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NATIONAL PROGRAMME ON
TECHNOLOGY ENHANCED LEARNING

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Quantum Information and
Computing

Prof. D.K.Ghosh
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Modul No.05

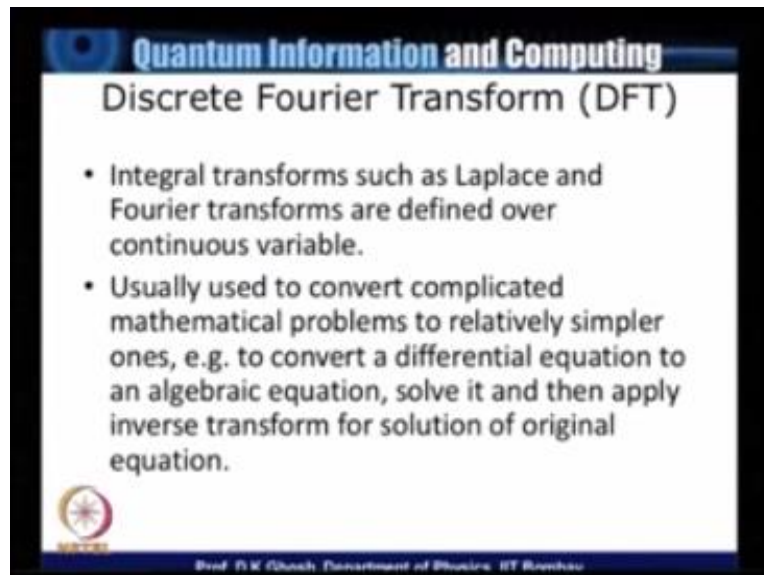
Lecture No.23

Quantum Fourier Transform

So far we have been talking about Grover's search algorithm and now I intend to switch over and discuss another major algorithm which is known as source factorizable, the reason behind discussing this algorithm is, this is one problem as we will see later, requires exponential time in classical computation but the quantum part of it can be completed in polynomial time. Now we will discuss it in detail as we go along in the next three or four lectures.

But before I can introduce you to the methods of source algorithm, I need to introduce some or acquainted with some mathematical preliminaries and today I will start with one of them and that is known as quantum Fourier transform.

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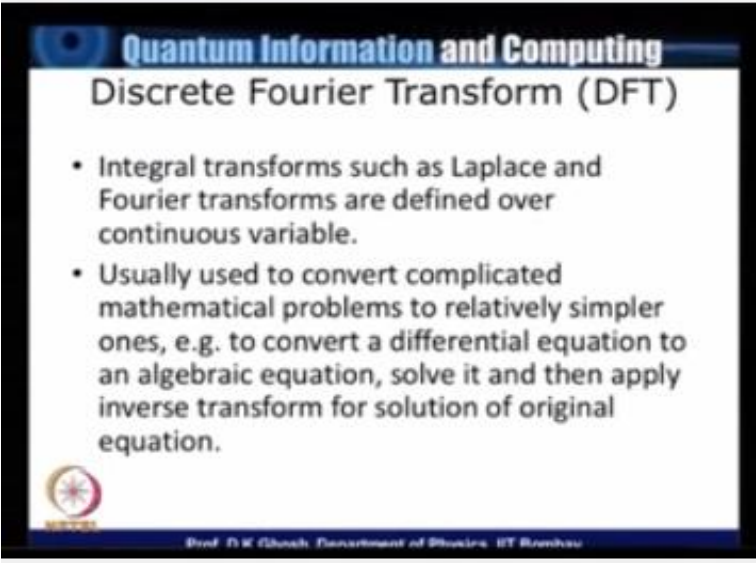


The quantum Fourier transform, if you all remember what is Fourier transform. Now normally in physics that we have or even in mathematics when we have been discussing various types of transforms, for example Laplace transform or Fourier transform etc, we have been doing it on a 2D as if, now in dealing with our quantum information and computation we are restricted to use discrete variables.

And so what we are going to do now, is to first introduce the concept general concept of a discrete integral transform and then having done that we will define a specific kernel for the quantum Fourier transform, remember what is the Kernel even in our standard definition of Fourier transform with or real variables the our definition was for example if I had a function of $X f(x)$ then we said that we define its Fourier transform as some function of $k f(x) = e^{ikx}$ integrated over $e^{ikf(x)}$ from minus infinity plus infinity.

And for example if you wanted a Laplace transform then instead of e^{ikx} you define $e^{-\lambda x}$ where λ is real. So this thing which multiplies the original function and then we integrate out that is what is known as the kernel of the transformer.

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Quantum Information and Computing

Discrete Fourier Transform (DFT)

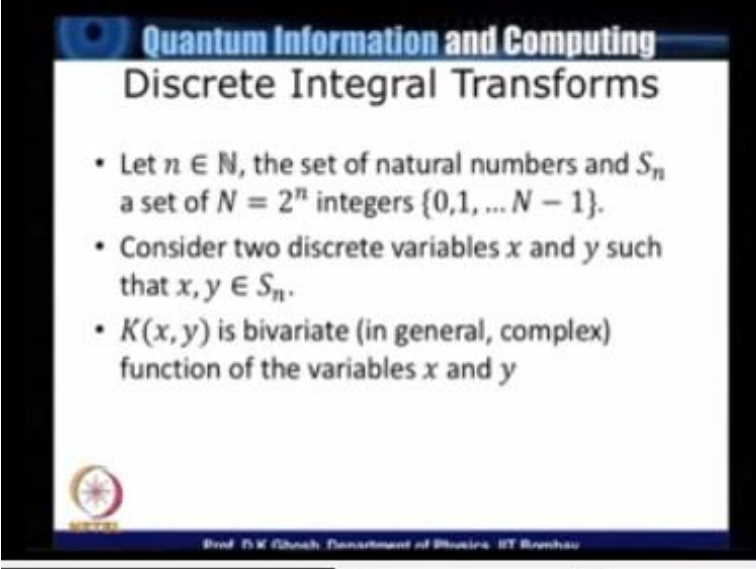
- Integral transforms such as Laplace and Fourier transforms are defined over continuous variable.
- Usually used to convert complicated mathematical problems to relatively simpler ones, e.g. to convert a differential equation to an algebraic equation, solve it and then apply inverse transform for solution of original equation.

