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Quantum Infromation and Computing

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Modul No.05

Lecture No.23

Quantum Fourier Transform

So far we have been talking about Grover's search algorithm and now I intend to switch over and discuss another major algorithm which is known as source factorizable, the reason behind discussing this algorithm is, this is one problem as we will see later, requires exponential time in classical computation but the quantum part of it can be completed in polynomial time. Now we will discuss it in detail as we go along in the next three or four lectures.

But before I can introduce you to the methods of source algorithm, I need to introduce some or acquainted with some mathematical preliminaries and today I will start with one of them and that is known as quantum Fourier transform.

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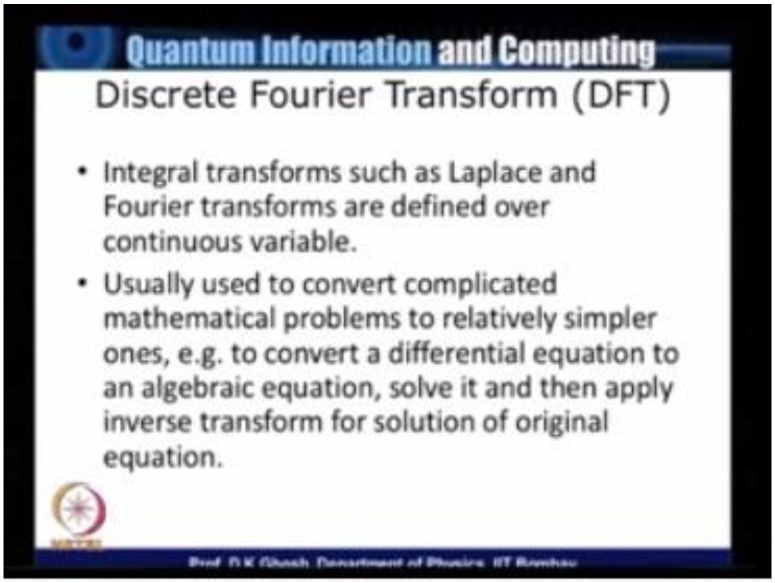

The quantum Fourier transform, if you all remember what is Fourier transform. Now normally in physics that we have or even in mathematics when we have been discussing various types of transforms, for example Laplace transform or Fourier transform etc, we have been doing it on a 20 was if, now in dealing with our quantum information and computation we are restricted to use discrete variables.

And so what we are going to do now, is to first introduce the concept general concept of a discrete integral transform and then having done that we will define a specific kernel for the quantum Fourier transform, remember what is the Kernel even in our standard definition of Fourier transform with or real variables the our definition was for example if I had a function of X f(x) then we said that we define its Fourier transform as some function of k $f(x) = e^{ik}$ x integrated over $e^{ikf(x)}$ from minus infinity plus infinity.

And for example if you wanted a Laplace transform then instead of e^{ikx} you define $e^{-\lambda x}$ where λ is real. So this thing which multiplies the original function and then we integrate out that is what is known as the kernel of the transformer.

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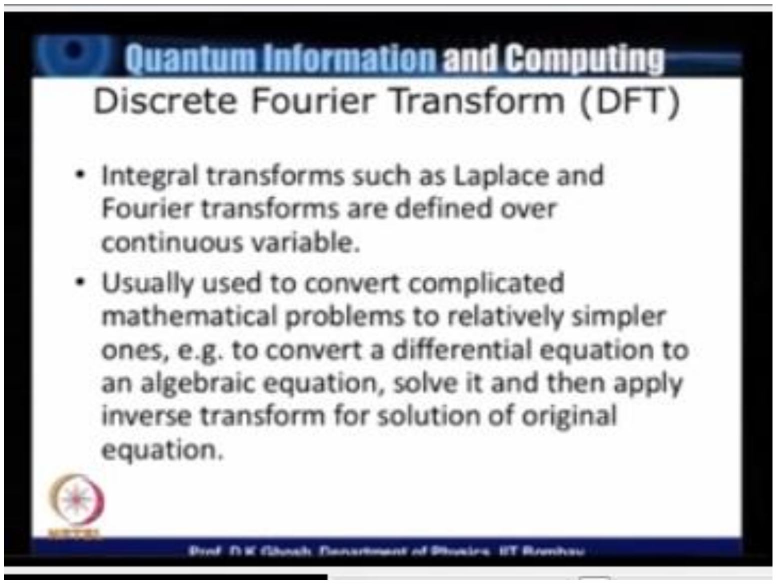

If you go to the slide what is the reason why you need it, so they are reused in many things where they Fourier transforms or Laplace transforms but typically their job has been to convert a mathematical problem to a relatively simple problem, for example you could be using it to solve differential equation and by taking a Fourier transform you could convert this to an algebraic equation which presumably is easier to solve.

And then of course since I need the original solution I apply an inverse transform so that I can get back the solution that I want.

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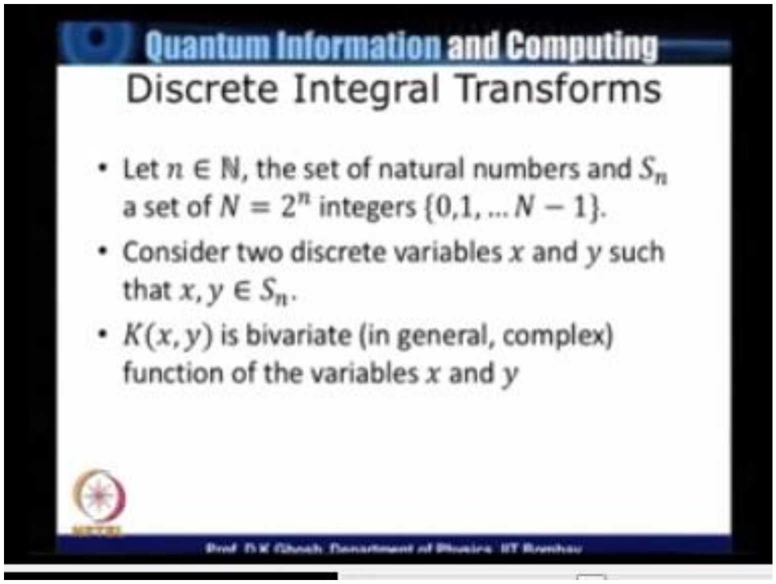

So let me first talk about a general discrete integral transform.

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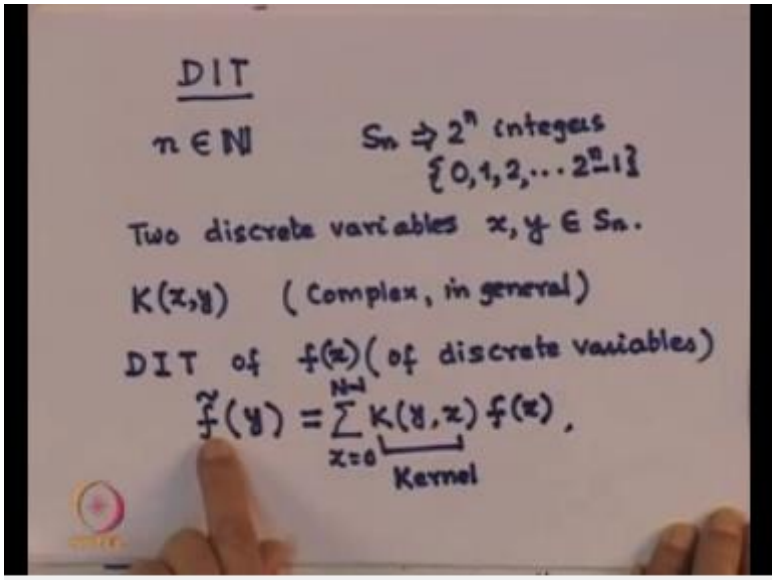

In short DIT, so let small m belong to the set of natural numbers and I will take S_n to be a set of numbers of which has 2^n integers, so this is the set consisting of 0, 1, 2 etc, etc upto 2^{n-1} now let me take P discrete variables belonging to the set, that we call one of them X and the other one Y each one of which can take these values 0, 1, 2^{n-1} So X, Y which belong to my S_n now I define K(x, y)as a bi-variate function.

