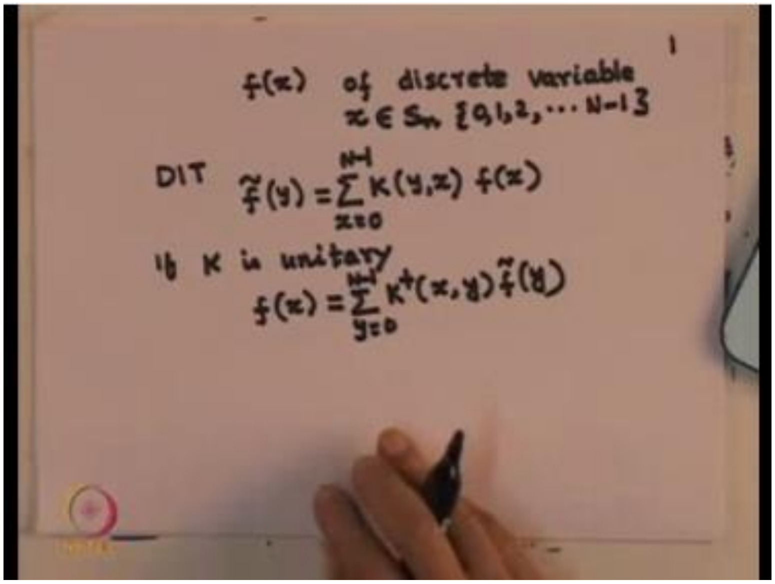

Suppose I have a function f(x) of discrete variable X which are belongs to as we define S_N which is the collection of numbers 0, 1, 2 etc upto n -1, we defined discrete integral transform of this function to be given by F tilde of y equal to sum over X equal to 0 to n-1 of a kernel which we wrote as K(y, x) times f (x) and we said that in case this K is invertible or in our particular case of interest if K is unitary.

We can define inverse integral transform which will be given as f(x) equal to sum over y equal to 0 to n-1 k^+ (x, y) and f tilde of Y. Now the way we defined a Fourier transform was to take a specific case of this kernel.

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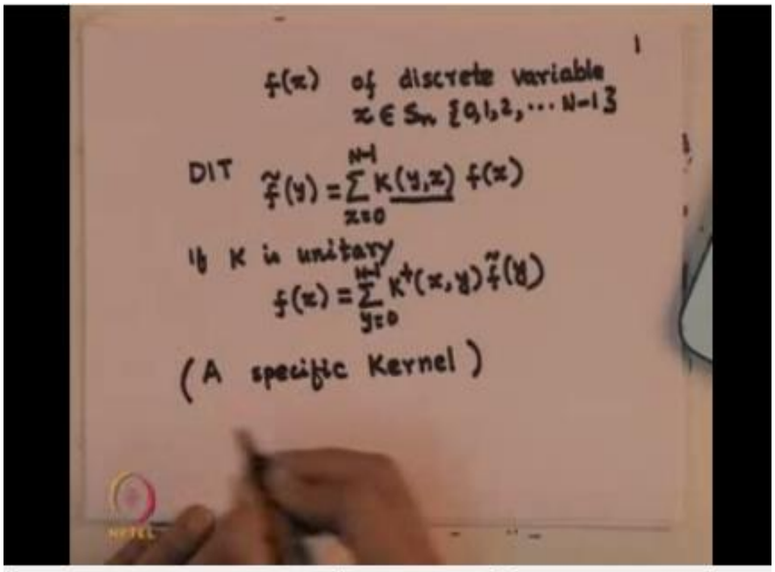

But before that what we did is to say that this definition enables us to generalize it or extend it to the case of a basis paid. Now so what we did is the following we said that supposing I have a basis set of basis States, let us say computational basis.

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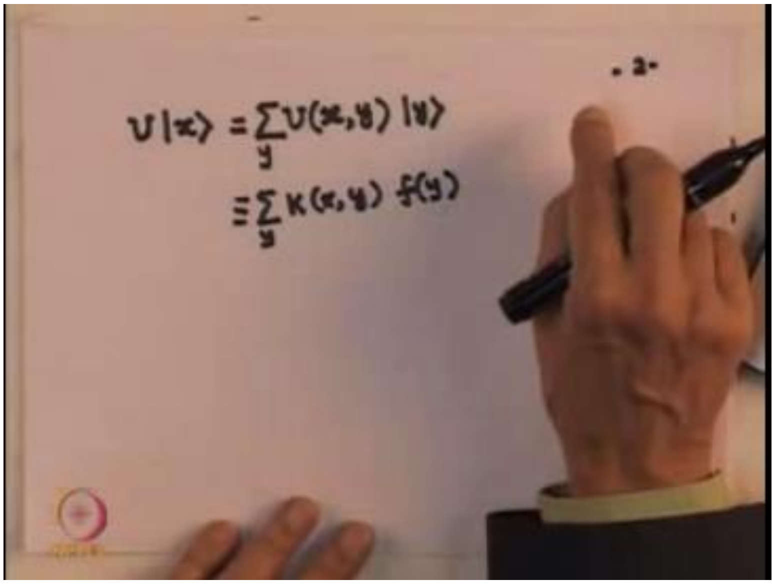

Then supposing U acts on one of those computational basis U is a unitary operator then by definition of a unitary operator this will give me a linear combination with matrix element of U in the state (x, y) |y> Now what we said is this, compare this with the expression of some over y K(x, y)|y> this tells me that I can essentially compute the Fourier transform of a basis in a similar way.

Provided the matrix elements of the unitary operator that I write down can be identified with the kernel that we talked about earlier. Now this has the great advantage in the sense that if we did it, it would mean that I can calculate the discrete integral transform of all the members of a basis set in one goal and this is of course as we have seen several times is because of the quantum parallelism that is inherent in any quantum computation.