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If you go to the slide what is the reason why you need it, so they are reused in many things where they Fourier transforms or Laplace transforms but typically their job has been to convert a mathematical problem to a relatively simple problem, for example you could be using it to solve differential equation and by taking a Fourier transform you could convert this to an algebraic equation which presumably is easier to solve.


And then of course since I need the original solution I apply an inverse transform so that I can get back the solution that I want.

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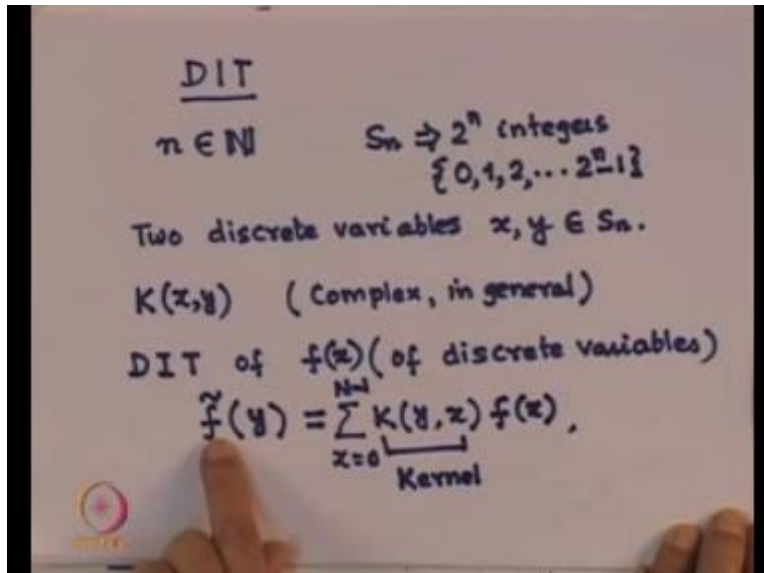
Quantum Information and Computing
Discrete Integral Transforms

- Let $n \in \mathbb{N}$, the set of natural numbers and S_n a set of $N = 2^n$ integers $\{0, 1, \dots, N - 1\}$.
- Consider two discrete variables x and y such that $x, y \in S_n$.
- $K(x, y)$ is bivariate (in general, complex) function of the variables x and y


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So let me first talk about a general discrete integral transform.

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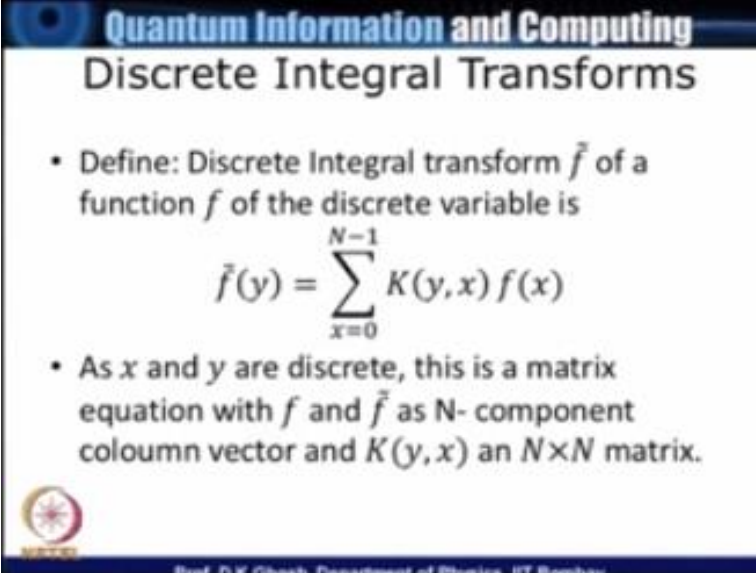
In short DIT, so let small m belong to the set of natural numbers and I will take S_n to be a set of numbers of which has 2^n integers, so this is the set consisting of 0, 1, 2 etc, etc upto 2^{n-1} now let me take P discrete variables belonging to the set, that we call one of them X and the other one Y each one of which can take these values 0, 1, 2^{n-1} So X, Y which belong to my S_n now I define $K(x, y)$ as a bi-variate function.

Which is in general complex as a function of x and y , the difference between what we are talking about here and what we are accustomed to talking about in our real variable theory or complex variable theory is, that these arguments here can only take discrete values but $k(x, y)$ could be any continuous function of these discrete variables as well, so I am not saying k has to be necessarily discrete.

Now let me define using these the discrete integral transform as a function f sum of F that this is a function of the discrete variables, so f is the function of X which is let us say discrete variable then $f \sim (y)$ which is the DIT of the function $f(x)$ is given by a sum instead of the integral with which we are familiar $x = 0$ to $n-1$ the complete set $k(y, x)$ this is that bi-variate Kernel I talked about.


So this is the Kernel times $f(x)$, now notice since x and y are both discrete this is actually a set of simultaneous equations.

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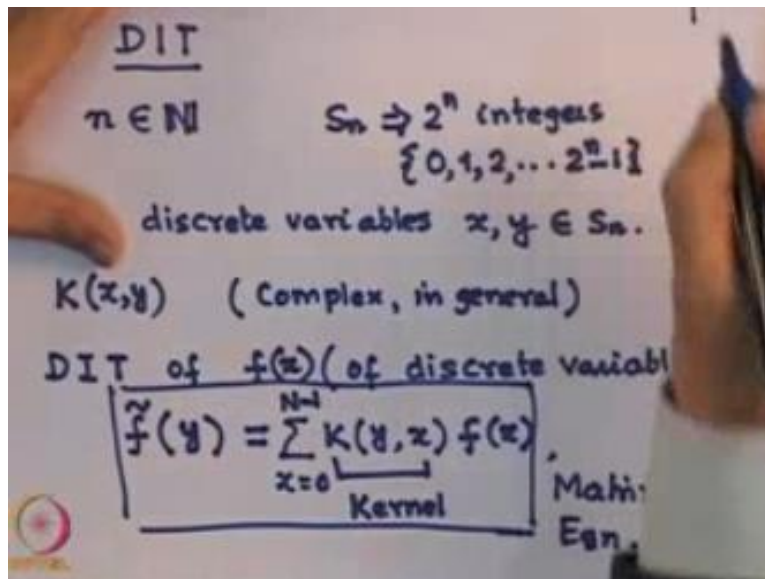
Quantum Information and Computing
Discrete Integral Transforms