Which is in general complex as a function of x and y, the difference between what we are talking about here and what we are accustomed to talking about in our real variable theory or complex variable theory is, that these arguments here can only take discrete values but k(x, y) could be any continuous function of these discrete variables as well, so I am not saying k has to be necessarily discrete.

Now let me define using these the discrete integral transform as a function f sum of F that this is a function of the discrete variables, so f is the function of X which is let us say discrete variable then f ~ (y) which is the DIT of the function f(x) is given by a sum instead of the integral with which we are familiar x = 0 to n-1 the complete set k(y, x)this is that bi-variate Kernel I talked about.

So this is the Kernel times f(x), now notice since x and y are both a discrete this is actually a set of simultaneous equation.

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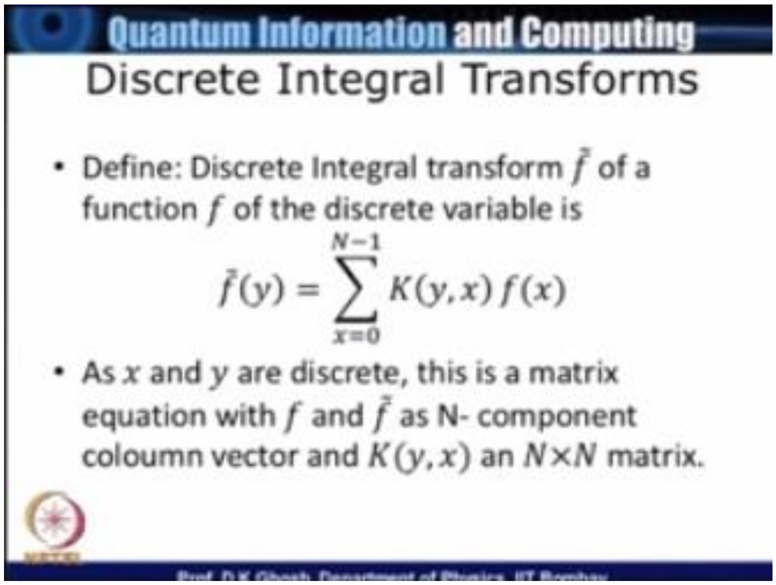

What we will do is this, that we will say that f(x) and f tilde (y) they are both column vectors.

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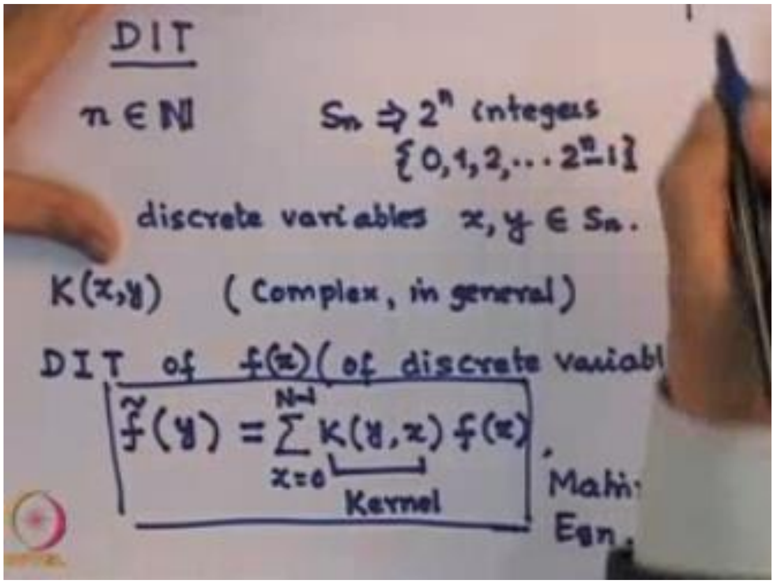

And then of course this becomes just a matrix equation this, the, the equation that I wrote down that becomes a, this is the matrix. Now let us suppose this kernel is inverted my general definition would also simply require it is invertible, but suppose I make an additional demand which is not required for this definition but is required when I talk about the kernel corresponding to the in quantum computation.

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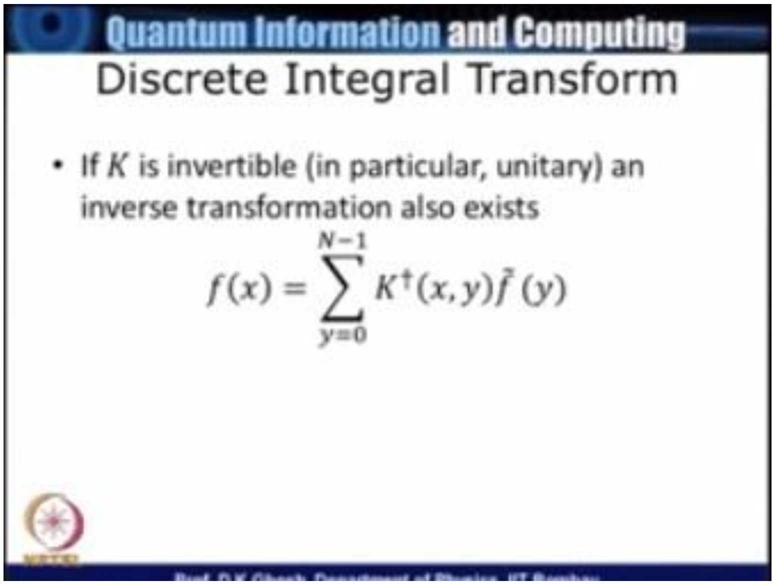

Supposing this K happens to be unit, so if K is unitary.

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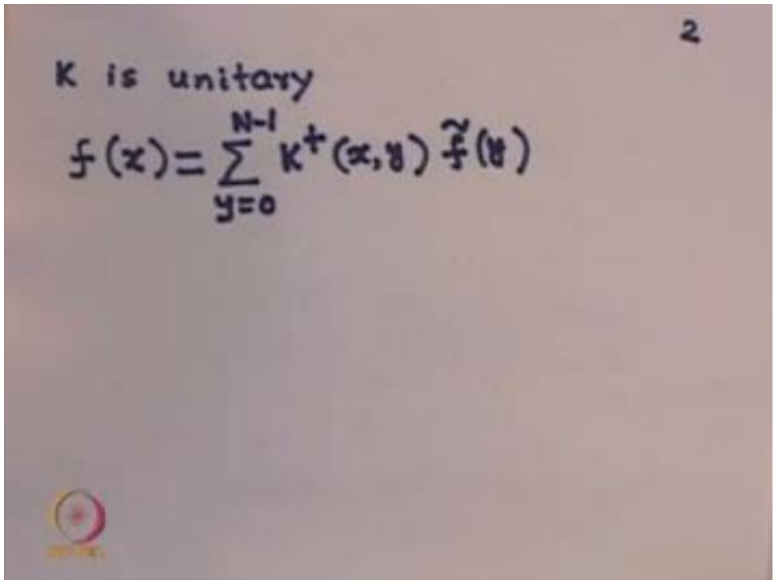

then I know that K^+ is defined and I can then talk about an inverse transformation that is starting from f tilde (y), y=0 to n- 1 again, and $K^+(x,y)$ f tilde (y), so this is my integral transform and I get back the function by applying and inverse integral transform. Now having discussed this let me.