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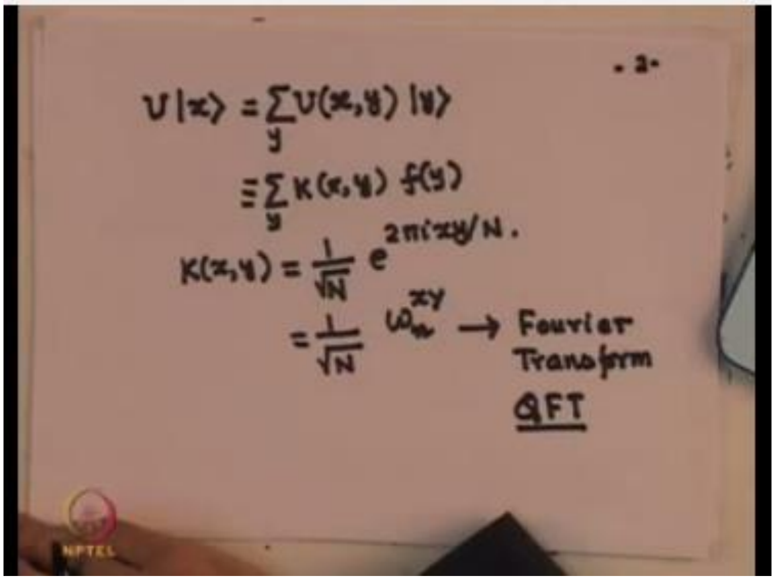

So what we said is this, that supposing I took this K(x, y) to be given by 1/ $\sqrt{N}e^{2\pi ixy/N}$ which is nothing but 1/ $\sqrt{N}$, the ^{nth} root of unity which is $\omega_m{}^{xy}$ this defines a Fourier transform. And we will call it simply QFT, the essential idea is the following that what we are doing is basically very similar to the way one calculates Fourier transform for a real or a complex variables in our mathematical analysis.

Accepting that our variables are discrete and we have been able to extend it to from the case of functions to the case of basis state. Now with this recapitulation of what we did last time, let me now continue with a four ire transform, now one of the applications that we will be making will be to find the period. So let me define what is meant by it but before I do that let me remind you that what we want to do ultimately is to get a factorization algorithm in place which was due to shore.

And what we are doing in the last and next couple of lectures is to get the components of source algorithm in place. One of the essential points of the source factorization algorithm is.

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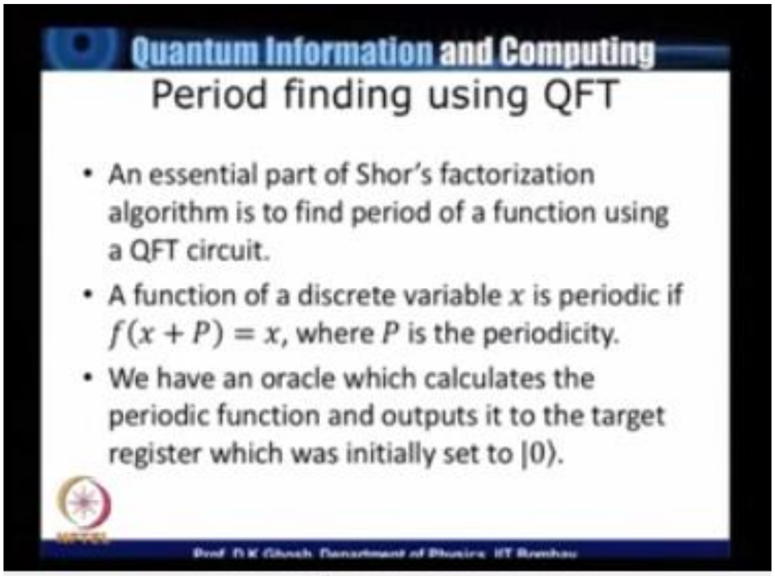

to find the period of a function and this using a quantum Fourier transform circuit. Now as is normally done, a function is said to be periodic.

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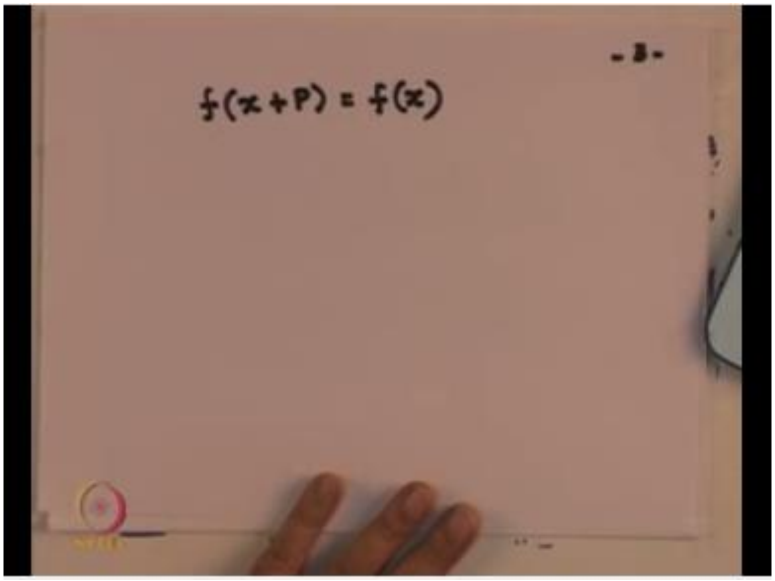

If $f(x + p) = f(x)$ we are familiar with several periodic functions for example trigonometric functions of the type of Sine, cosines are periodic functions with period 2π after which they are all repeated, the thing that we are talking about here is because our variables are discrete variables we are saying that P is a number.

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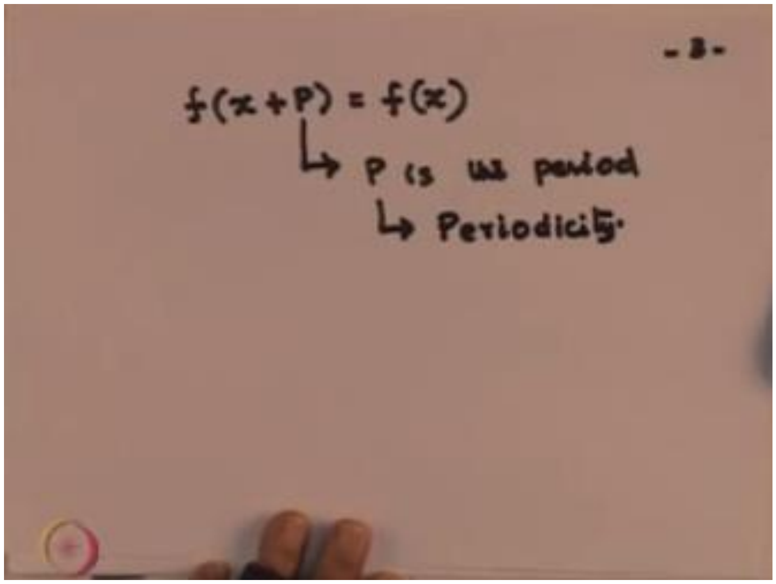

P is the period so P is the period after which the value of the function gets repeated, so P is a discrete number in this particular case and this is called periodicity. As with all quantum algorithms we will have an oracle which will calculate such a function.