- Define: Discrete Integral transform \vec{f} of a function f of the discrete variable is
$$\vec{f}(y) = \sum_{x=0}^{N-1} K(y, x) f(x)$$
- As x and y are discrete, this is a matrix equation with f and \vec{f} as N -component column vector and $K(y, x)$ an $N \times N$ matrix.


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What we will do is this, that we will say that $f(x)$ and f tilde (y) they are both column vectors.

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And then of course this becomes just a matrix equation this, the, the equation that I wrote down that becomes a, this is the matrix. Now let us suppose this kernel is inverted my general definition would also simply require it is invertible, but suppose I make an additional demand which is not required for this definition but is required when I talk about the kernel corresponding to the in quantum computation.


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Discrete Integral Transform

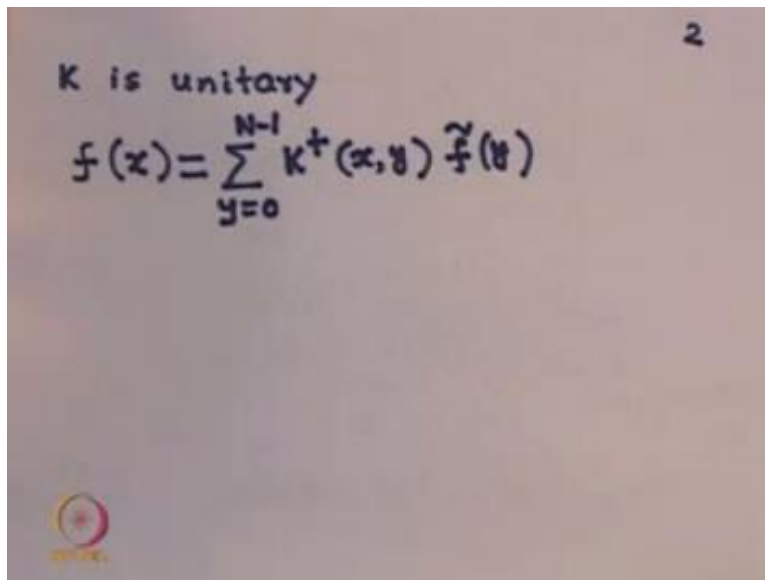
- If K is invertible (in particular, unitary) an inverse transformation also exists

$$f(x) = \sum_{y=0}^{N-1} K^\dagger(x,y) \tilde{f}(y)$$


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Supposing this K happens to be unit, so if K is unitary.

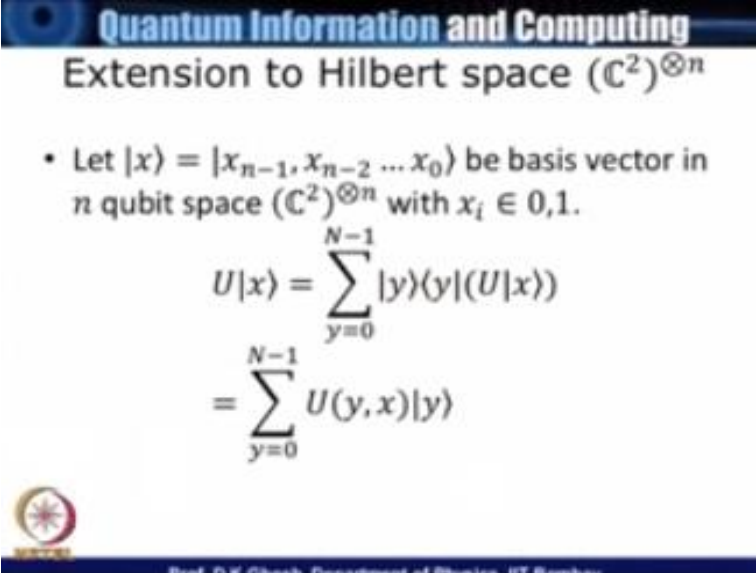
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A handwritten mathematical equation on a slide. The text "K is unitary" is written at the top left. The equation is $f(x) = \sum_{y=0}^{N-1} K^+(x,y) \tilde{f}(y)$. The number "2" is written in the top right corner. A small logo is visible in the bottom left corner of the slide.

then I know that K^+ is defined and I can then talk about an inverse transformation that is starting from $\tilde{f}(y)$, $y=0$ to $n-1$ again, and $K^+(x,y) \tilde{f}(y)$, so this is my integral transform and I get back the function by applying an inverse integral transform. Now having discussed this let me.


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Quantum Information and Computing

Extension to Hilbert space $(\mathbb{C}^2)^{\otimes n}$

- Let $|x\rangle = |x_{n-1}, x_{n-2} \dots x_0\rangle$ be basis vector in n qubit space $(\mathbb{C}^2)^{\otimes n}$ with $x_i \in \{0,1\}$.

$$U|x\rangle = \sum_{y=0}^{N-1} |y\rangle \langle y|(U|x\rangle)$$
$$= \sum_{y=0}^{N-1} U(y,x)|y\rangle$$


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Now extend this concept to a Hilbert space of n qubits, \mathbb{C}^{2^n} that we have been talking about. Now suppose by vector x .