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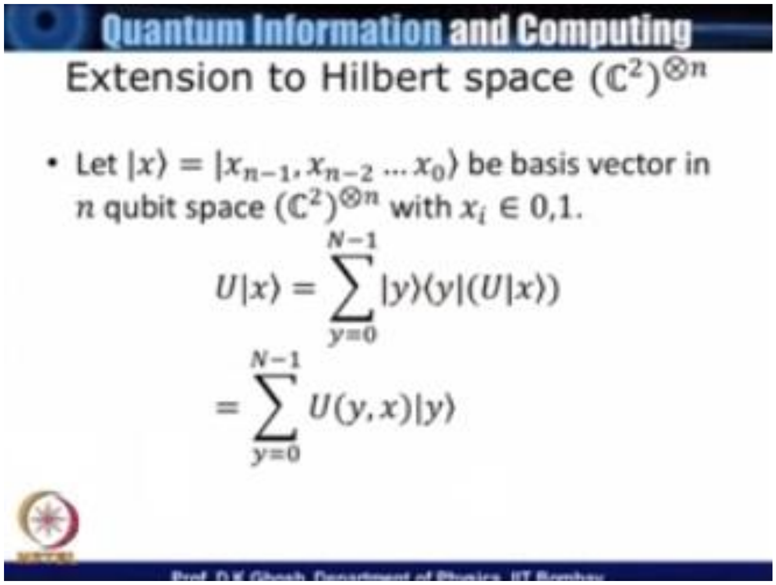

Now extend this concept to a Hilbert space of n qubits, C^2n that we have been talking about. Now suppose by vector x.

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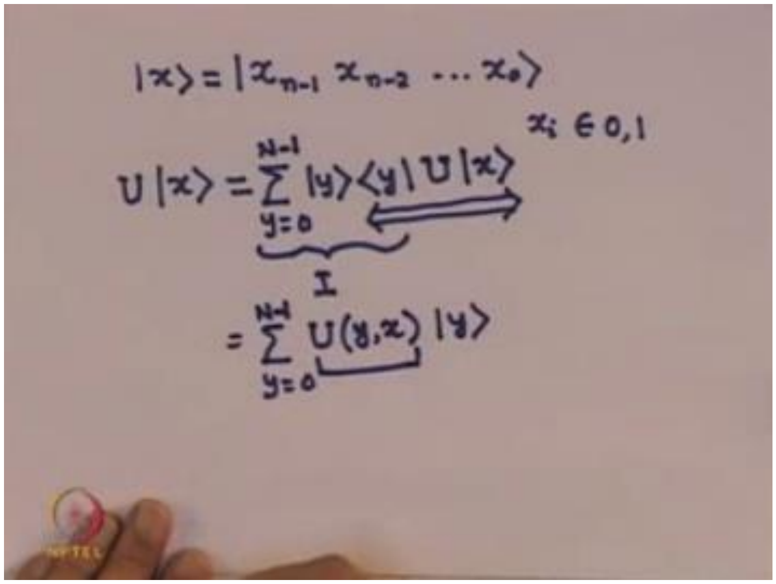

I indicated n qubit basis then it does not have to be basis, but let us say n qubit vector $x_{n\ -1}$, xn-2 extra up to x_0, where each of these numbers as we have seen x_i takes the value 0 or 1, then I would apply a unitary operator U|x> U you on x, now by completeness I can write this as y=0 to n-1, I introduce the identity operator here so this is my identity and then of course U|x>. Now you realize that this way of writing I have essentially written this as a matrix element of the unitary operator in my basis steps.

So this is nothing but ∑y=0 where N-1,U(y,x) which is the matrix element so this is the, this quantities definition acting on a vector one.

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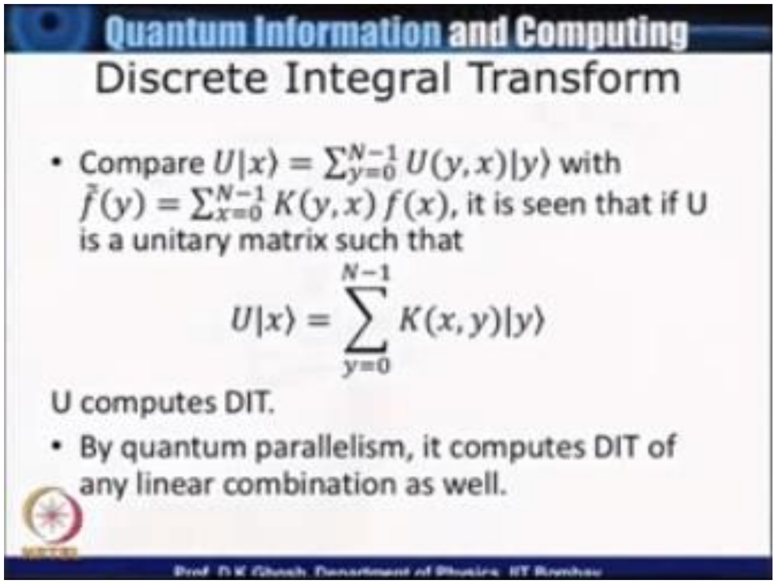

Now what I am going to do is this, I am going to take this statement.

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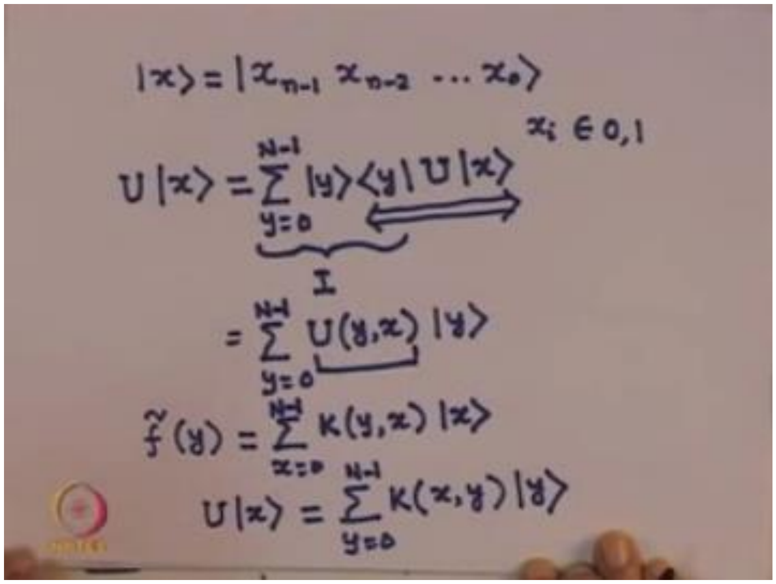

For the way and unitary operator work and I am going to ask to compare this with my definition of the Fourier transform or integral transform so this was equal to x=0 to N-1 K(y,x) |x> so you notice other them from some role change of x and y these two equations are essentially the same. Provided if I identify U to be a unitary matrix which gives me this, if U are acting on x gives me ∑y=0 to N-1 K(y,x) |y> then I can make a statement that this operator U the unitary operator U it computes that discrete integral transform of the distance the, the states that we have n qubit states. Now that is not all, the advantage that we have now is that because we follow a linear algebra and because of the quantum parallelism the DIT.