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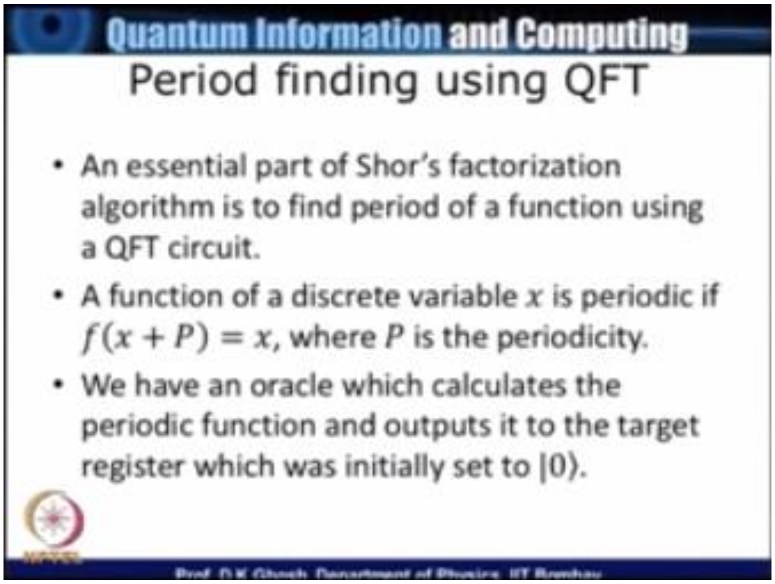

And output it to the target and as we know that if we had set our target register or the ancilla initially to 0 then of course the f(x) is calculated corresponding to all the different x in the input and I get simultaneously a linear combination of f(x) corresponding to every x.

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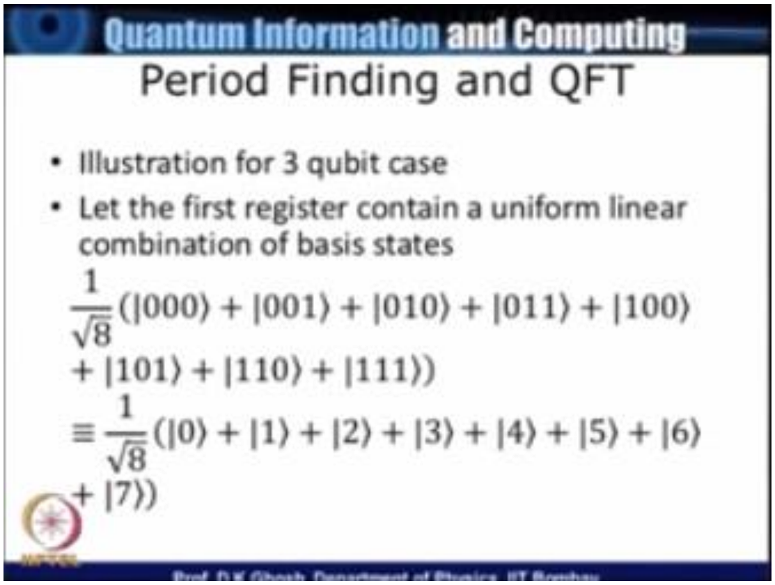

So let us see how does it work, what I am going to do however instead of doing it purely formally for an arbitrary periodic function, let me illustrate it for the case of 2 and 3 qubits in this particular case I am first taking a 3qubit case and try to find out what does this period mean and how is Fourier transform going to help us. So let me illustrate this for a 3 qubits case, so the first thing that we do is to let the input register contain a uniform linear combination of the 3 qubit basis state. Now what it means is my first register which is.

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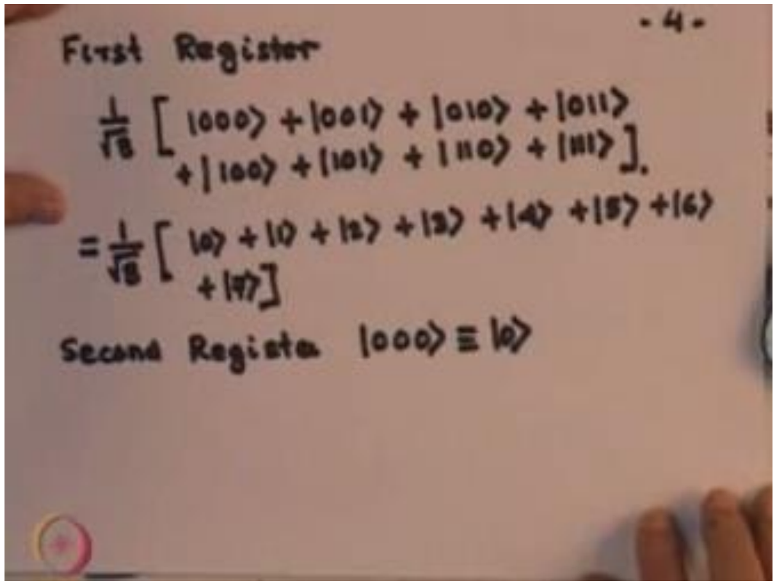

My input register will contain then 1/√8 [|000> 3 qubit state |001> I will just write down though there are several terms in it but is not impossible to write all of them down. Now instead of carrying out this, carrying on these 3 qubits which also is a bit clumsy we will go over to the decimal representation and write this simply as [|0>+|1>+|2>+|3>+|4>+|5>+|6>+|7>] and as we have said my target register the second register is said to |000> which is identical to |0> in the 3 qubit notation.