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$$\begin{aligned} |x\rangle &= |x_{n-1} x_{n-2} \dots x_0\rangle \\ U|x\rangle &= \sum_{y=0}^{N-1} |y\rangle \langle y| U|x\rangle \\ &= \sum_{y=0}^{N-1} \underbrace{I}_{\text{I}} U(y,x) |y\rangle \end{aligned} \quad x_i \in \{0,1\}$$

I indicated n qubit basis then it does not have to be basis, but let us say n qubit vector x_{n-1}, x_{n-2} extra up to x_0 , where each of these numbers as we have seen x_i takes the value 0 or 1, then I would apply a unitary operator $U|x\rangle$ on x , now by completeness I can write this as $y=0$ to $n-1$, I introduce the identity operator here so this is my identity and then of course $U|x\rangle$. Now you realize that this way of writing I have essentially written this as a matrix element of the unitary operator in my basis steps.

So this is nothing but $\sum_{y=0}^{N-1} U(y,x)$ which is the matrix element so this is the, this quantities definition acting on a vector one.

(Refer Slide Time: 10:37)

Quantum Information and Computing
Discrete Integral Transform

- Compare $U|x\rangle = \sum_{y=0}^{N-1} U(y,x)|y\rangle$ with $\tilde{f}(y) = \sum_{x=0}^{N-1} K(y,x)f(x)$, it is seen that if U is a unitary matrix such that

$$U|x\rangle = \sum_{y=0}^{N-1} K(x,y)|y\rangle$$

U computes DIT.

- By quantum parallelism, it computes DIT of any linear combination as well.

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Now what I am going to do is this, I am going to take this statement.

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$$\begin{aligned} |x\rangle &= |x_{n-1} x_{n-2} \dots x_0\rangle \\ & \quad x_i \in \{0,1\} \\ U|x\rangle &= \sum_{y=0}^{N-1} |y\rangle \langle y|U|x\rangle \\ & \quad \text{I} \\ &= \sum_{y=0}^{N-1} U(y,x) |y\rangle \\ \tilde{f}(y) &= \sum_{x=0}^{N-1} K(y,x) |x\rangle \\ U|x\rangle &= \sum_{y=0}^{N-1} K(x,y) |y\rangle \end{aligned}$$

For the way and unitary operator work and I am going to ask to compare this with my definition of the Fourier transform or integral transform so this was equal to $x=0$ to $N-1$ $K(y,x) |x\rangle$ so you notice other than from some role change of x and y these two equations are essentially the same. Provided if I identify U to be a unitary matrix which gives me this, if U are acting on x gives me $\sum_{y=0}^{N-1} K(y,x) |y\rangle$ then I can make a statement that this operator U the unitary operator U it computes that discrete integral transform of the distance the, the states that we have n qubit states. Now that is not all, the advantage that we have now is that because we follow a linear algebra and because of the quantum parallelism the DIT.

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Quantum Information and Computing


Discrete Integral Transform

- Compare $U|x\rangle = \sum_{y=0}^{N-1} U(y,x)|y\rangle$ with $\tilde{f}(y) = \sum_{x=0}^{N-1} K(y,x) f(x)$, it is seen that if U is a unitary matrix such that

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Of any function or any linear combination can be computed in one go.

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U computes DIT of any state

- $U \sum_x f(x)|x\rangle = \sum_x f(x)U|x\rangle$
 $= \sum_x f(x) \sum_y K(x,y)|y\rangle$
 $= \sum_y \left(\sum_x K(x,y)f(x) \right) |y\rangle$
 $= \sum_y \tilde{f}(y)|y\rangle$



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And to look at how it does that U acting on.

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$$\begin{aligned} U \sum_x f(x) |x\rangle &= \sum_x f(x) U|x\rangle \\ &= \sum_x f(x) \sum_y K(y,x) |y\rangle \\ &= \sum_y \left(\sum_x K(y,x) f(x) \right) |y\rangle \\ &= \sum_y \tilde{f}(y) |y\rangle \end{aligned}$$

$\sum_x f(x) |x\rangle$ what does it do, so this is by linearity I have $\sum_x f(x) U$ acting on $|x\rangle$ and just now we said U acting on $|x\rangle$ is so I have a \sum_x already $F(x)$ and U acting on $|x\rangle$ was identified with $K(y,x)$ actually it should have been $x,y |y\rangle$ I can because these are finite \sum I can without problem interchange them, and so therefore I have this thing $xK(x,y) f(x) |y\rangle$ but if you recall this was precisely my definition of the integral transform.

So this is equal to $\sum_y \tilde{f}(y) |y\rangle$ so with that I have defined what is.

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Quantum Information and Computing

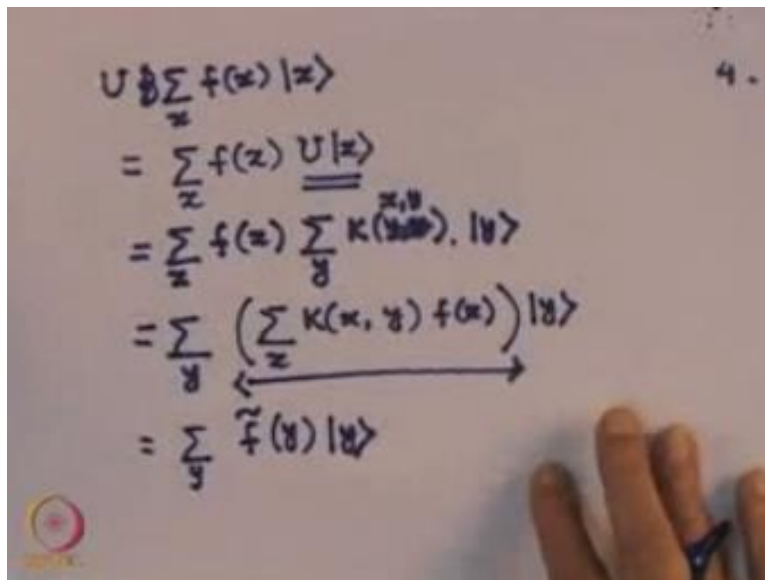
Quantum Fourier Transform(QFT)

- U computes DIT of 2^n states in one go.
- Fourier Transform is a special case of DIT with a specified kernel.
- $K(x, y) = \frac{1}{\sqrt{N}} e^{2\pi i xy/N} = \frac{1}{\sqrt{N}} \omega_n^{xy}$
 $\omega_n = e^{2\pi i/N}$
- Here x, y are usual decimal numbers and xy is usual multiplication (and not bitwise multiplication).