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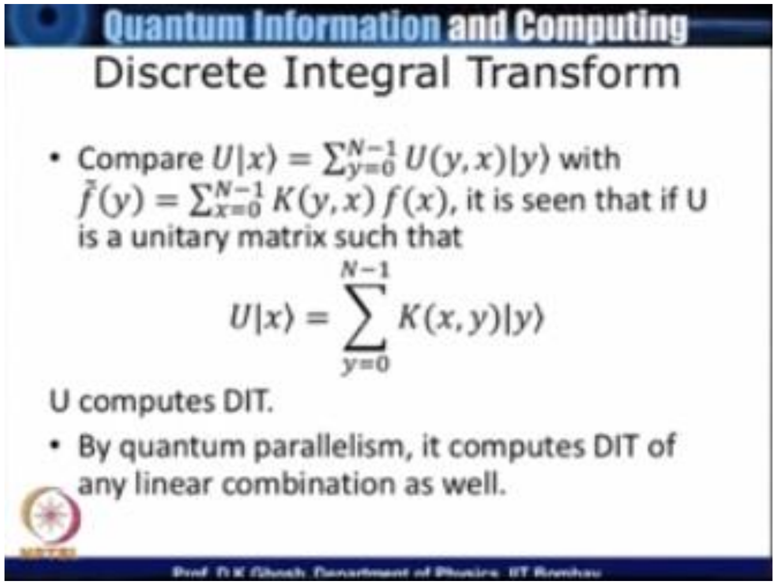

Of any function or any linear combination can be computed in one go.

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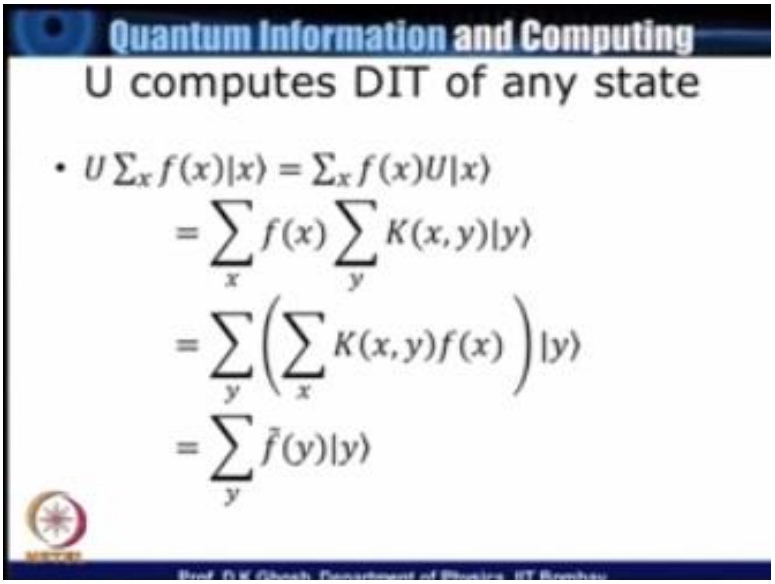

And to look at how it does that U acting on.

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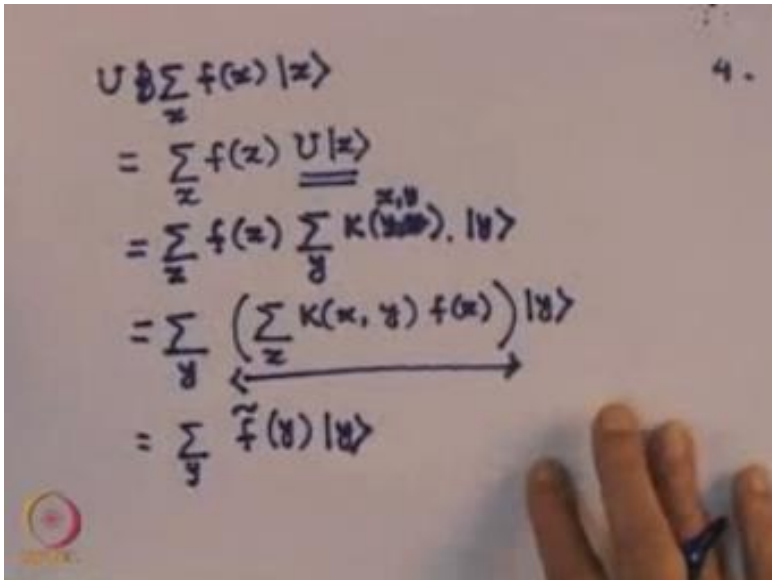

∑x f(x) |x> what does it do, so this is by linearity I have ∑x f(x) U acting on |x> and just now we said U acting on |x> is so I have a ∑x already F(x) and U acting on |x> was identified with K(y,x) actually it should have been x,y |y> I can because these are finite ∑ I can without problem interchange them, and so therefore I have this thing xK(x,y) f(x) |y> but if you recall this was precisely my definition of the integral transform.

So this is equal to ∑y f tilta(y) |y> so with that I have defined what is.

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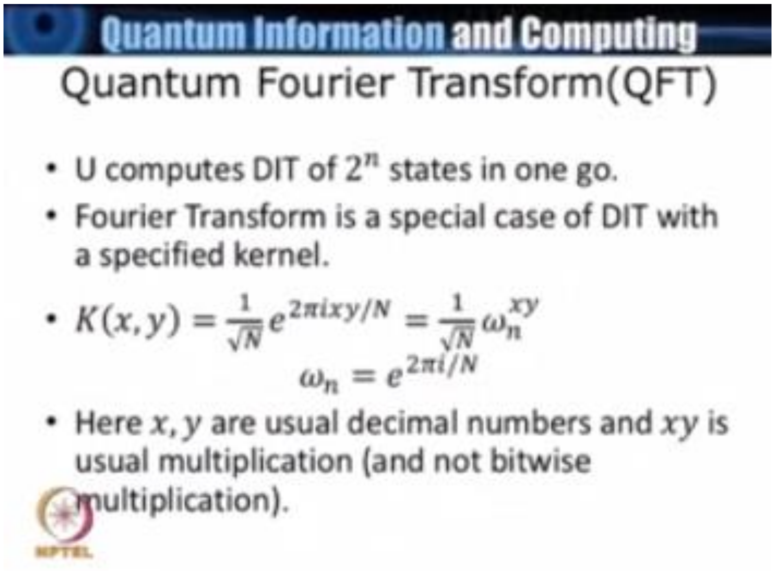

An integral transform.