We have several times pointed out the way to get the first register in this place is to start with this |000> and then pass each qubit through a Hadamard gate so that we get a linear combination of the various states 3 qubit basis. The oracle that we have, I am not reproducing that picture again this is routine I have done it several times. But what the oracle does is to compute f(x).

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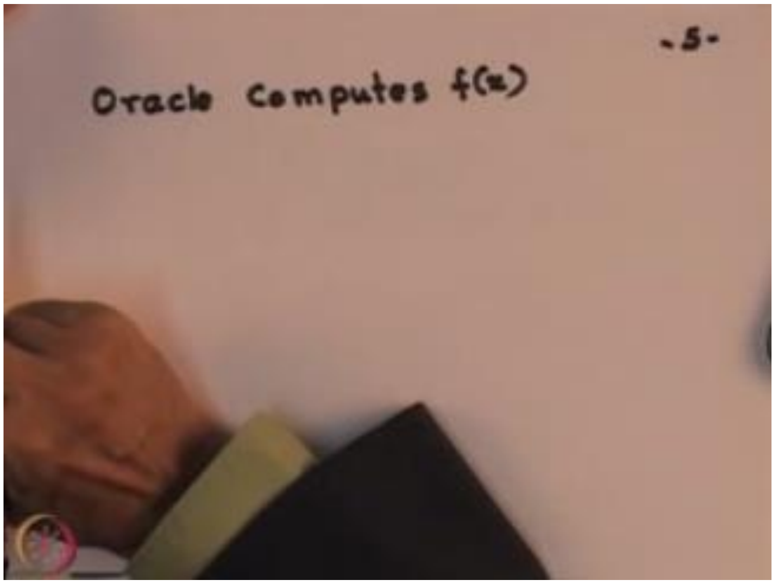

And since my second register was set to null the output of the second register will contain f(x). Obviously, I will not be able to write the output.

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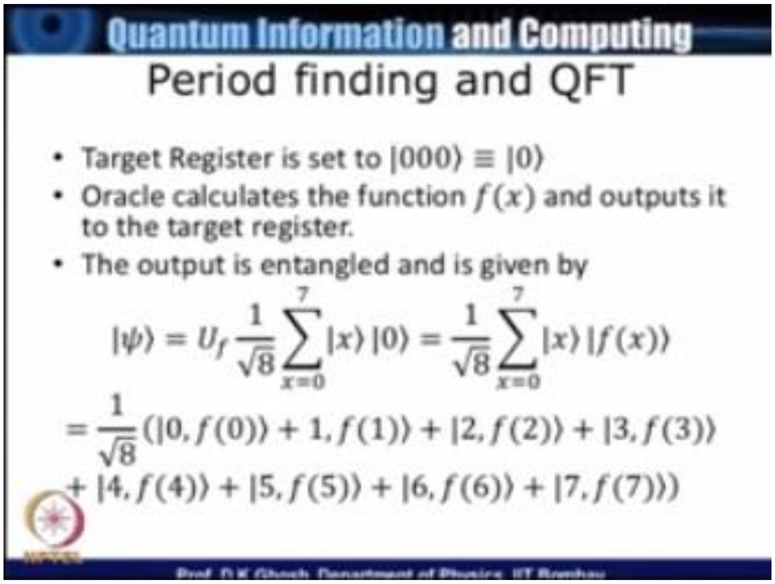

In a factored form because the output gets entangled, let me do this right this output.

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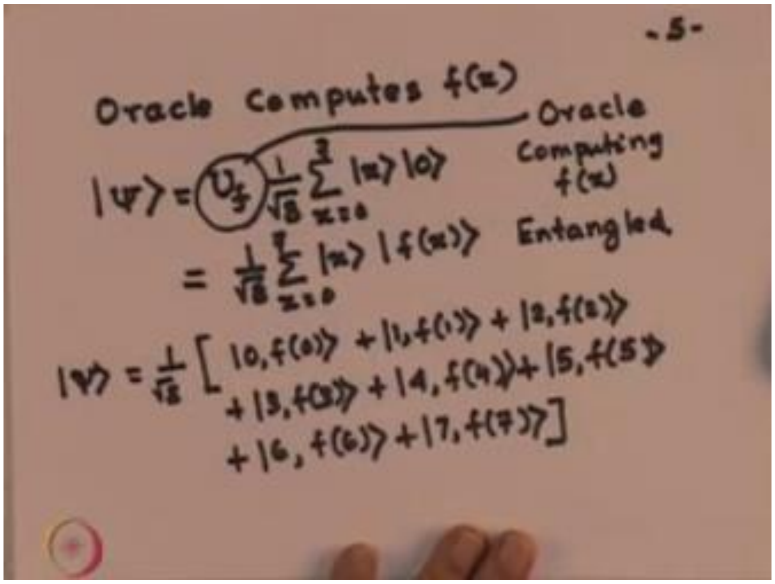

As $|\psi\rangle$ which is given by U_f that is my unitary operator acting on the input state which was $1/\sqrt{8}$ $\sum x=0$ to 7 $|x\rangle$ $|0\rangle$ this was my input and my output is then on when U_f acts on this, so this is x=0 to 7 $|x\rangle$ and f(x) so the output is entangled. Since I was just 8 terms it will help me if you just write this down so we will write down $\psi=1/\sqrt{8}$ $|0,f(0)\rangle$ I am writing it in both the states together instead of factored form $+|1,f(1)\rangle$ like this.

So there are 8 terms in it, so this, this is, this is what my oracle have given us, so this U_f is the oracle which computes the function. Let me write this clearly because there will be one more U_f that I will be talking about. Now at this stage what I will do is, I will apply QFT quantum Fourier transform on the first register, now of course we will see later in this lecture that corresponding to Quantum Fourier Transform I can find a unitary operator which would do this job. But let us just assume that this is so and proceed.