An integral transform.

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$$\begin{aligned} & U \hat{O} \sum_x f(x) |x\rangle \\ &= \sum_x f(x) \underline{U|x\rangle} \\ &= \sum_x f(x) \sum_y K(x,y) |y\rangle \\ &= \sum_y \left(\sum_x K(x,y) f(x) \right) |y\rangle \\ &= \sum_y \tilde{f}(y) |y\rangle \end{aligned}$$


(Refer Slide Time: 13:55)

Quantum Information and Computing

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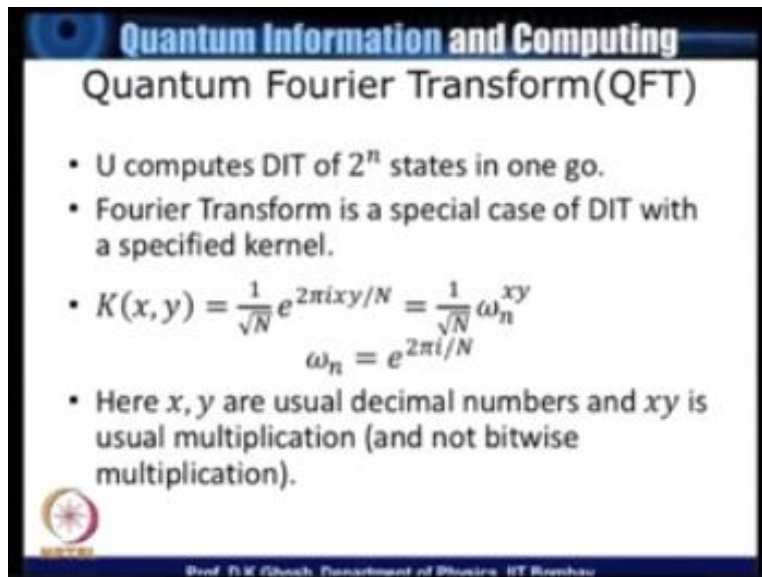
And now I specialize it to the case of a quantum Fourier transform which is simply extending this concept to a very special case where the kernel looks very similar to the way we had taken for the case of normal for Iran so what we do is this that quantum.

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The image shows a whiteboard with handwritten mathematical equations. At the top left, the word "QFT" is written and underlined. To the right, the number "5" is written. The main equation is $K(x, y) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i xy}{N}}$. Below this, it is simplified to $= \frac{1}{\sqrt{N}} \omega_n^{xy}$. To the right of this, a small note says $e^{\frac{2\pi i}{N}} = \omega_n$. At the bottom, a note says "xy is a usual decimal product". In the bottom left corner, there is a small logo with a red and yellow circle.

Fourier transform frequently written as QFT there my kernel $K(x, y)$ will be taken to be $\frac{1}{\sqrt{N}} e^{\frac{2\pi i xy}{N}}$ you notice that $e^{\frac{2\pi i}{N}}$ is nothing but N th root of unity and in usual complex variables we have been representing this as an ω_n so therefore this is $\frac{1}{\sqrt{N}} \omega_n^{xy}$ one thing I want to point out that in this case the product xy is a normal product usual decimal product what do I mean by that because frequently we have been talking about bit wise move now in this case we take x and y which have a decimal representation.

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Quantum Information and Computing

Quantum Fourier Transform (QFT)

- U computes DIT of 2^n states in one go.
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- $$K(x, y) = \frac{1}{\sqrt{N}} e^{2\pi i xy/N} = \frac{1}{\sqrt{N}} \omega_n^{xy}$$
$$\omega_n = e^{2\pi i/N}$$
- Here x, y are usual decimal numbers and xy is usual multiplication (and not bitwise multiplication).

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In the sense for example if you take 101 now then I know that, that 101 stands for a decimal number 5.

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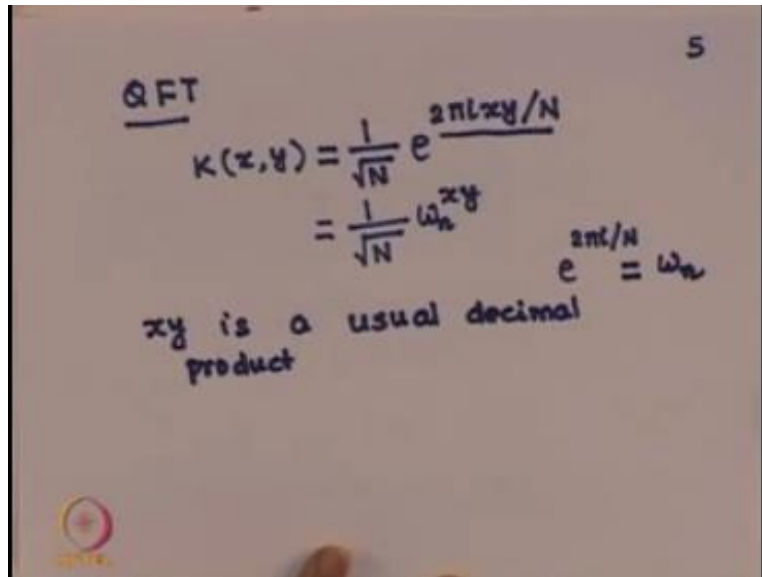
5

QFT

$$k(x, y) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i xy}{N}}$$
$$= \frac{1}{\sqrt{N}} \omega_N^{xy}$$

xy is a usual decimal product

$e^{\frac{2\pi i}{N}} = \omega_N$



So here the x and y are usual decimal numbers and $k(x, y)$ is taken to represent that.