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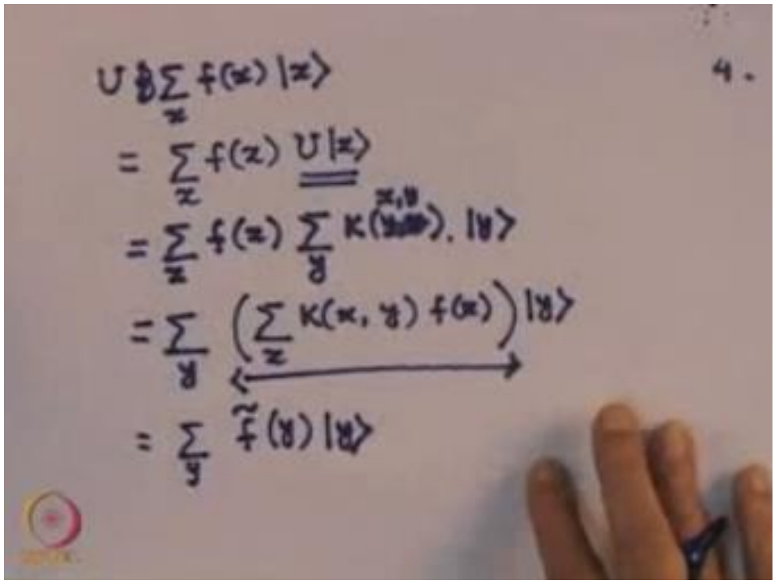

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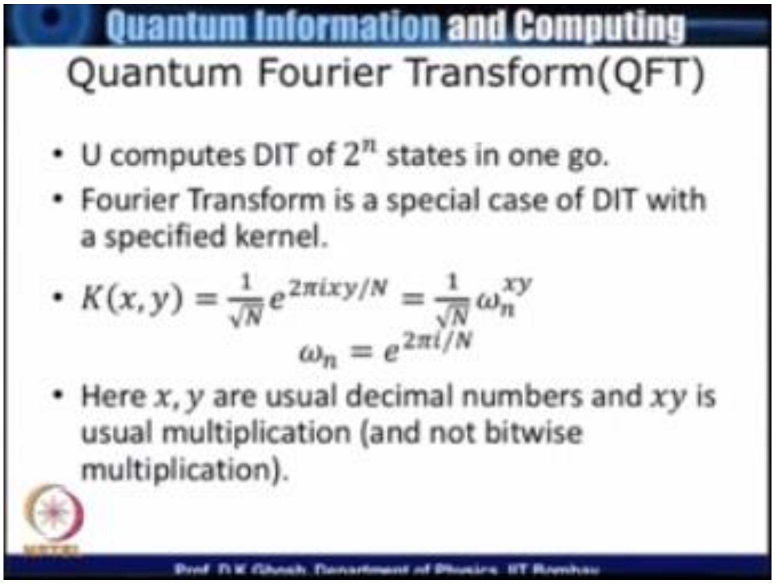

And now I specialize it to the case of a quantum Fourier transform which is simply extending this concept to a very special case where the colonel looks very similar to the way we had taken for the case of normal for Iran so what we do is this that quantum.

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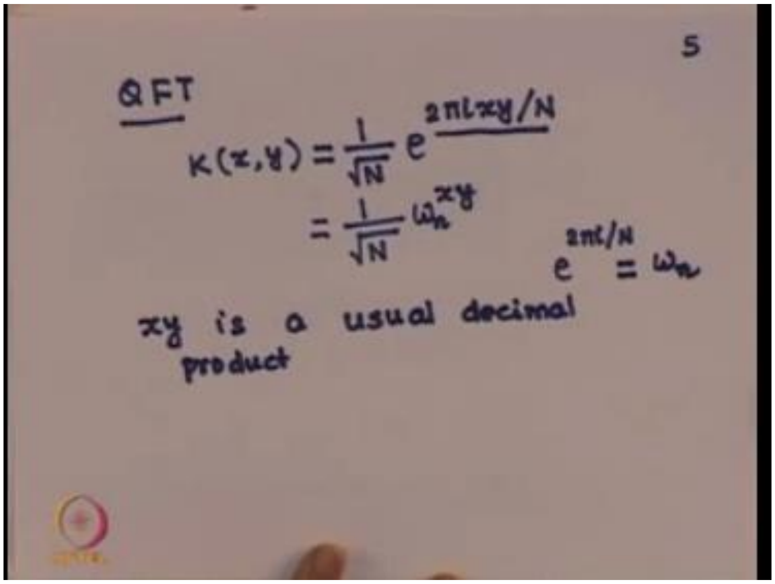

Fourier transform frequently written as QFT there my kernel K(x, y) will be taken to be 1 over $\sqrt{}$N $e^{2лxy}$/N you notice that e 2 ^{2л}i/N is nothing but n 8th root of unity and in usual complex variables we have been representing this as an Ωn so therefore this is 1 over $\sqrt{}$M $Ωn^{xy}$ one thing I want to point out that in this case the product xy is a normal product usual decimal product what do I mean by that because frequently we have been talking about bit wise move now in this case we take x and y which have a decimal representation.

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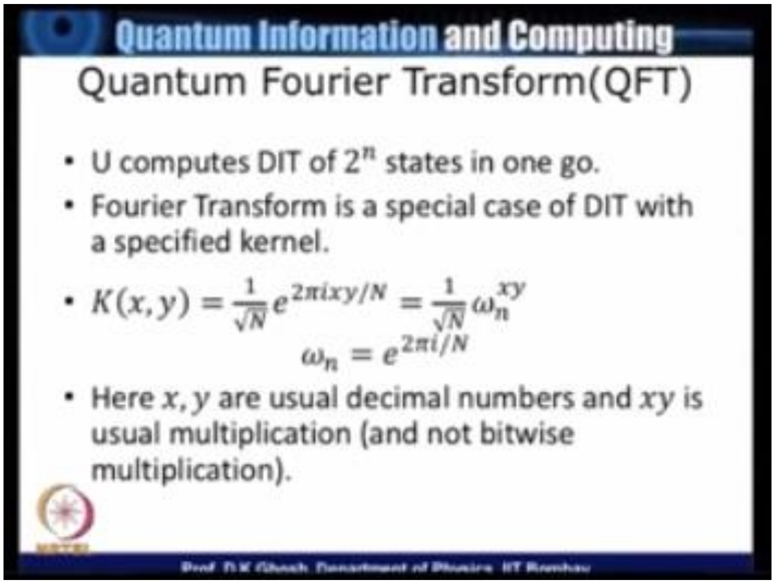

In the sense for example if you take 101 now then I know that, that 101 stands for a decimal number 5.

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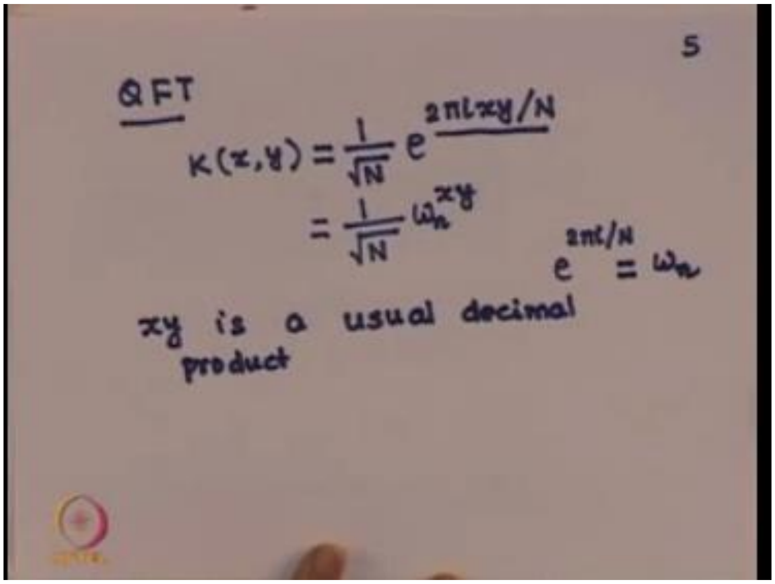

So here the x and y are usual decimal numbers and k(x , y is taken to represent that.