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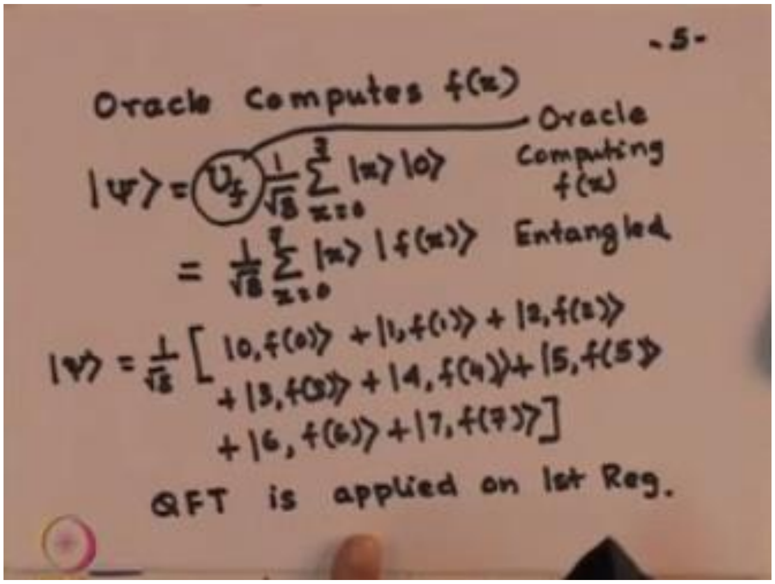

Notice one thing the if QFT is applied on the first register, since it is a tricky bit QFT each term gives me 8 terms. Since, I already have 8 terms when I apply QFT I will get 8x8=64 terms. Now obviously, I am not planning to write all of them down, but I would like you to have a feel of what these 64 terms how can they bring group together and things like that, so QFT on ψ.

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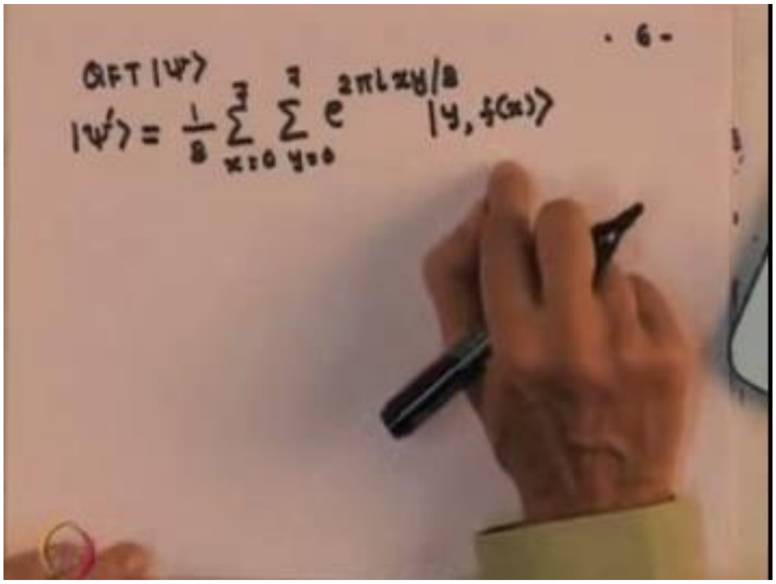

Let us call its if |ψ> FT prime, ψ′ so ψ′ then is given by I already had a 1/√8 in my definition of the input and this QFT gives me another √8 so I get 1/8 ∑x = 0 to 7∑y= 0 to7 e2лixy/8 y f (x) notice I left the second register untouched but applied the Fourier transform on the first register so first register we had x we had xfx so this now each x give me eight terms and I am left with 64 now do I have 64 terms what I am going to do is to group them and i will good them in the following way for each i will group.

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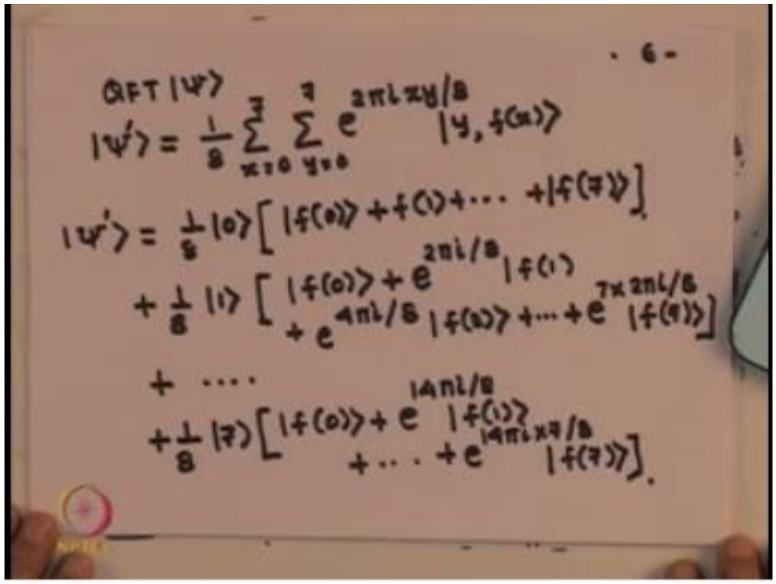

8 terms corresponding to each value of y so let me just show you how this should be done short let, let me rewrite Ψ so i have a 1 over 8 now supposing y = 0 in this then what do I have here now y = 0 this exponential calculation is easy irrespective of what is the value of your ex so each one the exponential to the power anything where there is a 0 that is = 1so therefore since y = 0 my sum is over 127 so therefore what I get here is f0 + f1up to f7 I will write down one more term when y is equal =1.

Remember that the y is being factored out so i will have only x varying from 0 to 7 so here why is being put equal to 1 so i will get the first term still equal to f0 because if x = 0 e^0= 1 but now i will start having this exponential factor i have y = 1 so i got 2лix = 1 y = 1 / 8 f1 +e to the power next time my x = 2 so I get 4лi y =1 by 8 f 2etc. And the last term will obviously be $e^{7 \text{ x } 2\text{лi} /8f(8)}$ it is impractical to write down the remaining terms so i will just put dot. Let us look at the last term also which is this so what you do here is to put.

The value of y = 7 x goes from 0 to 7 again so therefore i get f of 0is just a matter of multiplication $e^{14\text{лi} / 8}$ f1 + etc. $e^{14\text{лi}}$ x 7/ 8fnow notice that I have so far not utilized the property that i started with namely what i have is a periodic function so once we have group there is twenty sixty four terms in terms of what is the coefficient with which comes for y = 0 y = 1 up to y =7 what do we will show now is that if you assume a period in this example I will take the period to be equal to 2.