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Quantum Information and Computing

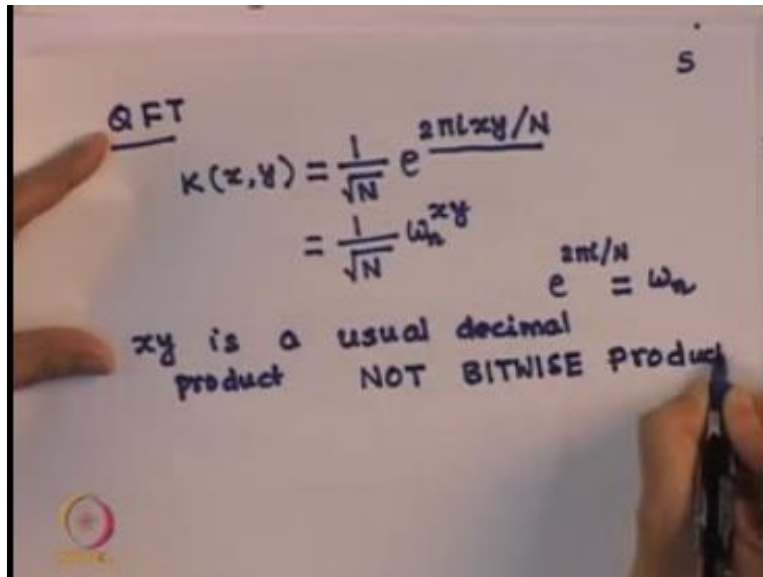
Quantum Fourier Transform(QFT)

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- Here x, y are usual decimal numbers and xy is usual multiplication (and not bitwise multiplication).

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So not bit wise multiplication.

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The image shows a whiteboard with handwritten mathematical equations and text. At the top left, the word "QFT" is written and underlined. To its right, the letter "S" is written. The main equation is $K(x, y) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i xy}{N}}$. Below this, it is simplified to $= \frac{1}{\sqrt{N}} \omega_N^{xy}$. To the right of this, the expression $e^{\frac{2\pi i}{N}}$ is equated to ω_N . At the bottom, the text reads "xy is a usual decimal product NOT BITWISE product". A small logo is visible in the bottom left corner of the whiteboard.

QFT

S

$$K(x, y) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i xy}{N}}$$
$$= \frac{1}{\sqrt{N}} \omega_N^{xy}$$

$e^{\frac{2\pi i}{N}} = \omega_N$

xy is a usual decimal product NOT BITWISE product

I will make it further emphasize it not bit wise.

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Quantum Information and Computing
Quantum Fourier Transform

- Example : $n = 1, N = 2$

$$K(x, y) = \frac{1}{\sqrt{2}} e^{2\pi i x y / 2} = \frac{1}{\sqrt{2}} (-1)^{xy}$$
$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is the matrix for Hadamard Gate. Thus Hadamard Gate is a QFT in \mathbb{C}^2

- QFT of $\alpha|0\rangle + \beta|1\rangle$ is its Hadamard transform

 Prof. D.V. Choudhary, Department of Physics, IIT Roorkee

Let me let me give you an example.

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The image shows a handwritten derivation on a whiteboard. At the top right, there is a small number '-6-'. The main derivation starts with $n=1$ and $N=2$. The kernel function is defined as $K(x,y) = \frac{1}{\sqrt{2}} e^{2\pi i xy/2}$. This is simplified to $= \frac{1}{\sqrt{2}} (-1)^{xy}$. To the right, it is noted that $e^{\pi i} = -1$. Below this, the values $x=0$ and $y=1$ are specified. The resulting matrix is given as $K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is identified as the Hadamard matrix.

Of this let me take $n = 1$, $N = 2$ and in two-dimensional space by definition my $k(x, y)$ is $1/\sqrt{N}$ which is $1/\sqrt{2} e^{2\pi i xy/2}$ because that was n now remember $e^{2\pi i}/2$ which is -1 so therefore this is I know what is this value so let us look at this -1^{xy} this is by De Moivre's theorem $e^{i\pi}$ equal to $\cos \pi + i \sin \pi$ since $x=0$ so that is the way it is so what will this matrix k remember that my x and y in this case can take values 0 and 1 so if I write a matrix this is $1/\sqrt{2}$ x equal to 0 y equal to 0 so it is 1 1 and then -1 recall this matrix this was the matrix which correspond it.

To Adam aggregate so this was Hadamard matrix in other words the Hadamard matrix implements.

(Refer Slide Time: 18:28)

Quantum Information and Computing
Quantum Fourier Transform

- Example : $n = 1, N = 2$

$$K(x, y) = \frac{1}{\sqrt{2}} e^{2\pi i x y / 2} = \frac{1}{\sqrt{2}} (-1)^{xy}$$
$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is the matrix for Hadamard Gate. Thus Hadamard Gate is a QFT in \mathbb{C}^2

- QFT of $\alpha|0\rangle + \beta|1\rangle$ is its Hadamard transform

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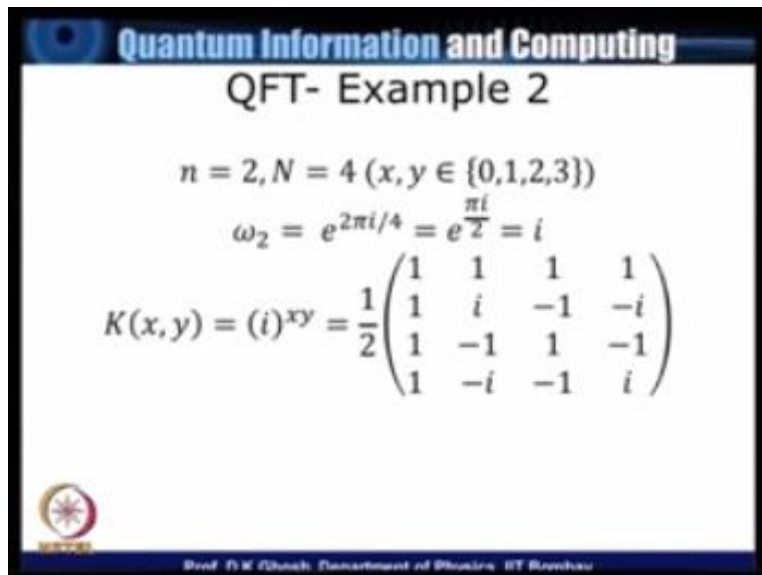
Quantum Fourier transform in situ so QFT.