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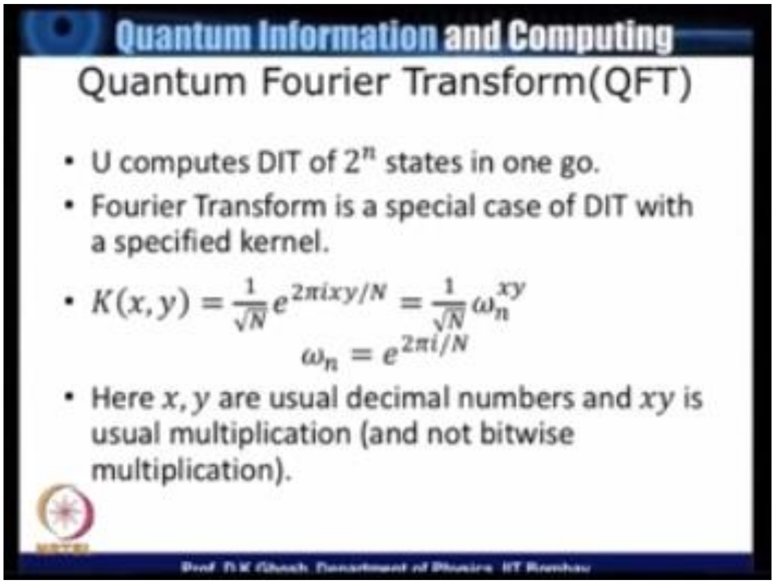

So not bit wise multiplication.

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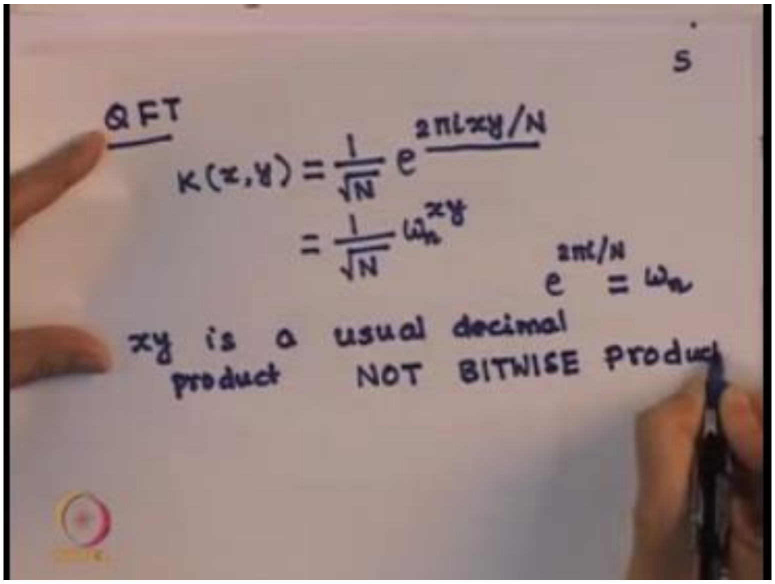

I will make it further emphasize it not bit wise.

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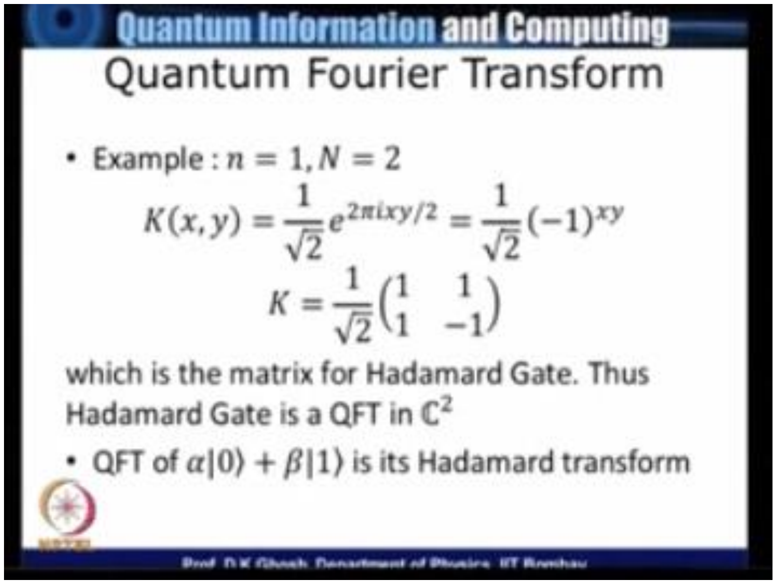

Let me let me give you an example.

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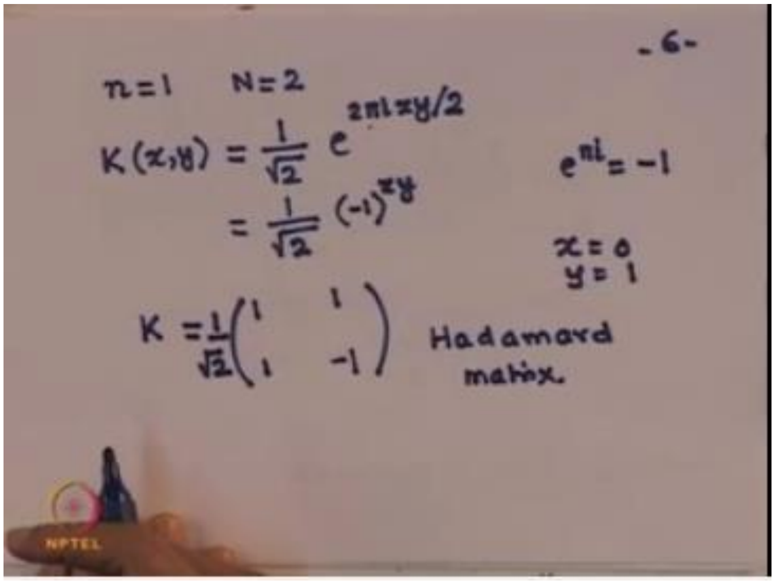

Of this let me take n = 1, 1 cubic so that capital N = 2 and in two-dimensional space by definition my k (x , y) is 1 over √N which is 1 √2 $e^{2лixy}$ / 2 because that was n now remember $e^{2лi}$ / 2 which is - 1 so therefore this is I know what is this value so let us look at this - 1^{xy} this is by D my first theorem e ^{лi} equal to $\cos^{лi}$ + i sinлi sine x 0 so that is the way it is so what will this matrix k remember that my x and y in this case can take values 0 and 1 so if I write a matrix this is 1 over √2 x equal to 0 y equal to 0 so it is 111 and then - 1recall this matrix this was the matrix which correspond it.

To Adam aggregate so this was Hadamard matrix in other words the Haddam at gate implements.

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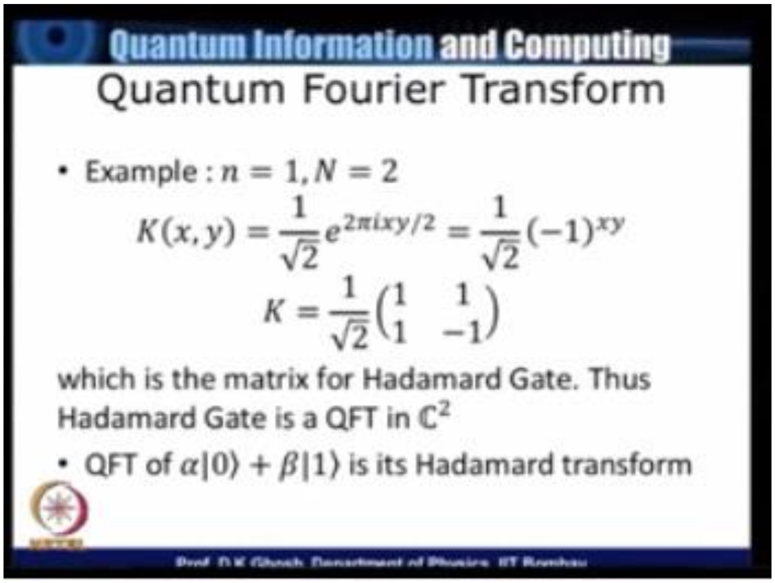

Quantum Fourier transform in situ so QFT.