If you assume a period many of the terms will vanish leaving you with certain terms which depend upon what the assumed value of period is so let us assume the period to be equal to 2 so that I take.

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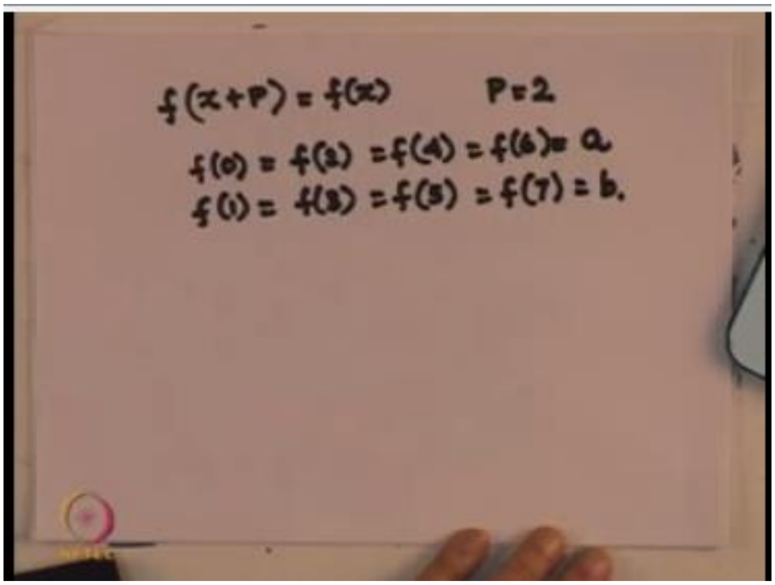

x + P the function of x + P = f of x and I will take p = 2 now if you take me to be equal to 2 then I have f0 = f2 = 4 = f6 = 8 + f 1 = f3 = f5 = f7 = b now out of these the first term.

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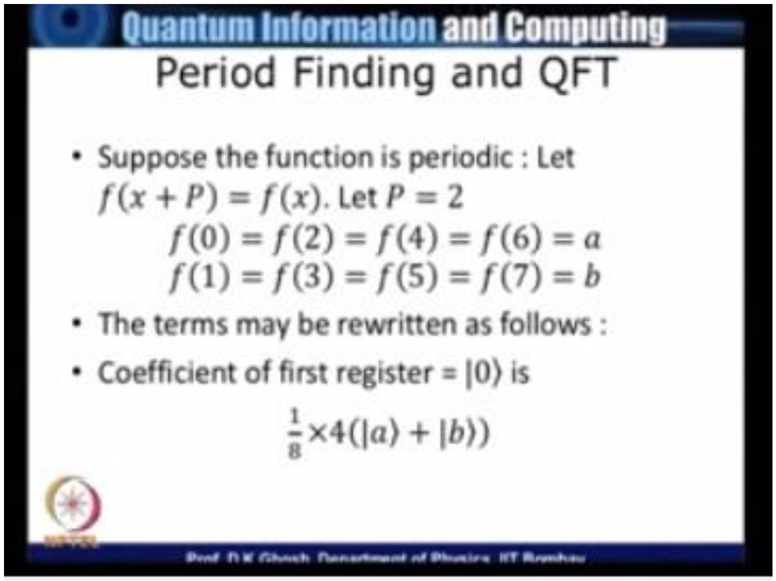

Is much easier.

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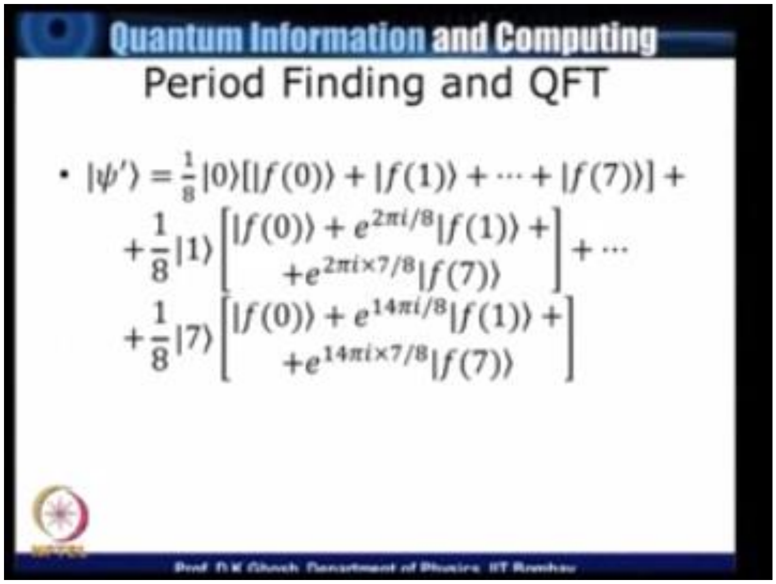

To understand because the first term.

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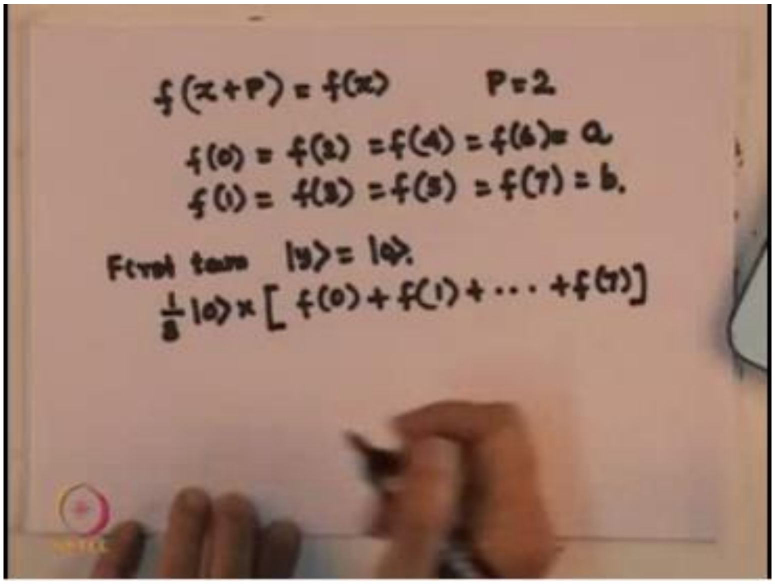

Which corresponds to y = 0 so what I have is 1 over 80 x remember this was f0 + f1 up to epsilon but we have said f0 = f2 =f4 = 6. For the state one what I get is this 1/8 again.