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The image shows a handwritten derivation on a whiteboard. At the top right, there is a small number '-6-'. The main derivation starts with $n=1$ and $N=2$. The kernel function is given as $K(x,y) = \frac{1}{\sqrt{2}} e^{2\pi i xy/2}$, which simplifies to $\frac{1}{\sqrt{2}} (-1)^{xy}$. To the right, it is noted that $e^{\pi i} = -1$. Below this, the Hadamard matrix is written as $K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. To the right of the matrix, it is labeled 'Hadamard matrix' and 'QFT in \mathbb{C}^2 '. At the bottom left, the state $\alpha|0\rangle + \beta|1\rangle$ is written. In the bottom left corner, there is a small logo for NPTEL.

In \mathbb{C}^2 is implemented by Hadamard gate and so therefore if you want to find out what is the QFT of $\alpha|0\rangle + \beta|1\rangle$ you could simply do it by means of whatever you have said just now and you can see that $|0\rangle$ becomes $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ becomes $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(Refer Slide Time: 19:01)




Quantum Information and Computing

QFT- Example 2

$n = 2, N = 4 (x, y \in \{0,1,2,3\})$

$\omega_2 = e^{2\pi i/4} = e^{\frac{\pi i}{2}} = i$

$K(x, y) = (i)^{xy} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$



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Let me continue with this example let me give the first nontrivial example.

(Refer Slide Time: 19:12)

$$n=2 \quad N=4 \quad x, y \in \{0, 1, 2, 3\}$$
$$\omega_2 = e^{2\pi i/4} = e^{\pi i/2} = i$$
$$K(x, y) = (i)^{xy}$$
$$K = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$n=2$ that means $N=4$ so that my x and y belong to the set $0, 1, 2, 3$ now I have got $\omega_2 = i$ which is $e^{2\pi i/N} = e^{\pi i/2}$ so this is $\cos \pi/2 + i \sin \pi/2$ which is $0 + i$ so this is equal to i you can write down the matrix K , $K(x, y)$ has elements given by i^{xy} x and y running like this and show that the matrix K is given by this one $\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$ summarizing the results.

(Refer Slide Time: 20:25)

The slide is titled "Quantum Information and Computing" and "QFT". It lists the following properties:

- K is unitary
- $\tilde{f}(x) = \frac{1}{\sqrt{N}} \sum_y e^{\frac{2\pi i x y}{N}} f(y)$
- $f(y) = \frac{1}{\sqrt{N}} \sum_x e^{-\frac{2\pi i x y}{N}} \tilde{f}(x)$
- $\langle x | K K^\dagger | y \rangle = \sum_{z=0}^{N-1} \langle x | K | z \rangle \langle z | K^\dagger | y \rangle$
 $= \sum_{z=0} K(x, z) K^\dagger(z, y) = \frac{1}{N} \sum_{z=0} e^{2\pi i z(x-y)}$

At the bottom of the slide, there is a small logo on the left and the text "Prof. T.K. Ghosh, Department of Physics, IIT Bombay" on the right.

The first thing is to realize that k is unitary let us let us see why the slideshows the proof so firstly you realize that this was my definition of a free transform and this is the definition of the inverse Fourier transform so that what is the matrix element of k , k^\dagger in the states x and y .

(Refer Slide Time: 20:59)

The image shows a handwritten derivation on a whiteboard. The steps are as follows:

$$\langle x | K^+ | y \rangle$$

$$= \sum_{z=0}^{N-1} \langle x | K | z \rangle \langle z | K^+ | y \rangle$$

$$= \sum_{z=0}^{N-1} K(x, z) K^+(z, y)$$

$$= \frac{1}{N} \sum_{z=0}^{N-1} e^{2\pi i z (x-y)/N}$$

Finite Geometric Sum

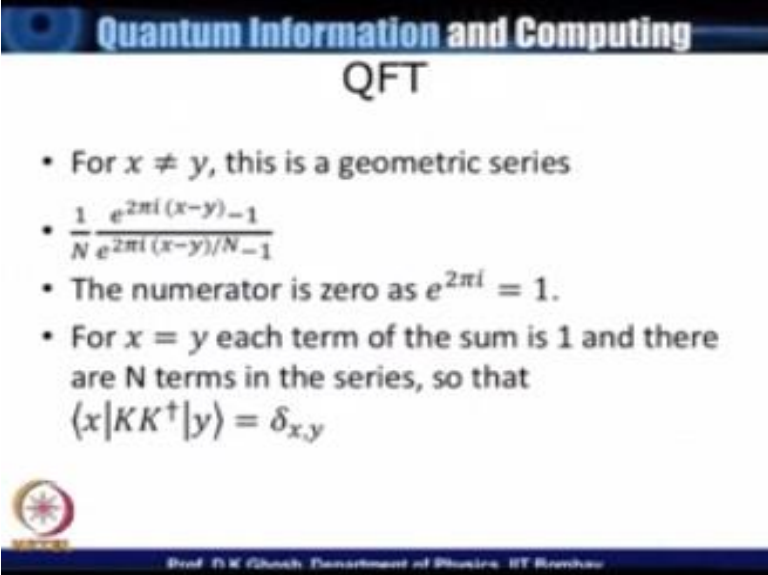
$$\frac{e^{2\pi i (x-y)} - 1}{e^{2\pi i (x-y)/N} - 1} = 0$$

So $x \neq y$, K together y now I evaluate it by the following I will introduce a complete set that is resolution of identity $\sum_{z=0}^{N-1} |z\rangle\langle z| = 1$. I have $\langle x | K | z \rangle \langle z | K^+ | y \rangle$ and since I know these matrix elements by our definition of Fourier transform $\sum_{z=0}^{N-1} K(x, z) K^+(z, y)$ so this is some $\sum_{z=0}^{N-1} e^{2\pi i z (x-y)/N}$ of z y and you can immediately write down that this is equal to $\sum_{z=0}^{N-1} e^{2\pi i z (x-y)/N}$. Now because of the fact that these are finite geometric sums, finite geometric sum.