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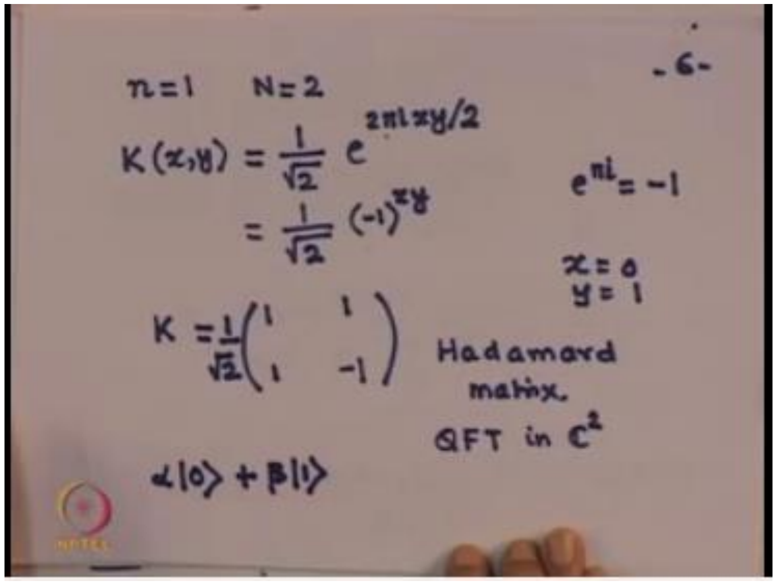

In C^2 is implemented by Hadamard gate and so therefore if you want to find out what is the QFT of α 0 plus β1 you could simply do it by means of whatever you have said just now and you can see that 0 becomes 0 + 1 by √2 one becomes 0 - 1 by √2.

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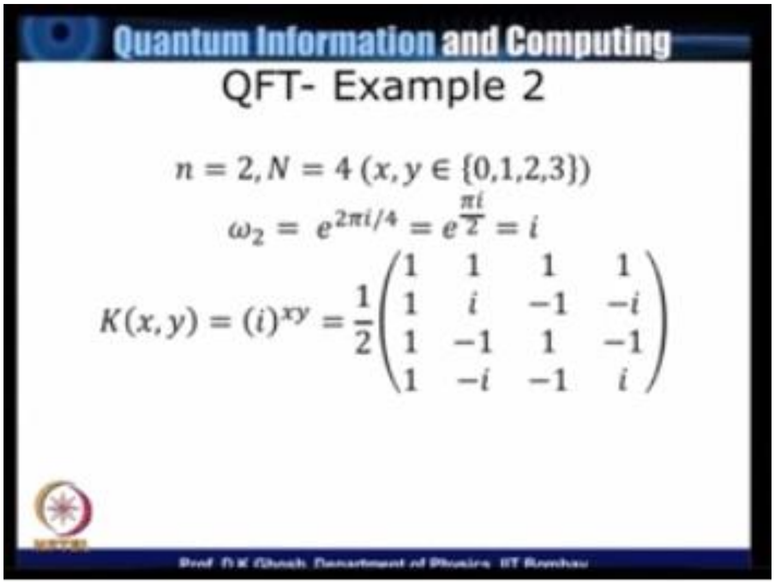

Let me continue with this example let me give the first nontrivial example.

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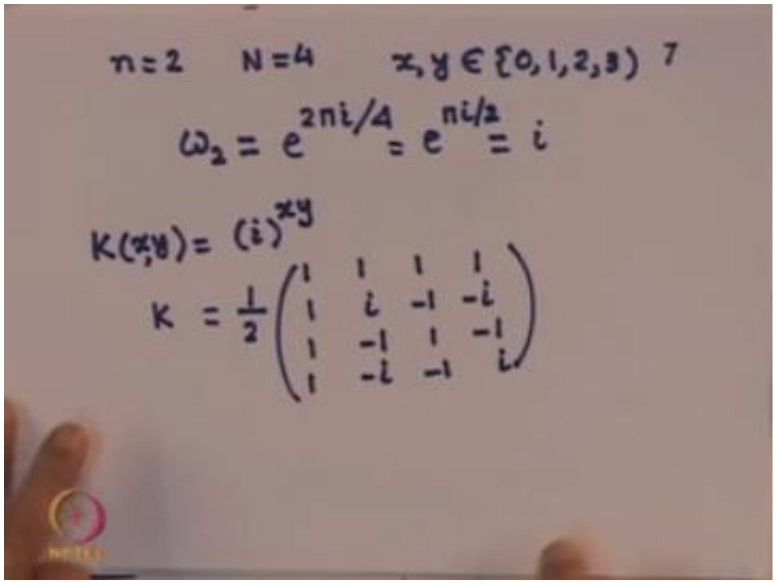

n =2 that means N = 4 so that my x and y belong to the set 0 1 2 3 now I have got w_2 = 2 which is e ^{2лi} /N = to 4 which is e лi / 2 so this is cosл by 2 which is 0 + i sine л by 2 so this is equal to I you can write down the matrix k, k (x , y) has elements given by i ^{xy} x and y running like this and show that the matrix k is given by this one 1111 I - 1 - i1 - 1 1 - 1 1- i - 1 summarizing the results.

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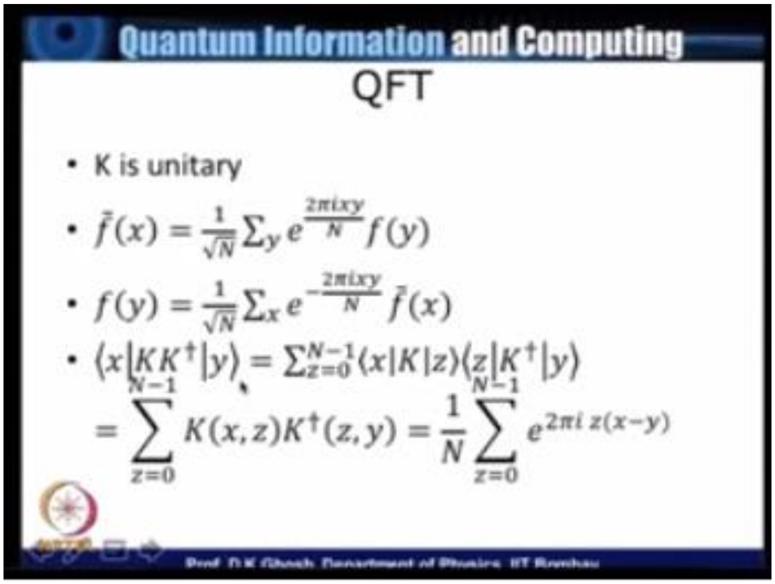

The first thing is to realize that k is unitary let us let us see why the slideshows the proof so firstly you realize that this was my definition of a free transform and this is the definition of the inverse Fourier transform so that what is the matrix element of k, k^+ in the states x and y.