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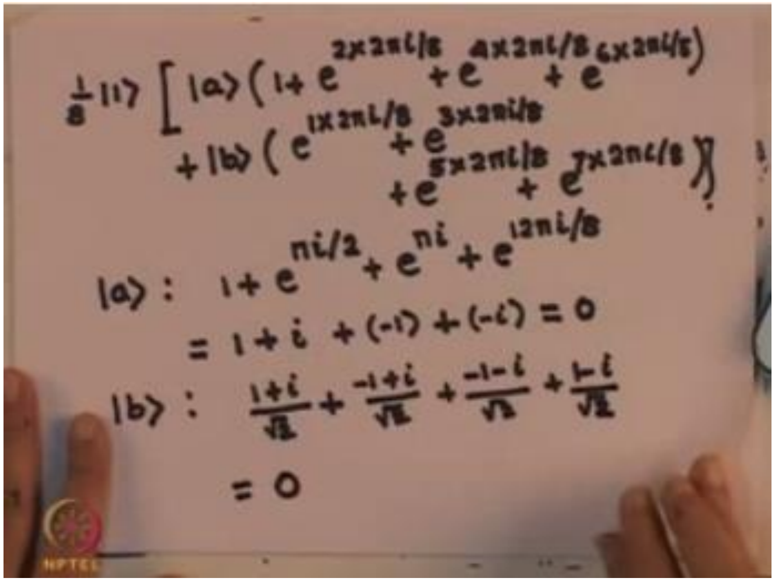

This is state 1 remember I have f_0 f 2_f 4 and f_6 of the same so therefore I put all of them together a x 1+ $e^{2x2\pi i/8}$ +$e^{4x2\pi i/8}$ + $e^{6x2\pi i/8}$ + b x $e^{1x2\pi i/8}$ just write it in a symmetric fashion $e^{3x2\pi i/8}$ + $e^{5x2\pi 8}$ and finally $e^{7x2\pi i/8}$. Now let us look at what do I have here for the coefficient of A what I have is one plus this is $e^{\pi i/4}$ so I get $e^{\pi i/2}$ + e to the power this is 4 x 2 so $e^{\pi i}$ + $e^{12\pi i/8}$.

So what do I actually get here so this is equal to 1 + $e^{\pi i/2}$ so which is $\cos^{\pi/2}$ +i $\sin^{\pi/2}$ so which is equal to 1 + i. Now this is $e^{\pi i}$ which is cosπ which is minus 1 plus i sinπ which is 0 so - 1 + $e^{12\pi i/8}$. which is 3/2πi which will simply give you – i so this adds up to 0. Now likewise you can do the coefficient for the state B now look at what do I get there I have $e^{\pi i/4}$ here so this is cosπ/4 + i sin/4 which is nothing but 1 +i/√2.

Do the next term you get -1+i/√2.because it is simply 3πi/4 we have to just be careful on which quadrant you are to find out the cosine and the sine value properly. So -1-i/√2and +1-i/√2which is equal to 0 number of i is will be equal to the number of –i and so also.

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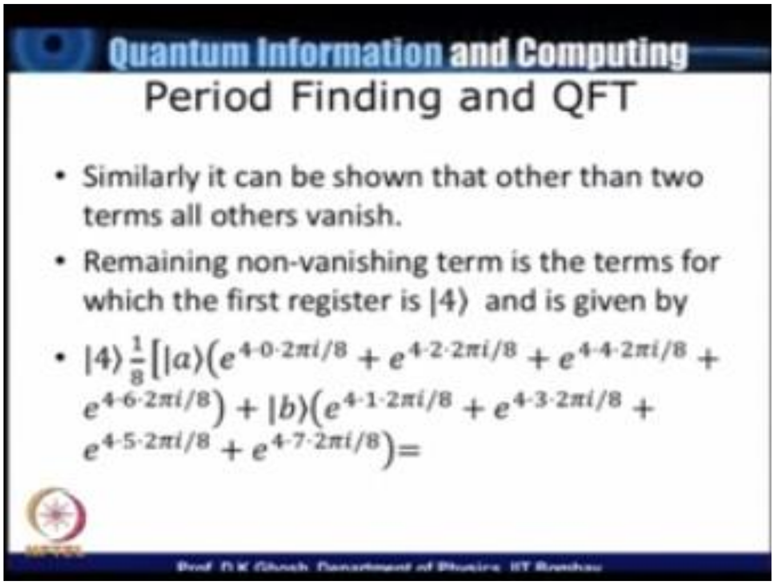

And likewise you can show that all other terms are also zero but that is not true of the state and which is the coefficient of 4.

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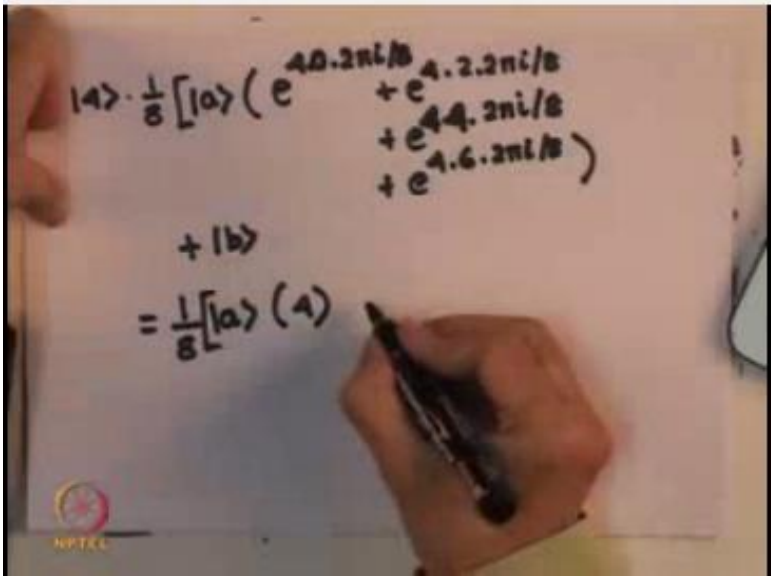