I can easily compute this sum but before I do that I have to realize that this term x should not be equal to y now if x is not equal to y so that this is not equal to 0 then my common ratio is $e^{2\pi i (x-y)/N}$ times whatever that quantity is. So that my sum will then be to the $e^{2\pi i (x-y)/N} - 1$ divided by $e^{2\pi i (x-y)/N} - 1$, since x is not equal to y the denominator is not equal to 0 but the numerator becomes 0 so therefore this sum becomes 0.


But on the other hand if x happens to be equal to y I had while writing is down add forgotten a factor $1/N$ so I needed to introduce that $1/N$ twice so therefore $1/N$ is there now so this is still equal to 0.

(Refer Slide Time: 23:46)



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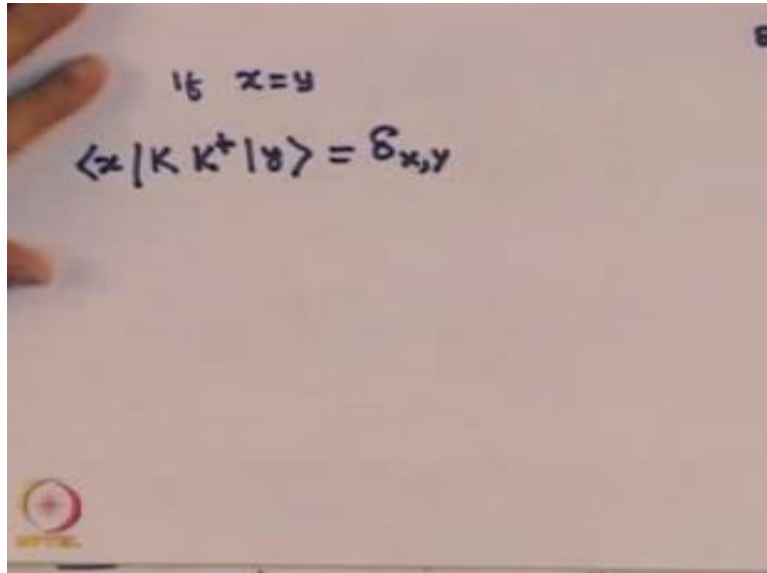
- For $x \neq y$, this is a geometric series
- $\frac{1 - e^{2\pi i(x-y)N}}{N e^{2\pi i(x-y)/N-1}}$
- The numerator is zero as $e^{2\pi i} = 1$.
- For $x = y$ each term of the sum is 1 and there are N terms in the series, so that

$$\langle x | K K^\dagger | y \rangle = \delta_{x,y}$$


Prof. D.K. Ghosh, Department of Physics, IIT Bombay

But if I have x is equal to y note that this term becomes 0 the, in because if it becomes zero then my $e^{2\pi i} = 1$ allowed. So I have got n number of terms each one of them equal to 1 so therefore I get n as a sum and I am a $1/n$ which gives me 1, so therefore if x is not equal to y I get 0 if $x = y$ I get 1.

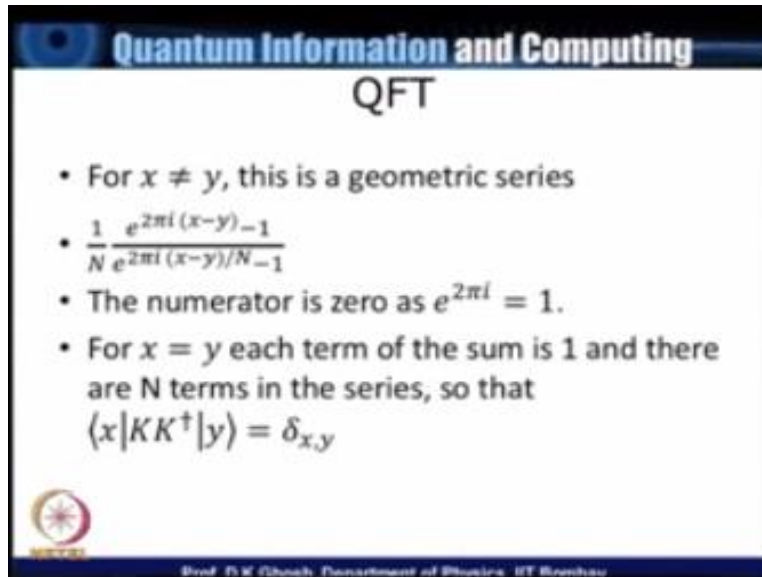
(Refer Slide Time: 24:28)



The image shows a whiteboard with handwritten mathematical expressions. The first line reads $\text{if } x=y$. The second line reads $\langle x | K K^\dagger | y \rangle = \delta_{x,y}$. A small number '6' is written in the top right corner. A logo is visible in the bottom left corner.

So therefore I will say that kk^\dagger matrix element X Y is simply given by the δ of x with y.

(Refer Slide Time: 24:46)



Quantum Information and Computing
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- For $x \neq y$, this is a geometric series
- $\frac{1 - e^{2\pi i(x-y)}}{N e^{2\pi i(x-y)/N} - 1}$
- The numerator is zero as $e^{2\pi i} = 1$.
- For $x = y$ each term of the sum is 1 and there are N terms in the series, so that
 $\langle x | K K^\dagger | y \rangle = \delta_{x,y}$

Prof. P. W. Shukla, Department of Physics, IIT Bombay

So what we have done today is to extend our concept of Fourier transform for real and complex variables continuous variables that we have learnt in physics and mathematics to the case where my variables are discrete because of the fact that we deal with sums rather than integration getting explicit representation of the column is fairly easy at least for those cases where the n value is small.

And we had seen that for n is equal to 2 in space C^2 my quantum Fourier transform is equivalent to doing a Hadamard gate operation, we will see that quantum Fourier transform has a significant role to play in our discussion of Shor's algorithm and as an exercise I will be giving you more examples of quantum Fourier transform in a separate session.

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