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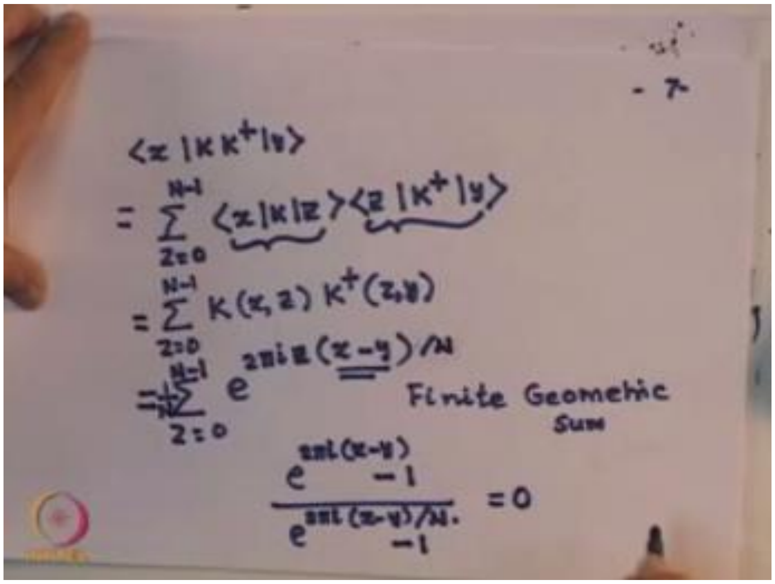

So x K, K together y now i evaluate it by the following I will introduce a complete set that is resolution of identity z = 0^{N-1} I have x k z z k^+ y and since I know these matrix elements by our definition of Fourier transform ∑so this is some ∑ z = 0 to n – 1 k XZ k^+ of z y and you can immediately write down that this is equal to z = 0 to n - 1 of $e2^{\pi i}$ z x – y. Now because of the fact that these are finite geometric sums, finite geometric sum.

I can easily compute this song but before I do that I have to realize that this term x should not be equal to Y now if X is not equal to Y so that this is not equal to 0 then my common ratio is $e2^{\pi i}$ times whatever that quantity is. So that my sum will then be to the $e2^{\pi i}$ X minus y minus 1divided by $e2^{\pi i}$ X minus y by n minus 1, since X is not equal to Y the denominator is not equal to 0 but the numerator becomes 0 so therefore this sum becomes 0.

But on the other hand if x happens to be equal to I had while writing is down add forgotten a factor1√N so I needed to introduce that1√N twice so therefore 1/N is there now so this is still equal to 0.

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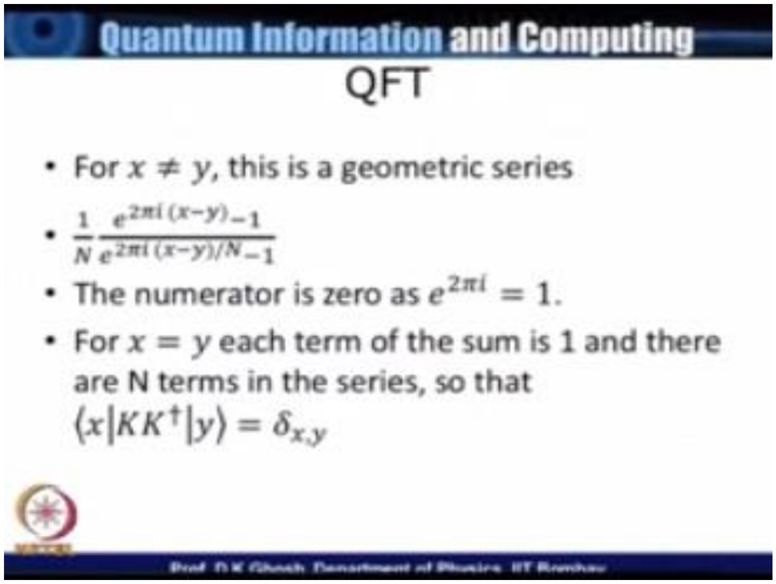

But if I have x is equal to y note that this term becomes 0 the, in because if it becomes zero then my $e2^0$= 1 allowed. So I have got n number of terms each one of them equal to 1 so therefore I get n as a sum and I am a 1/n which gives me 1, so therefore if x is not equal to y I get 0 if x = y I get 1.

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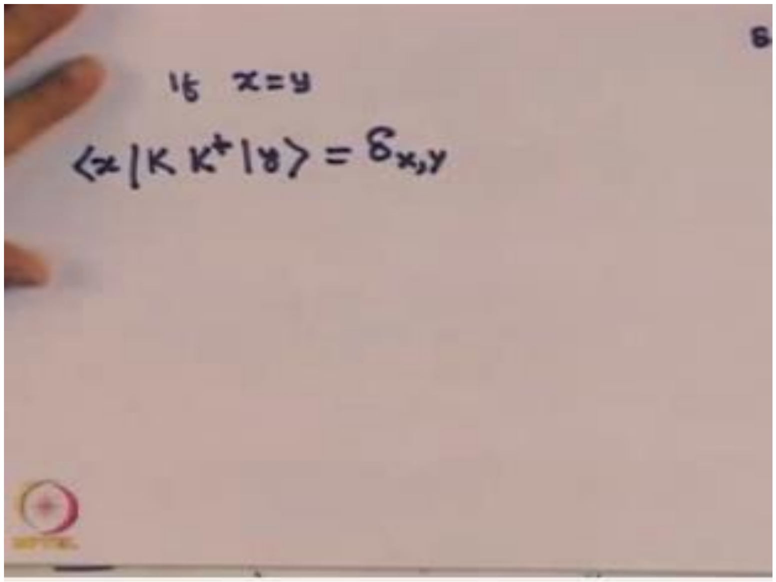

So therefore I will say that kk^+ matrix element X Y is simply given by the δ of x with y.

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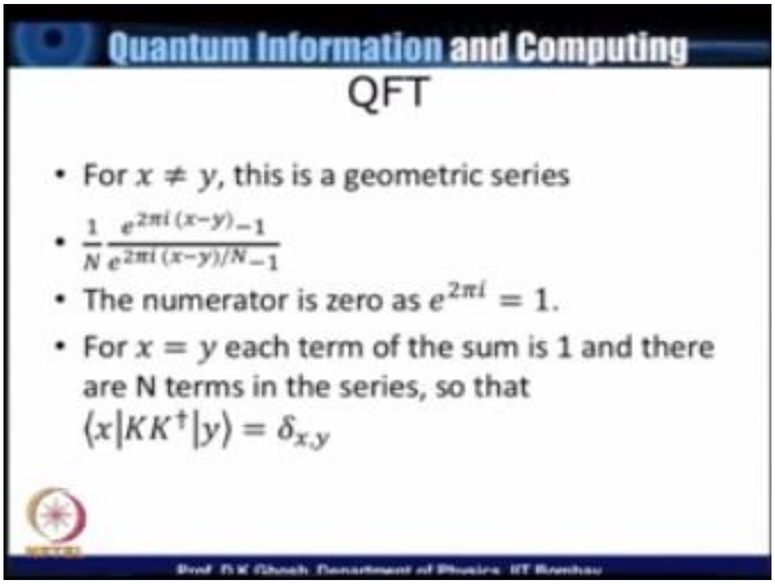

So what we have done today is to extend our concept of Fourier transform for real and complex variables continuous variables that we have learnt in physics and mathematics to the case where my variables are discreet because of the fact that we deal with sums rather than integration getting explicit representation of the colonel is fairly easy at least for those cases where the n value is small.

And we had seen that for n is equal to 2 in space c2 my quantum Fourier transform is equivalent to doing a Hadamard gate operation, we will see that quantum Fourier transform has a significant role to play in our discussion of source algorithm and as an exercise I will be giving you more examples of quantum Fourier transform in a separate session.

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