So in the coefficient of 4 what I get is 1/8 supposing I am just collecting the terms for A so I get $e^{4x0x2\pi i/8}$ this 0, 2, 4, 6, they will be running because they are the value of x that comes up, so $+e^4$ this 4 came from this 4 $e^{4x0x2\pi i/8} + e^{2x2\pi i/8} + e^{6x2x2\pi i/8}$ sorry, $e^{4x4x\ 2\pi i/8} + e^{4x6x2\pi i/8}$ and likewise for b. So it is 1/8 a x now notice this term is of course 1 because it is 0 what about this step this is $e^{16\pi i/8}$ but then both cos and sin of a periodicity of 2π so each one of these terms is equal to 1.

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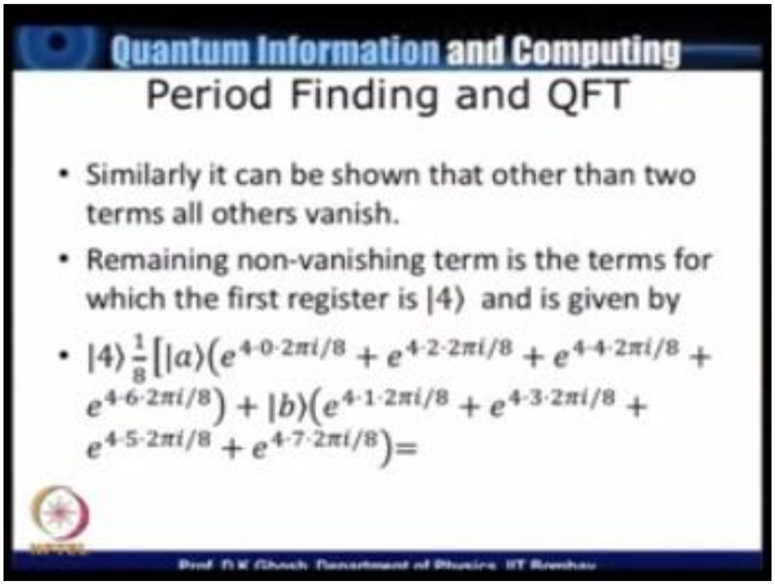

So this is 4.

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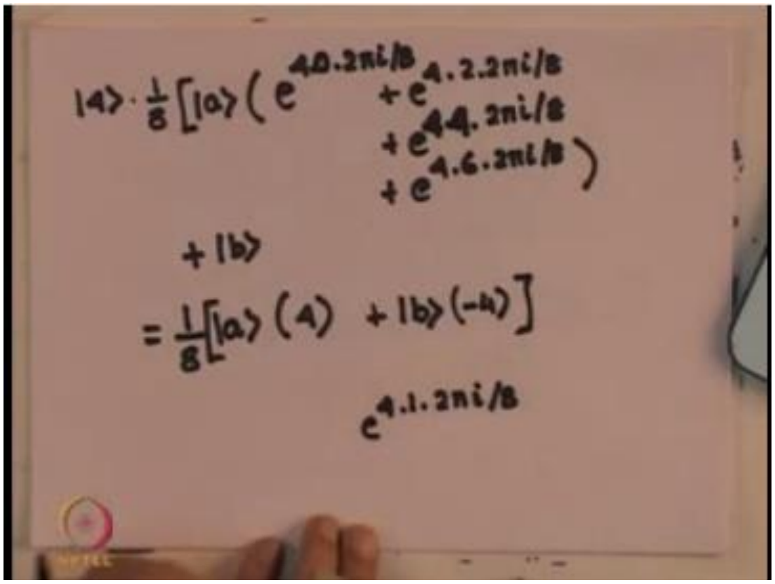

Similarly collect the terms for b I have given the details in the slide and you will find that each one of the terms happens to be equal to -1 because the first term there is $e^{4x1x2\pi i/8}$ which is $e^{8\pi i/8}$ which is $e^{\pi i}$cosπ + i sine πi this works out 2-1. Of course, the you can check that the each one of the terms will turn out to b - 1 and this will b - 1 but this overall sin is totally immaterial. So what I get when I do this is given in this slide.

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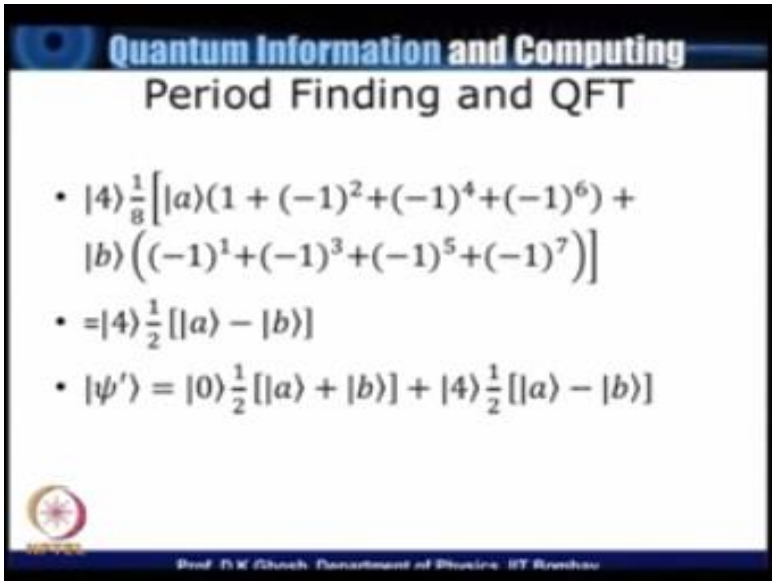

That I have for y equal to 0 I get half because there was a 1 / 8 outside A+B and the other one is 4 x a – b. So if I now measure the first register so if I get either 0 or 4 I can conclude that my periodicity was 2.

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